The Quantum Dice

An Introduction to Stochastic Electrodynamics

by

Luis de la Peña and Ana María Cetto

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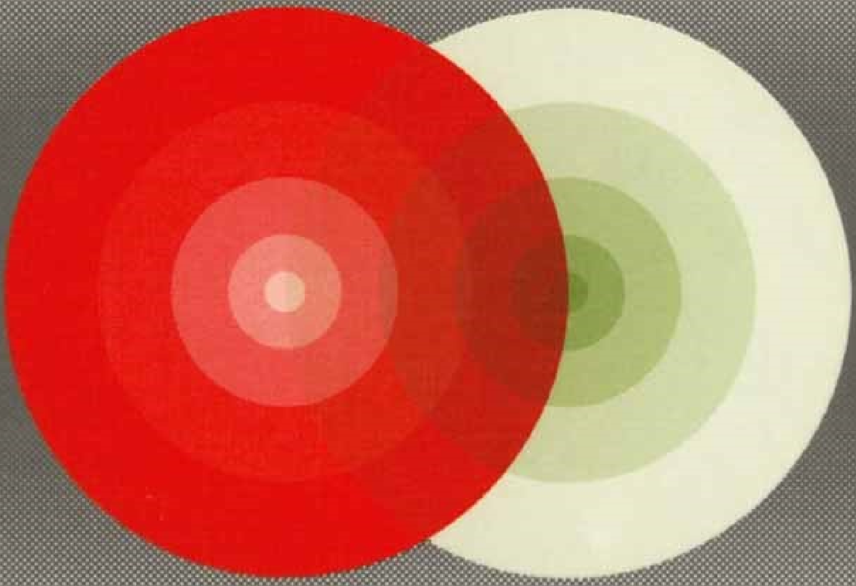

Fundamental Theories of Physics

The Quantum Dice

Fundamental Theories of Physics

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An Introduction to Stochastic Electrodynamics

by

Luis de la Peña

and

Ana María Cetto

Instituto de Física,
Universidad Nacional Autónoma de México,
México

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PREFACE

Physics in the twentieth century has developed towards a world view radically different from the one suggested by classical physics. The new conception has expanded its domains so broadly and deeply as to make it almost impossible to find a single corner of fundamental physics still untouched by the new conceptions. To look again into the physical world with classical eyes has become simply inconceivable, just as it is inconceivable to enjoy modern art from a classical perspective. The strength of this phenomenon can be perceived by recalling the dismissal of Einstein by so many of his mature colleagues, because of his stubborn opposition to accept the new quantum theory —to the establishment of which he himself so decidedly contributed— as a definitive representation of nature.

Now this transition to modern physics was certainly not a gradual process, but a rather tempestuous leap that resulted in new schemes suddenly occupying the place of the classical ones. Especially in the case of quantum mechanics the replacement was so successful that it entailed the sacrifice of those established principles of physics —and even of the philosophy of science— which were seen to be in contradiction with the emerging paradigms; only those clearly compatible with the new paradigms could survive.

It is therefore most surprising that amid the vast ocean of new laws and principles, a most fundamental one deeply rooted in classical conceptions and contrary to the quantum rendering of the world can still be found wandering about, at least in the conceptual discourse. We are referring to the postulate (it is indeed just a postulate) that the solutions of the (classical) Maxwell equations in the absence of sources correspond necessarily to a strict vacuum, to a space devoid of any field. On the other hand, to refer to the fluctuations of the vacuum is as normal and conventional in modern physics as could be, say, to speak of the fluctuations of the pressure of a gas in classical physics. But this means that the solution for the source-free field equations cannot be empty space, so that indeed one may infer that the classical Lorentz boundary condition 'no source – no field', is too narrow to accommodate the richness of nature. Some repair work seems to be required.

The consistent introduction of a vacuum (or zeropoint) radiation field represents a major step outside the classical framework, although it does not contravene the spirit or the foundations of classical physics. The postulate of its existence could therefore provide a possible clue —or the clue?— for a new understanding of microphysics, to which the qualifications of 'classical' and 'non-classical', if used in a broad sense, would equally apply. What sort of physics can it be? How much of it depends on the properties of the vacuum field, and how well can they be determined? How important is this electromagnetic vacuum with respect to other physical vacua?

These are just some of the questions addressed by stochastic electrodynamics. The zeropoint electromagnetic radiation field, which is the central element of the theory, is considered to be a purely random field (because its average is zero); the intensity (or average energy) of its fluctuations is measured in terms of Planck's constant, which appears thus as a new fundamental constant, representing the departure from classical physics. The new theory is therefore expected to differ essentially from the classical, field-free limit, by allowing for the occurrence of fluctuating phenomena characterized by this new constant. In this scheme, for example, the old analysis of the atomic electron spiralling towards the nucleus ceases to apply, because it does not take into account the energy absorbed by the particle from the field, which in principle may compensate for the energy lost by radiation. Is it not possible that we are here revealing the mechanism responsible for atomic stability? Some readers might object that we are pretending to explain atomic behaviour without resorting to quantum mechanics, at the end of the twentieth century! Quite the contrary, the intention is to get a deeper physical understanding of quantum theory itself. This is part of the program that stochastic electrodynamics —at least, at the moments of highest expectations— dreamed for itself long ago. And in saying this there is no exaggeration, since the embryo of the theory can be traced as far back in time as to Planck and Nernst.

In the early (and not-so-early) days of stochastic electrodynamics, optimism prevailed among its (always scarce in number) practitioners, and the hope of arriving at a happy end within a not too long time span was expressed on various occasions, with different degrees of confidence and enthusiasm. As is most often the case, however, reality happened to be more complex than predicted, and around the year 1980 it had become clear that if such program was to make any sense, some major changes in the approach were required. Thus, a period closed and another one, somehow more cautious, was initiated, in which new forms and branches of the theory arose and began to be explored.

Even if still under construction, the results achieved so far by stochastic electrodynamics are not meagre; but they are found scattered and barely systematized, so that a general review of them seems to be timely. At the same time it is true that the developments are still in such a stage, that to tell the problems that have already found lasting solutions from those still in demand of reflection, is a risky task that can have different outcomes.

So the present account is by necessity a very personal one, even if in writing it an effort has been made to reduce the arbitrariness in the selection of topics to a minimum. This book attempts to present an overview as fair as possible of most of the main developments up to the beginning of 1994, to characterize as clearly as possible the stage reached by the theory at the present moment, and to identify the problems that have been solved and how, as well as the ones that are still wanting for a solution and why. To keep the size of the book within reasonable limits has resulted in some topics being covered very briefly, and others touched upon only incidentally or even overlooked; as a partial remedy, the bibliography at the end includes all the works known to the authors that are directly related to stochastic electrodynamics. (In writing this work, the authors have discovered that a complete exposition of stochastic electrodynamics would demand at least two volumes the size of the present one.)

We discuss in detail how near —or how far, according to your enthusiasm for the theory— present-day stochastic electrodynamics is from the golden dream of making contact with modern theory, both quantum mechanics and the much more elusive spheres of quantum electrodynamics and quantum optics. And also, how far —or, again, how near— it is from the even more ambitious goal of offering new perspectives and tools that could help to build a theory free of those incredibly complex and resilient problems that entangle the contemporary quantum description.

Quite independently of personal prejudices in favor or against its relationship with the quantum world and the status of the present quantum theory, stochastic electrodynamics is in itself an interesting physical corpus; we would therefore like to consider this work also as an attempt to present an objective appraisal of it as a theory of nature, and of the possibilities to use it in exploring that part of the microworld which is dominated by electromagnetic interactions.

We hope to offer a systematic and detailed presentation that can serve as a general reference to researchers and as an introduction to graduate students, of the abundant material today scattered in hundreds of papers and a few —by now old— reviews or partial reviews. Certainly not all derivations and discussions given in this book are rigorous enough as to take them as final and able to compete with the elegant theoretical corpus displayed in some of the modern quantum treatises. But also it is true that, even if stochastic electrodynamics is not precisely a newborn theory, it is still in the making, so that the lack of rigour, rather than a shortcoming, is a transitory requirement that allows evading the risks of asphyxia which premature formal demands could bring about.

We recognize that this book is not for everybody. Those physicists convinced that contemporary quantum theory is the final word (the end of the road?), or that there is no reason nor need to dig deeper into the quantum world, will find it irreverent and superfluous. However, we are confident that some physicists —most probably among the younger ones who have not yet surpassed the elastic limit— and philosophers of science, will receive it as a stimulating and promising theorization. Anyhow, stochastic electrodynamics directs our attention to a simple and rich theoretical possibility for a new approach into the quantum world, and the only reasonable way to decide about its intrinsic value and potentialities as a theory of nature, is to try it. What would really be irreverent to Nature would be not to pay due consideration to each one of the possibilities She lays before us, as an invitation to probe further into Her mysteries.

In writing this volume we benefitted from the generous and invaluable help of many colleagues and friends; we would like to publicly express our gratitude at least to some of them. In the first place to Professors E.A. Power and D.G. Larman from the Mathematics Department of the University College, London, for their more than kind invitation and unlimited hospitality that made it possible for the authors to prepare the first draft of the book during a sabbatical leave, and to Dr. P.E. Hodgson of the University of Oxford, who in a most gentle manner took for himself the task to read the whole manuscript and expurgate it from its worst aggressions to the English language. Most significative was the interest of Professors A. Rueda (U. of California) and A. van der Merwe (U. of Denver), who gave us the indispensable final stimulus to decide to embark into such complex task as the present writing. A general reading of the first draft of the manuscript was kindly made by Prof. E. Santos (U. of Cantabria), with the result of a most valuable series of comments and suggestions; our colleagues I. Campos, S. Hacyan, R. Jáuregui and C. Villarreal added their suggestions on particular chapters, and the always witty comments of Dr. A. Kracklauer were highly stimulating. The drawings were produced with the help of the efficient computer program *Metagráfica* developed by our young colleague, the physicist A. Aguilar. Finally, we want to acknowledge the facilities always provided to us by the Instituto de Física, UNAM, as well as partial supports received from the Consejo Nacional de Ciencia y Tecnología and the Dirección General de Asuntos del Personal Académico, UNAM.

Luis de la Peña
Ana María Cetto

May 1995.

Part I

Prelude

CHAPTER 1

QUANTUM MECHANICS AND THE REAL WORLD

> *True, the finite interaction between the object and the measuring devices... implies... the necessity to renounce the classical idea of causality, and a radical revision of our attitude toward the problem of physical reality.* (Bohr 1928)
>
> *I cannot seriously believe in [quantum mechanics] because it cannot be reconciled with the idea that physics should represent a reality in time and space, free from spooky actions at a distance.* (Einstein 1947)

Quantum mechanics is a marvelous theory; its towering successes and amazing predictive power are beyond the slightest doubt. Yet we are still amid the great quantum muddle, as Popper (1967) put it. Physicists and philosophers become puzzled by the deep conceptual and philosophical problems that emerge as soon as they dig for a profound understanding of what quantum theory says about the world and how it says it.

In no other theory of physics do the formal elements of the description receive so many different and even contradictory meanings as the wave function has received; nobody really knows what it is, yet it is used day after day with guaranteed success to solve a remarkable variety of problems and situations ranging from the structure of elementary particles to the giant superconducting magnets. In no other domain of physics we are told that the description of the physical system does not really describe the system, but merely our knowledge of it, that the theory is about measurements and observables and not about *beables* [Bell 1976]. And moreover, that the awareness of our knowledge may suddenly and acausally 'actualize' a system that pervades the whole available space into a single point, thus promoting us from passive and external observers to active participators [Patton and Wheeler 1975]. Further, in no other part of physics does one find oneself in the middle of a complex calculation fighting with infinities that, once subtracted, lead to an answer which is correct up to eight or nine decimal places!

And yet, despite the many insights and renewing proposals that have entered along the years into the discussions on the fundamental problems of quantum mechanics, the basic issues remain as alive and unsolved as they were six decades ago. It is difficult to state it in better words than Bell did: quantum mechanics is an alright FAPP theory —alright *f*or *a*ll *p*ractical *p*urposes. But then one must ask: what is beyond FAPP? [Maxwell 1992].

This book is basically a review of a theory commonly known by the name of *stochastic electrodynamics* (SED, for short), though occasionally it has been called *random electrodynamics.* Although SED may be considered a legitimate physical theory by itself, its sources are so deeply motivated by the problematic features of present day quantum theory, that a brief review of some of the most relevant of those problems constitutes a natural preamble for the exposition of the theory proper. We start with the question of realism, for some a scientific conjecture overwhelmingly confirmed [Gardner 1989], for others and particularly after the advent of the Bell inequalities, a philosophical principle already disproved by experiment.

1.1. Realism and quantum mechanics

Let us consider a box that is divided into two smaller equal boxes L and R by means of a movable wall. Assume that inside the big box there is a (single) particle. This simple system has been used for purposes similar to the present one by many writers, Einstein among them, so that we will refer to it as the Einstein box.[1] A slight variant of the experiment, in which the two boxes can be separated at any convenient time, helps to frame the so-called de Broglie paradox, which will be recalled below. We ask a simple question: Where is the particle? Even though it would be difficult to pose a simpler question, physicists are imaginative enough as to have begotten a full range of answers to it; however, since our interest lies in the fundamental content of those answers, we may abstract the details and reduce them to just the two that catch the main tendencies. So, where *is* the particle?

A) *The conventional description.* A basic tenet of the conventional interpretation of quantum mechanics is that the wave function affords a complete description of each individual system.[2] In the case of the Einstein box,

---

[1] The example is given in a letter by Einstein to Schrödinger of June 1933, as recounted in Moore (1989), chapter 8. A detailed account of this classic problem can be found in Deltete and Guy (1990), so the present discussion will be rather schematic.

[2] It is normally accepted that every physical theory contains at least two components: 1) an *abstract formalism* and 2) a set of *semantic rules*, that may be called collectively *the interpretation* (or the semantics; they are also called operative definitions, epistemic correlations or rules of correspondence). The formalism is the logical skeleton of the theory; though it contains nonlogical, descriptive terms (such as mass, electric field, and so on), it is merely an abstract mathematical structure devoid of empirical meaning. It acquires physical meaning by means of the interpretation that correlates the nonlogical terms in the theoretical model with the empirical quantities or operations they are supposed to represent.

The usual textbook views on quantum mechanics are based on some variant of the Copenhagen or *orthodox* interpretation. Since this interpretation (and indeed, all interthis means that the wave function refers to the one particle inside the big box and the answer to the above question depends on whether we have observed the interior or not. Previous to any observation the state is *completely* described by stating that the probability of the particle being in any of the two boxes L or R is $\frac{1}{2}$; there is no more to that. Thus, the particle is in a state of indeterminate localization (delocalized) in the big box. By looking inside (making a measurement to know its whereabouts) we perturb the system and bring it into a new state, (objectively) localized either in box L or in box R.[3] The transformation of the wave function from the (pre-observation) indeterminate localization state to the (post-observation) determinate state constitutes the *reduction* or *collapse* of the wave function, brought about by the observation. Whether the particle ends up in box L or in box R after the measurement, is a matter of chance.

The assumption that the wave function refers to a single system thus has enormous consequences. Quantities such as Δx (uncertainties in the conventional language) become objective restrictions on the localization of the particle, meaning that there exist intrinsic limitations on the corresponding measurements. So, quantum mechanics goes as far as is possible and physicists must renounce once and for all the hope for a detailed description of the individual. Further, since the concept of probability is being applied to a single event and no sample space can be constructed, there is no consistent way of viewing the result as a property of the system, and it must be interpreted as an uncertainty of our knowledge. The observer slips thus into the description, and the fundamental principle that physics refers to the world rather than to our knowledge of it, is eroded. For some

---

pretations of quantum mechanics) contains in an essential way Born's (1926) probabilistic interpretation of the wave function, and in addition it was strongly influenced by Heisenberg, it would be more properly called *Copenhagen-Göttingen interpretation*, though Wigner (1963) proposed to apply the term 'orthodox' more specifically to the view adopted by von Neumann, as reshaped by London and Bauer (1939). One could also call the former the *customary* or *regular* interpretation, although it is not so clear that the present-day practicing physicist adheres to it in his daily endeavours as tightly as such names may fancy. In a broad sense we call it normally (but not necessarily) the *conventional interpretation*. One should bear in mind, however, that such terms do not refer to a sharp set of precepts, since an ample range of tenets with respect to some of the central interpretative issues can be distinguished among its practitioners. An introductory account of the different interpretations of quantum mechanics and their variants can be found in Bunge (1956), and more advanced expositions by professional philosophers of science are found, among others, in Bunge (1973) and Redhead (1987).

[3] The term *measurement* is common in this context within the conventional interpretation; however, it is an ill-defined concept. It may refer to a perturbation of the system, or to the creation of a result by the measurement, or to a measurement of what was already there. Sometimes it refers to a real physical action on the system, sometimes to a mere change in our knowledge of it; further, the measurement may or may not require an observer, or even his or her conciousness, and so on: *quot homines tot sententiae.*

people this situation is discomforting enough as to have led during recent years to some efforts towards a more objective rendering of the quantum phenomena, in which the observer is no longer needed for the reduction of the wave function, as will be briefly discussed in §1.4.2.

B) *The realist description.* The particle is assumed to have at each moment a set of well-defined, even if unknown, objectively real properties. The wave function ψ is a catalog of the different possibilities associated with the system, and assigns to each of them a certain probability. For instance, in the case of the Einstein box the probability for the particle to be in box L (or box R) is again $\frac{1}{2}$, assuming the boxes to be alike. However, a *complete* description should specify in which of the two boxes the particle is. Hence, the description afforded by ψ is incomplete: the theory contains no element whatsoever to predict in which of the two boxes, L or R, the particle *will be* found under an actual observation. The description maintains its objective character all the time, but it has become merely statistical. Also, since now the wave function refers not to an individual system, but to an ensemble of similar situations (the ensemble required to define the different probabilities), neither its reduction nor an external act of observation or measurement are required to explain the result of a particular observation, which merely shows what is actually happening in an individual instance.

We have thus got two mutually exclusive answers. Which is the correct one, if any? From an ontological point of view, what schools A and B above claim is the preexistence or not of the individual observed states. For the conventional school A, the position of the particle (either in box L or in box R) is materialized or brought into being, as it were, as a result of the observation. The behaviour of the particle recalls that of the cursor on the screen of a computer: it can go from one position to another without ever being in the intermediate positions, it may even not exist before the mouse is clicked. Since the size of box L can be made as small as desired, the localization of the particle by the measurement may be as sharp as desired (even as sharp as an eigenvalue, according to the laws of quantum mechanics). The values of the observables are objectively indeterminate prior to their measurement, and only probable values can be assigned to them; probabilities become fundamental and irreducible. The collapse of the wave function, which is the theoretical counterpart of the changes on the individual systems brought about by the active observer (the big intruder), becomes inevitable.

Let us recall de Broglie's modification of the experiment. We separate the two boxes, take box R without looking into it, to a laboratory three floors below, come back, peep into box L and (say) find the particle in it. According to school A, the wave function of box R, three floors, many meters and tons of building stuff away from L, collapses as soon as we look inside box L. This shows the collapse as a nonlocal and noncausal process, reminiscent of action at a distance in classical physics. By denying the preexistence of the measured properties, by assuming the probabilistic description to be complete and by including the (nonlocal and instantaneous) collapse of the state vector as part of it, the conventional rendering of quantum mechanics becomes a nonrealistic, indeterministic and noncausal interpretation.[4] All this was clearly recognized by Bohr (1928) in his famous Como Lecture of September 1927, a characteristic sentence of which we have used as epigraph to this chapter. To this list of qualifications, one would be tempted to add that of nonlocality, as revealed by de Broglie's box experiment and many other situations. We have however refrained from doing so because nonlocality —as is just the case with the intrinsic statistical nature of the theory— is a much deeper problem, inherent to the structure of the theory and thus common to all interpretations, although subject to different meanings (see section 1.3 below).

Now according to interpretation B, each individual system has a definite (possibly evolving) position, momentum and so on, even if these properties cannot be ascertained simultaneously by measurement on the same system.[5] Not all this information about the individual system is contained in the state vector, so that no complete prediction can be made. However, with the information contained in it, the state vector gives the probabilities for all possible outcomes. In other words, ψ refers not to the individual single system, but to a statistical collection of them or *ensemble* (of similar systems, in the sense of Gibbs and Einstein, usual in statistical physics).

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[4] Realism is a philosophical term to which there correspond many nonequivalent notions; an idea of the rich variety of its meanings may be obtained from Harré (1986). In its broad ontological meaning, (objective) realism postulates that independently of our theories and prior to them, there is an objective reality; in other words, it posits the existence of an independent reality which precedes any effort to disclose it. The task of scientific endeavour is just to disclose the nature of this reality and the laws of behaviour of its things. On the epistemological plane, realism opposes subjectivism; however, there is a rich variety of epistemologic versions of realism. The empiricist viewpoint (adopted by the great majority of writers of conventional texts on quantum mechanics) postulates that our knowledge of the external world originates exclusively in our sensorial perceptions, and that it is not possible to go beyond them (and make inferences on unobservable entities); so, the whole question of an objective reality evaporates as mere speculation. Logical positivism (also known by other names, such as logical empiricism) is represented by a specific empiricist school (the Vienna Circle) that had a deep and extended influence in the development of the interpretation of quantum mechanics. It holds that only the propositions analysable with the tools of logic into elementary propositions that are either tautological or empirically verifiable, are meaningful. (We comment that rational knowledge should not be reduced to a mere logical process, since the creative acts of the mind more often than not fall outside the domain of logic; thus the advance of knowledge rests frequently on (rational) ideas and innovations, the genesis of which cannot be the object of logical analysis. The fact that several logics coexist —traditional, fussy, multivalued, various brands of quantum logic, and so forth—, all of them rational, shows clearly that what is rational may not be logical.)

In the text we use a restricted notion of physical realism which originates in the famous EPR paper [Einstein, Podolsky and Rosen 1935], namely, that the values determined (for the elements of reality) without disturbing the individual system exist prior to the determination. It is a realism of possessed values, according to which the individual systems (to be taken as the fundamental concern of physics) are at all times in well-defined objectively real states [Deltete and Guy 1990]. Thus, for instance, individual systems have objectively real trajectories, even if unknown, and their space-time description should be possible in principle. The meaning given to the terms determinism and causality is discussed in §1.1.1.

Once again, chance enters into the picture in a fundamental way, and the theory becomes essentially statistical, in the sense that it gives a picture that allows merely for statistical relationships among the variables of interest, it does not "represent things themselves, but merely the probability of their occurrence" [Einstein 1933, slightly adapted]. In its essentially statistical nature lie both the strength and the weakness of this *ensemble* (or *statistical*) interpretation.[6]

By recognizing that quantum theory is merely statistical[7] and thus incomplete, this school allows for the possibility of understanding the indeterminism as due to this incompleteness, without the need to give it a more fundamental meaning, as would be an ontological or irreducible indeterminism. This leaves the door open to further studies at a deeper level, a most important point for us. For a realist, who believes that each individual system has always a real state and that the task of physics is to describe such real individual states, an essentially statistical theory is at most provisional.

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[5] Needless to say, traces of quantum particles are observed in cloud chambers and photographic films, from which the trajectories can be determined; but even these are then interpreted as produced by a measurement. To overcome this objection, experiments specifically designed to verify the existence of trajectories have been suggested [see, e.g., Cufaro-Petroni and Vigier 1992].

[6] The ensemble interpretation was first adumbrated by Slater (1929) in the very beginnings of quantum theory; a considerable contribution to its development was given by Einstein from 1935 on [Einstein 1949, 1953a, 1953b]. Among other physicists who embraced it are Langevin (1934), Blokhintsev (1953, 1965), Margenau (1958, 1978), Mott (1964), Lamb (1969, 1978), etc. Expositions of it at different levels may be found in Ballentine (1970), Belinfante (1975) —where it is called 'objective interpretation'—, Ross-Bonney (1975), Newton (1980); a more recent, detailed and comprehensive discussion of the ensemble interpretation is Home and Whitaker (1992). The great majority of textbooks on quantum mechanics are written following the conventional interpretation, mostly implicitly. Among the rare exceptions are the above mentioned book by Blokhintsev (1965), the Russian edition of *Quantum Mechanics* by Sokolov et al. (1962) (in the English translation the relevant parts were omitted), de la Peña (1979), and Ballentine (1989). Despite its conceptual advantages (resolution of the known paradoxes, disappearance of the collapse and of the measurement problem, etc.), but probably due to its limitations (to be discussed below), the ensemble interpretation is frequently neglected by philosophers of science [see, e.g., Hanson 1959, Putnam 1965, Fine 1973]. Others instead take it seriously, as, e.g., Powers (1982) and Harré (1986). A detailed analysis of the conventional interpretation is given in Stapp (1972). A discussion on why Einstein considered quantum mechanics an essentially statistical theory, as opposed to statistical physics, is given in Einstein and Infeld (1938), p.280.

[7] In the EPR article it is shown that the requirement that quantum mechanics give a local realistic description of nature implies that it is incomplete. Essentially the argument goes as follows. The definition of completeness used is that each pertinent element of reality should have its counterpart in the theory. Since, according to the criteria used to identify the elements of reality, in the EPR gedankenexperiment both the position and the momentum of the particle are real, they should have well-defined simultaneous values in the description, which is obviously not the case in quantum mechanics. In an immediate answer to this paper, Bohr (1935) concludes that the EPR results are inadmissible because they are based on a conceptual scheme that violates the quantum

The two pictures A and B differ so widely —they even exclude each other— that at a first glance it would seem a simple matter to demonstrate empirically the fallacies behind one or the other. But almost seventy years have elapsed since the advent of quantum theory and the dichotomy is still around, notwithstanding the endless discussions on the subject.[8] The root of the question at issue is, as should be by now clear, that it cannot be reduced to bare physics and subject to direct empirical tests, due to its strong philosophical content. In Planck's words [Planck 1954], to embrace one of the two alternatives demands from us an act of faith, just the one needed to decide between embracing a positivistic or a realistic philosophy. Without recourse to such an act of faith, or inner voice, or better, without a metaphysical choice, there is no quantum theory, but bare formal mathematical cryptography. Thus, for a realist interpretation A is implausible, to say it mildly (other terms such as aberrant, bizarre, absurd, etc., have been used), while a moderate orthodox considers interpretation B full of unnecessary metaphysics, or just dogmatic; for a more radical orthodox it lacks the space needed for other elements demanded by his world vision, such as the observer and perhaps his mind.

The pragmatic (FAPP) physicist argues that theory A has been used successfully for many years without a single failure, which is a proof of its correctness, and we should therefore derive from it our vision of the world and not the other way round. He therefore expects us to renounce our basic categories of physical thought in order to be able to understand physics [Tambakis 1994], on the basis of a 'quantum syllogism', similar in nature to that used to give theological support to the theory of the epicycles, as Jaynes (1993) puts it. Further, the conventional physicist might add that quantum mechanics describes what can be described, and that importing

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rules. In contrast, Schrödinger (1935a, b) takes the paradox further and even transfers it to the macroscopic scale with his cat paradox (§1.4.1), to show the nonrealistic nature of the conventional description. The subject has been discussed with unending interest since the initial publication of the EPR paper, and the related literature is huge; digested accounts are given in Jammer (1974) and Selleri (1988).

[8] Reviews or reprints of important work, as well as ample lists of references to papers dealing with this subject matter, can be found in De Witt and Graham (1971), Belinfante (1973), Jammer (1974), Nilson (1976), Wheeler and Zurek (1983) and Ballentine (1988).

into the quantum domain knowledge originated in the classical world is dangerous and may lead to contradictions and difficulties. We are told, for instance, that the paradoxes of quantum mechanics come about because we look at the quantum world with classical eyes [Lévy-Leblond 1973], as Bohr alerted us since 1935.

As the reader has presumably done, it is clear that we also have made our choice. We have tried to write this book as consistently as possible within the framework of picture B, but recognizing that, notwithstanding its conceptual advantages, there are still basic questions in quantum mechanics that remain unanswered in this framework. We will find that an advantage of looking at it from the B side, is that one recognizes that there still are many fundamental things to learn about the quantum world and its mysterious workings.

1.1.1. A DIGRESSION ON DETERMINISM AND CAUSALITY

In view of the prodigious variety of meanings that are ascribed in the physical and philosophical lore to the concepts of determinism and causality —so important for the foundations of quantum mechanics—, it seems convenient to define more precisely the meaning that will be attached to these words in the present volume.[9]

Several not entirely consistent notions are frequently blended under the heading of determinism; it refers variously to the type of predictions rendered by a theory, to Laplacian determinism, to philosophical determinism, to instantaneos actions at a distance, and even to the causal links within a system. Here we will understand by physical determinism a property of the *description* of a physical system, not of the system itself. To see what we mean, consider a volume of gas; several physical descriptions of it can be made in principle, with different results as regards determinism, namely,

a) a complete and detailed Newtonian description: totally deterministic;

b) according to classical statistical mechanics, with all fluctuations taken into account: indeterministic;

c) a thermodynamic description in which all fluctuations are suppressed and no reference is made of the mechanical model: this is again a deterministic rendering, though of an entirely different nature.

Laplacian determinism is bidirectional in time, as is model (a) for the gas; it allows for both predictions and retrodictions, which it looks at as related by an inessential time-reversal symmetry operation. This kind of determinism, however, does not correspond in general to reality, which far from being just a reversible machinery is full of unidirectional and irreversible processes. In the real world, myriads of initial states may correspond to a given final state and the possibility of retrodiction is lost. More generally, only a unilateral form of determinism is compatible with random and irreversible processes.[10]

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[9] For our discussion we follow the treatment given in Brody (1993), where the present points of view are developed in more detail. Other points of view, arbitrarily selected from among the vast and contradictory literature on these subjects, may be seen in Bunge (1959, 1973), Bhaskar (1975), Powers (1982), Lucas (1984), Harré (1986), Sosa and Tooley (1993).

In its turn, an extreme form of determinism, associated to full predictability and called philosophical determinism, occurs when the notion of (bidirectional) determinism is extended to the entire universe. This extension leads to the concept of an object without surroundings, something foreign to science and of an entirely unknown nature. Moreover, since the notion is to be applied to the universe itself, and the description should involve and account for the model (including ourselves), self-reference problems arise with such a formulation.

Although different meanings are frequently ascribed also to causality, this term refers to a more precise and direct connection among the elements of the description. The meaning we are giving to it can be exemplified as follows. In the example of the gas, physical relations such as $pV = kNT$ and the like are not causal connections by themselves, but general laws that apply to the given system. Now, when one selects one of the variables entering into the relation and makes on it an arbitrary change (a change not described by the given theory), to be identified as the cause, it is said that the relation gives a causal connection for the corresponding effect of the change. For example, an arbitrary change δT in the temperature causes a change in the pressure (assuming the volume is held constant) $\delta p = \frac{kN}{V}\delta T$; then δp is the effect of δT under constant volume (and constant number of molecules).

It is clear that the general law contains all possible causal links that may be effective in the given system. Since the cause never occurs before the effect and this time ordering is preserved by a Lorentz transformation, it seems more appropriate to consider that causal connections refer to an essential property of the underlying physical reality, rather than merely to a description of it. Thus, causality is better described as an ontological property of the system.[11]

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[10] The determinism of modern digital computers refers to a unidirectional procedure, since in the memory operations previous values are replaced by new ones in an irrecoverable form. This is why they can be used to generate series of random numbers, using only causal processes. With a Laplacian machine this would be inconceivable.

Also, in the physical world, even for relatively simple (but nonlinear) systems, governed by strictly causal laws, the possibility of prediction is lost more frequently than not. Although it has become customary to talk in such cases of deterministic chaos, it would perhaps be more correct to call it 'causal chaos'.

[11] Some supporters of the conventional interpretation of quantum mechanics have considered an undermined notion of causality, referred to as *statistical causality*. Rosenfeld

1.1.2. CAN QUANTUM MECHANICS BE COMPLETED?

A question that arises naturally is whether it is possible to complete the quantum description by embedding it in a deeper deterministic (subquantum-level) theory, so that the quantum states correspond to probability distributions on the set of the subquantum states. The variables that would be required to accomplish such purpose, and that do not appear in the present description, are currently known under the name of *hidden variables.* They have never been very popular (representing a sort of return to superseded classical models), and there are results establishing that no hidden-variables theory can reproduce all the predictions of quantum mechanics, the most well-known ones being the (already obsolete) von Neumann theorem and the (now fashionable) Bell theorem.[12]

For a realistic physicist convinced that the conceptual problems in the quantum domain must have a solution, all this suggests that the more complete theory (if it exists at all) should be a much deeper thing, and that very probably despite the tremendous successes of quantum theory, changes in our present description of the quantum world are required. Looked at from this perspective, a naive hidden-variables program seems clearly insufficient and doomed to failure; what is required is not a mere completion, but the construction of a more fundamental theory that in some limit, or approximation, or averaging, or some other sort of simplification, leads to the present-day quantum description. The following illustrative and well-known remarks by Einstein seem to reflect a similar conviction [for complete references see Deltete and Guy (1990)]:

> "Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the 'Old-one'. I, at any rate, am convinced that He is not playing at dice." [Letter to Born, 1926, in Born (1971), p. 91]
>
> "If in quantum mechanics we consider the psi-function as (in principle) a complete description of a real physical situation, we thereby

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(1953) even asserts that the statistical causality that he sees in quantum theory is already present in thermodynamics. Some realistic physicists have expressed analogous ideas, e.g., Bitsakis (1983) under the name of *statistical quantum determinism* or Selleri (1987, p.125), who calls it *predictability with some degree of inductive probability.* These concepts represent attempts to reintroduce causality into the conventional quantum view, indeed, but at the price ofinterpreting limitations of a description as ontological properties.

[12] Though not absolutely devoid of sense, the term *hidden variables* is not the most fortunate one [Bell (1987) calls it 'a piece of historical silliness']. The subject has been amply reviewed in the literature, some well-known monographies being Belinfante (1973), Jammer (1974) and Nilson (1976). Related theorems are those of Gleason (1957) and Kochen and Specker (1967). In section 13.5 some questions concerning the related Bell theorem are studied.

imply the hypothesis of action-at-distance, a hypothesis that is hardly acceptable. If, on the other hand, we consider the psi-function as an incomplete description of a real physical situation, then it is hardly to be believed that, for this incomplete description, strict laws of temporal dependence hold." [Einstein 1948]

"I think it is not possible to get rid of the statistical character of the present quantum theory by merely adding something to the latter, without changing the fundamental concepts about the whole structure." [Letter to A. Kupperman, 1954]

1.2. What sort of statistical theory is quantum mechanics?

The conclusion that the quantum description should be considered statistical can be reinforced with a few simple considerations. One is that an eigenstate of the Hamiltonian cannot give a description of the individual particle in the classical limit, because it is stationary; neither does a wave packet, since it acquires an unlimited spatial dispersion as it evolves. Indeed, it is well known that except for a few cases (such as a coherent packet of harmonic oscillators or the Airy packet of Berry and Balasz (1979)), the quantum wave-packets are dispersive, which makes it difficult to put them into correspondence with classical particles that retain their identity. However, in the statistical interpretation the problem is naturally solved by ascribing to the wave packet a dispersive ensemble. Since the operation of taking the limit cannot change the nature of the problem, one concludes that in the classical limit the quantum mechanical description corresponds to a statistical ensemble. This argument was advanced for the first time by Einstein (1953a) —triggering, by the way, a painful and unforeseen discussion with Born, in whose honour the paper had been written— and can be found today scattered in the literature (e.g., the above cited paper by Berry and Balasz) and even in textbooks that embrace the conventional interpretation [Messiah 1959, vol. 1, chapter VI]. The argument has been used to assert that the classical limit of quantum mechanics is not classical mechanics, but classical statistical mechanics [Yvon 1967, Note I; Ballentine 1989].[13]

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[13] As is well known, the classical limit is defined in quantum mechanics as the result obtained when $\hbar \to 0$ or when the quantum numbers $n \to \infty$. However, frequently both limiting processes are simultaneously required under constraints such as that $J = n\hbar$ remains fixed and equal to its classical value. For example, the formula for the energy levels of the H atom, $E_n = -Z^2me^4/2\hbar^2n^2$, goes to the classical solution $E = -Z^2me^4/2J^2$ in the last instance, but to ∞ or 0 in the first two cases, respectively. This is conventional quantum folklore; reality has proven to be more complex, and which is the proper classical limit remains as yet unclear. It is possible, for instance, to show that the chaotic features of classical nonlinear systems cannot be recovered from the corresponding quantum systems by a limiting procedure. An illustrative example of such behaviour has been

Of course, by accepting the ensemble interpretation one leaves aside any pretension to use quantum mechanics for a complete description of the individual behaviour; the theory is simply not designed for that purpose. This resembles the situation occurring in the theory of Brownian motion, say, where the use of statistical methods does not undermine the utility and efficacy of the theory, but rather enhances them by affording a usable description.

1.2.1. A SECOND DIGRESSION: ON THE NATURE OF PROBABILITY

Several notions of probability coexist, all in use in the physical literature and subject to much philosophical discussion. In view of the basic role played by the concept of probability in quantum theory, it seems appropriate to make another detour from the main course and give a brief account of its most important interpretations found in the physical literature. The discussion that follows is much inspired by that of Brody (1993), where the reader can find further details and references.[14]

a) *The formal viewpoint.* The notion of probability is restricted to the mathematical formalism (usually, the set of Kolmogorov axioms), without further assumptions. Although relatively common among physicists, and quantum theorists in particular, it is obviously insufficient, since the axioms and ensuing theorems constitute a yet incomplete formulation that assigns neither meaning to the concepts nor numerical values to the probabilities in each particular instance of application; they do not even specify when the notion of probability has sense. One can even argue that calling such description an *interpretation* demands some stretching of the word.

b) *The subjective interpretations.* Probability is attributed not to events happening in the real world, but to propositions, and its value corresponds to the intensity of their acceptability. Some (Keynes (1921) among them) demand that the acceptance of the proposition be rationally defendable and common to all those who consider the problem. For others (as de Finetti (1974)) the very fact that the proposition is dubious makes it impossible to look for a general agreement about it, and thus its acceptability should be individual and not subject to verification or revision through rational argumentation. One is allowed to assign probabilities to single events, such as 'the probability that my preferred team wins the next football game'. It is not clear, however, why our beliefs and opinions should satisfy Kolmogorov's axioms, nor how a subjective but convincing concept of conditional probability, for instance, can be formulated.

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given by Mantica and Ford (1992).

[14] Other discussions can be found, e.g., in Bunge (1970), Lucas (1970), Rédei and Szegedi (1989), Home and Whitaker (1992).

c) *The frequentist* or *objective interpretation.* According to this interpretation, proposed and developed by von Mises (1957) and Reichenbach (1949), among others, a series of (experimental) observations is made and the relative frequency of an event is determined; its probability is taken as the value attained in the limit when the number of cases in the series tends to infinity. Now we are dealing with events, not with propositions or opinions, and the determination of the relative frequency is a physical, objective process. Unfortunately, there are still problems: if experimental frequencies are used, the infinite limit is unattainable and does not exist; if the relative frequency is a theoretical estimate, then the limit is probabilistic and the frequentist definition becomes circular. Moreover, the theoretical structure lacks an apparent experimental counterpart: why should the experimental relative frequencies correspond to the theoretical estimates?

d) *The ensemble interpretation.* There are still other interpretations and varieties of probability, as those of Carnap (*b* and *c* above are two different probabilities p_1 and p_2 that coexist and are both required, depending on the case), Popper (probability as a *propensity*), Heisenberg (probability as an Aristotelian *potentia*), etc. However, a very important view on probability that is much extended among physicists and to which we turn now our attention, is the ensemble interpretation [Brody 1975, 1993, particularly chapter 10].

Let us first recall the usual concept of *ensemble.* Each theoretical model of reality should be in principle applicable to all cases of the same kind, i.e., to all cases where the properties of the system considered by the model are equal; the factors neglected by the model may fluctuate freely, but in consistency with the physical laws. The set of all these cases constitutes an ensemble. The notion of ensemble as a set of theoretical constructs can thus be established without recourse to the concept of probability, and can be structured so as to possess a measure, which is then used to define averages over the ensemble; this makes it possible to introduce the ensemble concept of probability as follows. Let A be a property of interest and let χ_A be the indicator function of A, i.e., $\chi_A(\omega) = 1$ if member ω of the ensemble has it, $\chi_A(\omega) = 0$ if not. Then the probability of A is the expectation over the ensemble (the average) of $\chi_A(\omega)$,

$$\Pr(A) = \int_\Omega \chi_A(\omega) d\mu(\omega), \tag{1.1}$$

where $\mu(\omega)$ is the measure function for the ensemble, usually normalized over Ω, the range of the events ω. It is possible to show that this definition satisfies all the axioms of Kolmogorov (1956), so that indeed the ensemble can become the basic tool for probabilistic theorization.

The experimental counterpart of this probability is the relative frequency as measured in an actual (and of course finite) series of experiments. If the relative frequencies thus measured do not correspond to the theoretical estimates, the ensemble (the measure) should be redefined until agreement is reached through the appropriate research work. Here there is no global solution. Of course, as is the case with any other physical quantity, theoretical probabilities and their experimental values need not necessarily be exactly the same. Also here it is impossible to apply the notion of probability to a singular case. Thus, for example, the old philosophical problem of the probability of a given theory being true, becomes meaningless, as meaningless as the notion of a wave function for the entire Universe —unless one believes that there are infinitely many equivalent universes. The most interesting aspect of the ensemble notion of probability is its direct correspondence with the concept used by physicists in their daily undertakings, so that we adhere to it in the present volume.

1.2.2. PROBLEMS WITH THE ENSEMBLE INTERPRETATION

The acceptance of the ensemble interpretation of quantum mechanics leads to a further question, namely: is the quantum mechanical description really a statistical one? Unfortunately the ensemble interpretation is far from being free of difficulties, and there persist some unresolved problems on a very fundamental level. Not only does a coherent and finished statistical formulation of quantum mechanics not exist, but objections have been advanced to the extent that it has recently become fashionable to consider such a formulation as empirically disproved (about this there is much to say later on; see section 13.5). For example, results such as those of Kochen and Specker (1967) show that although each observable can in itself be considered as a classical random variable, two incompatible observables (corresponding to noncommuting operators) cannot be viewed simultaneously as classical random variables defined on the same space of events, independently of the specific context. A consequence of this is the nonexistence of a (context-independent) joint distribution of such variables [Suppes and Zanotti 1981]. Some examples can serve to further illustrate this kind of difficulties.

To avoid the use of involved mathematics, let us retake a simple example considered by Mermin (1990), which despite its simplicity reveals the nature of the problem under discussion.[15] Consider a dynamical variable $f(\hat{A}, \hat{B})$ that is a function of two commuting observables $\hat{A}$ and $\hat{B}$. It is well known that if there exists a simultaneous eigenstate of $\hat{A}$ and $\hat{B}$, so that $\hat{A}\psi = a\psi$ and $\hat{B}\psi = b\psi$, then for such state one can assign to f the value $f(a, b)$, in the sense that $f(\hat{A}, \hat{B})\psi = f(a, b)\psi$. One might naively want to extend this property to any state by assuming that a) in any state one may assign numerical values a and b to the commuting variables $\hat{A}$ and $\hat{B}$ (as if the corresponding measurements were performed), and b) these values satisfy the same functional relationship $f(a,b)$ fulfilled by the operators. To show that the combination of these two assumptions is in general incompatible with quantum mechanics, Mermin makes use of a pair of spin $\frac{1}{2}$ particles in any possible spin state, and thus of a four dimensional Hilbert space (the conclusion holds in general for Hilbert spaces of dimension three or higher). Consider the following nine operators

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[15] For particularly clear detailed discussions of these important topics see Mermin (1993) and Peres (1993).

$$\begin{array}{ccc} 1 \otimes \sigma_z & \sigma_z \otimes 1 & \sigma_z \otimes \sigma_z \\ \sigma_x \otimes 1 & 1 \otimes \sigma_x & \sigma_x \otimes \sigma_x \\ \sigma_x \otimes \sigma_z & \sigma_z \otimes \sigma_x & \sigma_y \otimes \sigma_y \end{array} \tag{1.2}$$

each one with eigenvalue +1 or −1. The array is such that the three operators of each row and of each column commute, and furthermore, each operator is the product of the other two, except for a minus sign in the last item of the third column, $(\sigma_z \otimes \sigma_z)\,(\sigma_x \otimes \sigma_x) = -\ \sigma_y \otimes \sigma_y$. This minus sign makes it impossible to assign to each one of the nine entries definite numerical values (+1 or −1, to be interpreted as results of possible measurements) obeying the same multiplication rule as the operators themselves. Thus, only a proper subalgebra of the dynamical variables of the quantum system can possess actual values at a given moment, the rest of the observables remaining indeterminate.

One concludes from this example that in quantum mechanics the meaning of sentences such as 'the result of a measurement of $\hat{A}$' depends in general not only on $\hat{A}$ and the system under measurement, but on the full context of the problem. Or, in even simpler terms, that observables are not beables. Unfortunately, we do not really know what they really are.

Of course, by allowing for noncommuting operators things become only worse. Consider for instance the linear correlation C_{QP} of two noncommuting observables $\hat{Q}$ and $\hat{P}$.[16] Quantum mechanics contains no rule to fix it uniquely, so that no generally accepted definition of C_{QP} exists for arbitrary observables $\hat{Q}$ and $\hat{P}$. It is sometimes defined by analogy with its classical counterpart as $C_{QP} \equiv \frac{1}{2} < \hat{Q}\hat{P} + \hat{P}\hat{Q} >$ [see, e.g., Bohm 1951], but it can be defined more generally as $\frac{1}{2} < \hat{Q}\hat{P} + \hat{P}\hat{Q} > + i\eta\frac{1}{2} < \hat{Q}\hat{P} - \hat{P}\hat{Q} >$, with η a real constant; this expression has the correct classical limit and satisfies all statistical requirements, but it remains arbitrary. Now consider, in particular [Claverie and Diner 1976a], a system with only two eigenstates of energies E_0 and E_1, connected by transitions, and let $\hat{Q} = \hat{Q}(t_1)$ and $\hat{P} = \hat{Q}(t_1 + t)$, where $\hat{Q}$ can only take the values $+\frac{1}{2}$ and $-\frac{1}{2}$; then it follows that $C_{QP}(t) = \frac{1}{4}\cos\left((E_1 - E_0)\,t/\hbar\right)$. In usual statistical terms, such a correlation function corresponds to sample functions that are continuous sine curves, which is contrary to the assumption that the variables Q and P are dichotomic. Thus, either the definition does not correspond to a true correlation function, or the quantum description is not ruled by statistical laws.

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[16] A more complete discussion can be found in Cohen and Zaparovanny (1980).

Despite all these problems, statistical descriptions of the quantum system in phase space exist and are of real value, the best known one being the Wigner function.[17] Due just to the said problems, it happens that such phase-space functions are generally not everywhere positive (they are hence frequently called pseudo-probabilities). In particular, for one-dimensional pure states the Wigner function is nonnegative for Gaussian states only [Urbanik 1967, Hudson 1974, Piquet 1974], and the same has been shown to apply to the n-dimensional case [Soto and Claverie 1983a, b]. Further, Cohen (1966) and Wigner (1971) demonstrated that positive phase-space distributions bilinear in the wave function (which is the case with all usual distributions) and compatible with some very basic rules of quantum mechanics, do not in general exist. However, if the requirement of bilinearity is lifted, positive distributions do exist and can be generated at will [Cohen and Zaparovanny 1980].

It is important to recall that each quantum phase-space description is associated with a well-defined *correspondence rule*, which means that a specific ordering rule for the noncommuting operators $\hat{x}$ and $\hat{p}$ must be used to translate the classical functions of the phase variables x and p (c-numbers) into the corresponding quantum observables [see, e.g., Shewell 1959, Cohen 1966]. But, as we know, unique correspondence (association) rules are not part of quantum mechanics, and thus there is ample room for arbitrariness. The acceptance of a correspondence rule may even demand abandoning other more fancy quantum rules, to avoid inconsistencies. As an illustration, let us refer to an example that will appear several times in this book.

According to usual quantum rules the eigenstates of any Hermitian operator $\hat{F}$ correspond to well-defined (dispersion-free) values of F; for instance, the ground state of a harmonic oscillator of frequency ω has a *fixed* energy of value $\mathcal{E}_0 = \frac{1}{2}\hbar\omega$. Now, the Wigner function for the ground state of the harmonic oscillator is everywhere positive and leads to a fully acceptable statistical description; it predicts of course the same average energy E_0 and the same Gaussian marginal probabilities in x and p spaces as the usual description, but it also predicts a significant dispersion of the energy, of the order of $\mathcal{E}_0^2$. Which of the two is the right result?

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[17] The first phase-space description of a quantum system was made by Weyl (1927); the Wigner function was introduced some years later by Wigner (1932). The theory of the latter was substantially developed by Moyal (1949); ample reviews of it can be seen in Tatarskii (1983) and Hillery et al. (1984). General formulas for the quantum phase-space distribution (which apply to the Wigner function as a special case) are given in Cohen (1976) and Cohen and Zaparovanny (1980). Some aspects of the theory of quantum distributions (mainly from the point of view of quantum optics) are discussed in chapter 13.

The problem of the sign of the Wigner distribution is sometimes avoided by judiciously (but arbitrarily) smearing it over phase-space cells of volume of order $\hbar$; a Gaussian coarse-graining leads to the so-called Husimi distribution [Husimi 1940; Cartwright 1976]. However, the time evolution of the resulting wave packets may be at variance with some quantum laws [see, e.g., Nauenberg and Keith 1992].

A possible answer is that they are both right, but refer to different things. The Hilbert-space formulation contains the rule $\hat{F}^n = (\hat{F})^n$, so that, in particular, $<\hat{H}^2>=<\hat{H}>^2$. By contrast, in the Wigner formulation all operators are to be ordered according to Weyl's rule of correspondence $q^n p^m \rightarrow \frac{1}{2^n}\sum_{l=0}^{n} C_{nl}\hat{q}^{n-l}\hat{p}^m\hat{q}^l$ with C_{nl} a binomial coefficient; this gives for $\hat{H}^2$ an operator that differs from $(\hat{H})^2$ and assigns to the energy a dispersion different from zero, even for the ground state. Note that this change in the *definition* of $\hat{F}^n$ does not produce any change of the wave function (which determines the Wigner function); only the *meaning* of (the powers of) the operators is changed. A direct calculation shows that for the ground state of the harmonic oscillator the quantity $\langle \hat{x}^2\hat{p}^2 + \hat{p}^2\hat{x}^2 \rangle$ entering in the evaluation of $< \hat{H}^2 >$ is negative (!), which explains quantitatively the difference between the two results.

Unfortunately the problem has not received a final answer; so that the use of one formalism or the other is a matter of opinion and preference, even if the dominant use of the Hilbert-space approach, added to the difficulties of the phase-space formulation with positivity, have contributed to favour the former.[18] Indeed, the dispersionless eigenstates of the Hamiltonian are among the most celebrated quantum predictions; just look at the emission spectra of atoms and see the wonderfully sharp lines! This is true, of course, but it only shows that the frequency involved in a transition is (almost) dispersionless, and does not necessarily mean that the energy of each state is in itself a fixed quantity. If, for instance, atomic transitions involve highly correlated states, or are due to sharp resonances, one might get, at least in principle, neat emission spectra, even with dispersive energy states.

Negative probabilities appear in quantum mechanics not only in connection with phase–space distributions, but already as a result of the superposition principle, according to which amplitudes of probability, and not probabilities themselves, are to be superposed. They can interfere destructively and give rise to negative contributions to the probability densities, of a highly nonclassical nature. These results have led to a widespread ac-

[18] Julg (1988) has remarked that the dissociation energy of rigid diatomic molecules depends on the dispersion of the ground-state energy, which might allow for an empirical decision between the different ordering rules.

ceptance of negative probabilities as a necessary evil in quantum theory.[19] What we conclude is that neither the usual Hilbert-space formulation nor the phase-space (Wigner) formulation are fully consistent with a probabilistic interpretation. Thus we face yet another one of the astonishing quantum puzzles: even if quantum mechanics is an essentially statistical theory, it is not a genuine statistical theory.

1.3. Locality and separability

A most important question is that of quantum locality and separability. Also here, the usage of the terms is far from universal; they are sometimes treated as synonyms, at other times a distinction is made between them, but what is understood as locality in one place is called separability in other places; further, different notions of locality have been given, as is discussed in detail in Redhead (1987). It seems thus convenient to point out the meaning to be given here to these terms. By *separability* we refer to the notion that spatially separated individual systems have their own objective real states, each one independent of anything that happens to another one at a given time [Deltete and Guy 1990]. *Locality*, in its turn, means that any mutual influence between spatially separated individual systems propagates with a speed less than or equal to the speed of light. This notion of locality recalls the field-theoretical notion of microcausality, i.e., the postulate that space-like events cannot affect each other. Pre-relativist physics violates locality since it allows for actions at a distance, the most conspicuous one being gravity.[20] Further, we refer to the joint notion (locality+separability) as Einstein locality, though when no confusion arises or the distinction is unimportant, it will be simply called locality. Notice that under the assumption of separability, any prediction that can be made about one individual system by looking at another one is strictly due to correlations established in the common past.

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[19] The acceptance of negative probabilities implies a fundamental change in the axioms of probability theory. Since "they are well-defined concepts mathematically, which like a negative sum of money ...should be considered simply as things which do not appear in experimental results" [Dirac (1941); see also Feynman (1982, 1987) and the detailed discussion in Mückenheim (1986), where they are called *extended probabilities*], they tend to be pragmatically accepted, even if this renders the meaning of probability unintelligible. Once this door is open, anything may happen; thus, for instance, imaginary probabilities have been considered to reconcile quantum theory with locality [Ivanovic 1978].

[20] Field theory, from Faraday onwards, was the answer of theoretical physics against actions at a distance; as Maxwell puts it (1873): "Faraday saw a medium where [the mathematicians] saw nothing but distance". The introduction of the notion of field was perhaps one of the most revolutionary steps of 19th century physics. Analogously, it was the deep dissatisfaction with actions at a distance in the Newtonian theory of gravity that gave Einstein his strong drive to look for a more convincing theory of gravitation, thus eradicating all kinds of actions at a distance from mechanics. One could speculate that the requirement of absence of actions at a distance could have been the cornerstone for Einstein's construction of special relativity. Their resurrection by the conventional interpretation of quantum mechanics was one of Einstein's strong motivations against this interpretation (see epigraph to the chapter).

The question whether in quantum mechanics locality as relativistic causality can be violated, has a long history; a recent example is the following. A now traditional test of quantum mechanics against faster-than-light effects, proposed by Fermi as far as 1932, consists in considering two atoms A and B, separated a distance R, so that at time $t = 0$, A is in an excited state and B in its ground state. Causality (locality) demands that the deexcitation of A does not affect B before a time $t \geq R/c$; a standard calculation shows, however, that if the effects are described by the corresponding transition probabilities, the state B begins to be altered by the deexcitation of A practically from $t = 0$ [Hegerfeldt 1994]. This conclusion has been challenged by Buchholz and Yngvason (1994) on the ground that the transition probability test is not adequate for a thorough discussion of causal effects, and what is required instead is a consideration of the expectation values of appropriate localised observables. It is startling to observe the deep concern about quantum causality (rightly) manifested by quantum theorists, taking place simultaneously with the widespread (almost victorious) claims that the violation of locality by quantum systems by many standard deviations has been experimentally verified with pairs of photons produced by atomic cascades, as will be discussed in section 13.5.

On quantum nonseparability

In quantum mechanics the property of separability does not hold in the normal sense of probability theory. The usual example to illustrate this is the singlet spin state vector of a system of two identical spin one-half particles, given by

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}\Big[|+\rangle_1\,|-\rangle_2 - |-\rangle_1\,|+\rangle_2\Big], \tag{1.3}$$

where the first ket refers to particle in position x_1 (to the left, say) and the second one to particle in position x_2 (to the right, perhaps far away from the first one). This is an example of an entangled state vector describing a correlated system, characterized by the fact that it cannot be factorized, and broadly used in connection with the Pauli principle.[21] In the first term in equation (1.3) the particle at x_1 carries spin up and the particle at x_2 carries spin down; in the second term the spin orientations are interchanged. Let us assume that both particles are so far apart from each other that they do not interact any more, and so the spatial wave function factorizes. Then all the entanglement comes from the spin factor, which carries with it the correlations established in the common past of the two particles, when they constituted a composite system. One can extract some consequences of this entanglement by studying the corresponding density matrix, which is (leaving aside the spatial dependence, as is usually done),

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[21] The notion of entangled states was introduced by Schrödinger (1935a) during his search for a realistic description of the quantum world; he found them particularly distressing. Accessible discussions on the present status of the problem of entanglement are given in Greenberger et al. (1993) and Mermin (1994).

$$\hat{\rho}_P(1,2) = |\Psi_0\rangle\langle\Psi_0| = \hat{\rho}_M(1,2)$$

$$-\tfrac{1}{2}\Big[(|+\rangle\langle-|)_1\,(|-\rangle\langle+|)_2 + (|-\rangle\langle+|)_1\,(|+\rangle\langle-|)_2\Big], \qquad (1.4)$$

where

$$\hat{\rho}_M(1,2) = \tfrac{1}{2}\Big[(|+\rangle\langle+|)_1\,(|-\rangle\langle-|)_2 + (|-\rangle\langle-|)_1\,(|+\rangle\langle+|)_2\Big]. \qquad (1.5)$$

The operator $\hat{\rho}_M(1,2)$ alone is the density matrix describing a 50-50% *mixture* of two independent particles, one with spin up and the other with spin down; it represents what one would expect in a classical statistical description when each subsystem has well-defined properties (particles in x_1 in state up and particles in x_2 in state down in the first term, and vice versa in the second one, with both possibilities having equal chances). On the other hand, the extra contributions within square brackets in (1.4) show the surprising quantum mechanism describing the correlations between the partner particles, which is a result of working with amplitudes rather than probabilities and has no classical analog. Within the conventional interpretation these terms illustrate the impossibility of assigning a definite value to each of the (relevant) properties of *each* particle. In each of these two terms the particles at every position appear as having simultaneously both spin up and down. One can only assign a well defined property to *anyone* of the component subsystems (*one* particle has spin up, *the other* spin down, but one cannot identify them [Ghirardi et al. 1977]), and, of course, also to the whole system (the total spin projection is zero).

If one holds fast to the statistical meaning of the state vector, things become less catastrophic because nothing in the description refers to individual particles, but only to correlated subensembles. It is quite admissible to consider simultaneous subensembles with their elements having different properties (of course, formed with different particles), but one should not draw conclusions for the individual from the statistical description of the whole ensemble. For systems in the pure state described by equation (1.3), if a measurement of the spin projection of one particle gives spin up, say, this means that the other particle would show spin down projection; we know this for sure and instantaneously, even if the partners are 1000 km apart, and without in any way affecting the partner particle.[22] It is clear that the only possible way to get a realistic (separable) understanding of this description is by using the ensemble notion: in some instances (indeed, half the times in the long run) the particle at x_1 has spin up and the partner has spin down, and in the rest of the cases the spins are interchanged. Something similar happens in the case of the mixture, but without the possibility of associating the particles into composite systems, i.e., pairs of partners. The difference can be observed only through the study of variables sensitive to the correlations, such as $(\vec{\sigma}_1 \cdot \mathbf{a})(\vec{\sigma}_2 \cdot \mathbf{b})$ for the spin case, where **a** and **b** are unit vectors in arbitrary directions. With the discovery of the Bell inequalities, this peculiarity of quantum mechanics has been at the center of the debate on the foundations of quantum theory for the last 25 years, as discussed in section 13.5. But this does not mean that we have got a better understanding about the origin or meaning of the strange correlations behind the entanglement.

We verify the correctness of the above statement by noting that the density matrix for particle 1, say, which can be derived by taking the partial trace of the two-particle density operator (1.4) with respect to particle 2, is

$$\begin{aligned}\hat{\rho}(1) \equiv \mathrm{tr}_2\hat{\rho}(1,2) &= \langle +|_2\, \hat{\rho}(1,2)\, |+\rangle_2 + \langle -|_2\, \hat{\rho}(1,2)\, |-\rangle_2 \\ &= \tfrac{1}{2}\left[(|+\rangle\langle +|)_1 + (|-\rangle\langle -|)_1\right]. \qquad (1.6)\end{aligned}$$

This density operator shows that indeed particles at x_1 have equal chances of being in each one of the two possible states $|+\rangle_1$ and $|-\rangle_1$. Incidentally, the example serves to illustrate the general rule that the partial trace of a density matrix describing a pure state corresponds to a mixture.

More on quantum nonlocality

The question of locality in quantum mechanics has still other facets. Nonlocalities enter into conventional quantum mechanics through different gates, and one can distinguish at least three sources for them, each one giving rise to a particular form of nonlocality, or apparent nonlocality. The simplest one is almost obvious; it comes from the fact that the values assigned to observables are average values, and these depend on the value of the wave function in the entire available space. Thus $\hat{H}$ specifies the system, but $\langle H\rangle = \int \varphi_n^*(x)\hat{H}\varphi_n(x)dx$ characterizes the state and depends on the values of $\varphi_n(x)$ for all x. Something similar occurs of course in any statistical description. Consider for example the expression $m\left(d^2\langle x\rangle/dt^2\right) = \langle F(x)\rangle$, belonging not only to classical statistical mechanics, but also to quantum mechanics, where it is called Ehrenfest theorem; it says that the acceleration of a 'particle' at point $\langle x\rangle$ depends on the force at all points in space. Such 'nonlocalities' are clearly a product of the description and do not reflect any nonlocal physical phenomenon; they become unfathomable if one insists in considering statistical expressions as referring to individuals. For instance, the opening of a door at the far end of the room, by affecting the density ρ, modifies the average force 'acting' on such a 'particle'.[23]

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[22] Recall that this is just the criterion proposed in EPR to identify the elements of reality. So the spin down of the partner particle is identified as an element of reality. However, neither $|\Psi_0\rangle$ nor $\hat{\rho}_P$ allow one to predict this value, but only to assign to it a probability $\frac{1}{2}$. Thus, the description provided by the wave function is incomplete, and we come back to the substance of the EPR argument.

A second and more interesting example is the following. In the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi \tag{1.7}$$

we set

$$\psi=Re^{iS/\hbar}, \tag{1.8}$$

with R and S real functions of x and t. After separating real and imaginary parts and writing $\rho=|\psi|^2=R^2$, one gets

$$\frac{\partial\rho}{\partial t}+\frac{1}{m}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla}S)=0, \tag{1.9}$$

$$-\frac{\partial S}{\partial t}=\frac{1}{2m}(\boldsymbol{\nabla}S)^2+\left[V-\frac{\hbar^2}{2m}\frac{\nabla^2R}{R}\right]. \tag{1.10}$$

With the further substitution $\mathbf{j}=\rho\boldsymbol{\nabla}S/m$ and hence $\mathbf{p}=\boldsymbol{\nabla}S$, the first of these equations may be recognized as the continuity equation describing the local conservation of particles. The second equation is a little more tricky to identify, but a frequent procedure is to compare it with the Hamilton-Jacobi equation for Hamilton's principal function S, with its time derivative giving the energy and its gradient giving the linear momentum of the particle. For this to agree with the usual description of a classical particle [see, e.g., Goldstein 1980], all one needs is to consider the term within square parentheses as an effective potential, given by the sum of the external potential V plus another one, of obvious quantum nature, equal to $\left(-\hbar^2/2m\right)(\nabla^2\sqrt{\rho}/\sqrt{\rho})$.

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[23] The operation of averaging changes the nature of the quantity being averaged; thus, the average citizen of a country has a noninteger number of parents, of brothers and sisters, and even of eyes and legs. Upon averaging, the kinetic energy of the molecules of a gas becomes a temperature, a concept that cannot be applied to an individual molecule and is even foreign to mechanics. The process of averaging may even change the nature of the theory itself, giving way to new quantities and concepts and new laws relating them. In the case of the gas, for instance, it leads to the concepts of equilibrium, pressure (average momentum transfer), temperature (average kinetic energy), and so on, and then to the relations between them (ideal gas law, law of distribution of velocities, etc.). Even if quite obvious, this fact is frequently neglected, thus giving rise to paradoxical conclusions.

This extra term, called *Bohm's potential* or *quantum potential*, is clearly of a nonlocal nature, since it depends on ρ, a quantity determined by the whole setup, including boundary conditions and the like. Thus the resulting 'classical' particle is at each point in space affected by all its surroundings. This is the reason for saying that Bohm's (1952) interpretation of quantum mechanics is (intrinsically) nonlocal.[24] Once again we see that nonlocalities are present as long as one insists in applying the quantum description to the individual system.

Now, quite independently of the interpretation adopted there is an irreducible nonlocal element in quantum mechanics, clearly exhibited in the experiments of diffraction and interference with electrons, neutrons and other particles (including atoms or even molecules). For instance, the observed properties of *all* the particles impinging on the detector of an interferometer can be modified by introducing changes along the path of a *single* one of the interfering beams [Rauch 1986; see also Greenberger 1983]. Nonlocal phenomena of this kind obviously have to do with the wavelike behaviour of the quantum system, and therefore only by understanding the source of the undulatory phenomenon one can hope to explain them without ad hoc phenomenological recourses such as self-interference and the like. We will have opportunity to revisit this most important subject from within SED in chapter 12.

A simple way to show that even single-particle quantum systems may exhibit nonlocalities is by resorting to the Wigner description. The Wigner function $W(\mathbf{x},\mathbf{p})$ satisfies a complicated integro-differential equation that can be cast in the form [Hillery et al. 1984]

$$\frac{\partial W}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla W-\int d^N p' W(\mathbf{x},\mathbf{p}')\,\tilde{V}(\mathbf{x},\mathbf{p}-\mathbf{p}')=0, \tag{1.11}$$

$$\tilde{V}(\mathbf{x},\mathbf{p})=\frac{i}{\pi^N\hbar^{N+1}}\int d^N r\,[V(\mathbf{x}+\mathbf{r})-V(\mathbf{x}-\mathbf{r})]\,e^{2i\mathbf{r}\cdot\mathbf{p}/\hbar}. \tag{1.12}$$

The integral term contains contributions to the $W(\mathbf{x},\mathbf{p})$ coming from all phase space points where the potential is not zero, thus giving rise to the nonlocalities under discussion. For potentials of the form $V_0 + \mathbf{A} \cdot \mathbf{x} + B\mathbf{x}^2$ (linear forces) the equation reduces to the (local) Liouville equation

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[24] We are of course referring to the causal or 'hidden-variables' interpretation developed by Bohm from his famous paper of 1952, based on the set of equations just derived. The quantum potential is usually a rapidly varying and sensitive function of the coordinates, which results in a complex (and uncontrollable) motion of the otherwise classical particle. A most valuable outcome of the theory was that it served as an explicit counterexample against the impossibility proof of von Neumann (1932). For an introductory account see Albert (1994), and for recent discussions on the subject see Bohm and Hiley (1993) and Holland (1993). In section 2.2 a set of equations similar to that characteristic of Bohm's theory will be obtained from a statistical perspective and within a phenomenological description, according to which the quantum potential represents an energy due to the diffusive motions.

$$\frac{\partial W}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla W - \nabla V \cdot \nabla_{\mathbf{p}} W = 0, \tag{1.13}$$

and W not only becomes a local function, but remains positive for all $t > 0$ if it is positive at $t = 0$, whence it can be legitimately interpreted as a probability density. For more complicated potentials the equation contains derivatives of higher order [see, e.g., Hillery et al. 1984], so that both problems, nonlocality and nonpositivity, appear and give rise to the accompanying conceptual difficulties.

1.4. Quantum collapse and measurement theory

We have seen that the conventional interpretation requires for completeness the reduction postulate; and indeed, the proposal that a measurement induces a reduction or collapse of the wave function into an eigenfunction of the observable being measured, was introduced as a quantum postulate by von Neumann (1932) and Pauli (1933). Formally, the collapse can be described as a (nonunitary) transformation of an initial state vector describing a pure state, into a final state represented by a mixture. For instance, consider the initial pure state described by the following vector, where $|k\rangle$ is an eigenket of a maximal set of commuting observables which includes the measured observable $\hat{F}$,

$$|\varphi\rangle = \sum_k c_k |k\rangle ;$$

the corresponding density matrix is

$$\hat{\rho}_{\text{in}} = |\varphi\rangle \langle\varphi| = \sum_{k,k'} c_k c_{k'}^* |k\rangle \langle k'| = \sum_k |c_k|^2 |k\rangle \langle k| + \sum_{k \neq k'} c_k c_{k'}^* |k\rangle \langle k'| . \tag{1.14}$$

The reduction postulate assumes that a measurement of the observable $\hat{F}$ produces one of its eigenvalues and leaves the system in the corresponding eigenstate with probability $|c_k|^2$; therefore, the set of all possible outcomes is described by the mixture

$$\hat{\rho}_{\text{fin}} = \sum_k |c_k|^2 |k\rangle \langle k| . \tag{1.15}$$

A comparison of equations (1.14) and (1.15) shows that the measurement process eliminates all interference terms, i.e., it transforms a coherent superposition into an incoherent one; the measurement 'decoheres' the state.

Now, a unitary transformation is unable to perform such a kind of reduction and since the evolution controlled by the Schrödinger equation is unitary, the process of reduction is not described by such equation; it violates the quantum rules of (causal) evolution.

Quite apart from the problems associated with nonlocality and noncausality inherent in the notion of a sudden collapse of the wave function, the introduction of the reduction postulate makes measurement theory appear to be an essential part of the quantum formalism. This is of course hard to accept, if quantum mechanics is supposed to be a fundamental theory of nature, and as such it should not need an external observer. A theory of measurement is far from being fundamental; it refers to methodological questions related to our observations and knowledge of the world, not to the world itself, and its introduction transforms the fundamental theory into one about the relations of the observer with the world. Indeed, right from the beginning the proponents of the reduction postulate pointed to the possibility that the observers' conciousness might find its way into the description through the measurement process. Moreover, the assumption of a (classical) measuring apparatus as part of the quantum description generates a logical inconsistency, since at the same time, to avoid a regression, the measuring instrument cannot be made to behave quantum mechanically, so that the theory cannot hold for *all* systems.

As has been already remarked, in the literature two different kinds of reduction can be distinguished. One is the nonlocal and noncausal collapse previously discussed, according to which a 'measurement' performed on one particle collapses the wave function of its far away partner, as is the case with the de Broglie box. On other occasions, the collapse refers to real physical processes produced by physical interactions. For example, when photons from an entangled state pass through a polarizer, some are absorbed but those that pass change to a polarized (reduced) state. In these cases, the measuring apparatus affects the microsystem and destroys the coherence of the superposing amplitudes.[25] Unfortunately both situations are frequently treated as equivalent, which adds to the already confusing situation.

1.4.1. SCHRÖDINGER'S CAT PARADOX

The cat paradox is one of the outcomes of the studies on the foundations of quantum mechanics carried out by Schrödinger (1935a, b, 1936) in the aftermath of the EPR paper and following an interesting interchange of letters with Einstein on these matters.[26] As Schrödinger himself puts it, he resorted to a "burlesque case" in an attempt to translate to the macroscopic domain the counterintuitive description made of entangled states by the conventional interpretation. The situation is the following. Put inside a box a radioactive nucleus, a detector of the products of the eventual decay of the nucleus, a cat and, finally, an apparatus that poisons the cat if triggered by the activation of the detector; close the box and wait. The quantum description is an entanglement of two possible states, namely, the nucleus has not decayed (state vector $|I_0\rangle$) and the cat is alive (state vector $|C_L\rangle$), or the nucleus has decayed ($|I_1\rangle$) and the cat is dead ($|C_D\rangle$); thus, the state vector is

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[25] A most important example of this kind is the state preparation process, in the sense given to this term by Margenau (1963). For a discussion see, e.g., Home and Whitaker (1992).

$$|\Psi\rangle = |I_0\rangle\,|C_L\rangle + |I_1\rangle\,|C_D\rangle\,, \tag{1.16}$$

with $\langle\Psi|\,\Psi\rangle = 1$. The conventional interpretation regards the radioactive nucleus as being in an undefined state until it is determined by the observation. Since the state of the cat is causally connected to the state of the nucleus, it follows that it also is undetermined before the observation; one thus gets a whimsical superposition of alive and dead cat. But of course, once the cat is observed, it is definitely alive or dead. This corresponds to the reduction of the state vector carried out by the observation. Assuming the cat is found dead, for instance, the state function collapses (instantaneously) into

$$|\Psi\rangle = |I_1\rangle\,|C_D\rangle\,. \tag{1.17}$$

In practice the cat has been transformed into a new sort of detector. With this instrument at hand, Schrödinger asks himself: when did the cat die? All sorts of answers to this question have been given, some of them being conveniently encoded in assertions as: "No elementary phenomenon is a phenomenon until it is an observed phenomenon" [Wheeler 1979]; "no reduction of the wave packet occurs until some pertinent mental event occurs" [analysed in Stapp 1994]; or even "ask a foolish question and you will get a foolish answer" [Feshbach and Weisskopf 1988]. And what if nobody looks into the box?[27]

As is the case with other similar paradoxes, it vanishes by adopting the ensemble interpretation. The fact that an individual observation is made does not change the initial ensemble that contains both dead and alive cats, described by the vector (1.16). The observation does change our information and with this extra piece of information one can construct a different ensemble corresponding to the new situation, which is a quite normal statistical procedure; the vector (1.17) describes the new ensemble referring to those cats that have been observed dead. As is always the case with entangled states, the wave function provides merely probabilistic predictions and not precise knowledge about the state of the constitutive parts, as was discussed in section 1.3.

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[26] The episode is narrated in detail in Moore (1989), chapter 8. To explain his point of view to Schrödinger, Einstein used the example of a mass of gunpowder that would probably explode spontaneously in the course of the next year, so that during this interval the wave function would describe a superposition of exploded and unexploded gunpowder. One fancies here the seed of the cat paradox. Jammer (1974), chapter 6, contains a more technical account of Schrödinger's work on the EPR argument.

[27] Other characteristic antirealistic views nourished by the conventional interpretation can be seen in Rigden (1986) and Adler (1989).

1.4.2. NATURAL DECOHERENCE

The description of the measurement process given above is now recognized as being too simple and inadequate, and it has been elaborated in several directions during the last years, in an attempt to generate a more realistic quantum picture of nature that may allow to get rid of the reduction postulate. In these more sophisticated approaches the measurement is considered a physical process that involves a coupling between the system measured and the measuring device, allowing for random elements that act as the source of both the randomness of the selected result and the dissipation that accounts for nonunitarity.

Let us illustrate this matter with an example. Consider a complex axial molecule; the quantum description of it in terms of wave functions combines the constituents of the molecule with all possible orientations and relative positions. However, practicing quantum chemists know that individual axial molecules are well-defined structures with well-defined orientations; they treat them almost as classical objects. Something similar occurs with chiral molecules —for which quantum theory predicts a linear superposition of both states, although only one of them occurs naturally—, or with massive cosmic rays, which traverse enormous distances without losing their localized corpuscular structure, and so on. Obviously, to account for such 'spontaneous localization', some elements not contained in the usual description must be introduced. For example for the axial molecular problem, Claverie and Jona-Lasinio (1986) suggested as a possible mechanism of localization, the instability (in the semi-classical regime) of the delocalized states under small perturbations, such as those due to the reaction field (to the average local atomic magnetic moments) created by the molecular environment.

A similar situation is met in the quantum theory of measurement, which faces the need to provide a mechanism that brings about a definite outcome for each measurement, even when the measured variable may not have a definite pre-measurement value; the problem becomes even more acute when the passage to the classical limit is considered. In this context the riddle has been called the *objectification problem* [Busch et al. 1991], and the opinions about its meaning range from being insoluble for those who strictly adhere to orthodox quantum mechanics, to nonexisting for those who consider this theory to be incomplete by nature.

In the new approach, which seems to be rooted in the conviction that "the whole idea of collapse as a substitute for a more realistic treatment is an absurdity" [Fearn and Lamb 1993], the system is assumed to be coupled to a certain (random) 'environment' that makes it evolve quickly towards a mixture. The resulting theory is of course not a hidden-variables theory (the description of the quantum behaviour remains untouched), but an extra-variables theory, designed to achieve the required chain of localizations or decoherences, in such a form that the description of microscopic systems remains essentially unaltered, but in the limit of big systems, it goes naturally into the classical limit.[28]

The cat paradox affords a very simple means to exemplify the main idea. Consider a state vector (1.16) in the form

$$|\Psi\rangle = a(t)\,|C_L\rangle + b(t)\,|C_D\rangle\,. \tag{1.18}$$

Now for a macroscopic and complex system such as the cat, this wave vector is too naive; a slightly better description should be attained by considering the random, unavoidable disturbances on the system by the surroundings. These are taken into account in the simplest possible way by means of an extra random phase, whereby (1.18) transforms into

$$|\Psi_\alpha\rangle = a(t)\,|C_L\rangle + b(t)e^{i\alpha}\,|C_D\rangle\,. \tag{1.19}$$

In practice all possible values of the phase α will be realized, so that one should sum over all possibilities to get the total probability; assuming a uniform distribution of the phase,

$$\hat{\rho} = \frac{1}{2\pi}\int_0^{2\pi} d\alpha\,|\Psi_\alpha\rangle\,\langle\Psi_\alpha| = |a(t)|^2\,|C_L\rangle\,\langle C_L| + |b(t)|^2\,|C_D\rangle\,\langle C_D|\,. \tag{1.20}$$

This density matrix corresponds to a mixture, so no more problems arise with fanciful cats or wave-function collapses. Of course, for simple systems such as isolated atoms equation (1.18) continues to hold, since no interaction with any 'environment' not included in the original Hamiltonian can be added. So the mechanism of decoherence becomes effective in principle only for large enough systems, just as is required to account for the sharp meter pointer values.

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[28] An ample and commented selection of papers on the traditional measurent problem is found in Wheeler and Zurek (1983); a particularly strong defense and detailed study of the reduction postulate is given in Primas (1983), section 3.5. For more detailed studies of natural decoherence and related topics, the interested reader may consult the popular account in Zurek (1991) or the reviews by Zurek (1982), Walls et al. (1985), Busch et al. (1991), etc. Two recent reviews on the subject from different viewpoints are Chan-Pu (1993) and Pearle (1993).

Proposals for this kind of state-vector reduction appear in the literature under various forms and different names, as decoherence, consistent histories approach, continuous spontaneous localization (CSL), etc. The noisy environment may go unspecified, but in some instances it has been identified with the thermal bath [see, e.g., Zurek 1991 and references therein], with gravitational fluctuations [see, e.g., Karolyhazy 1966, 1986, Diosi 1987] and even with the quantum vacuum [Pfeifer 1981, Santos 1994b]. This general approach has been criticized on the basis that it solves the problem in practice but not in principle [Milburn 1991]. In other formulations decoherence is achieved by modifying the Schrödinger equation [Ghirardi et al. 1986, 1990, Ghirardi 1993], so that the evolution of many-body systems is radically different from the predictions of conventional quantum theory with collapse, but basic agreement with this theory is preserved for strictly microscopic systems. Still other authors avoid both the objectification problem and hidden variables by means of specific additional assumptions such as the use of restricted families of observables to describe the 'consistent history' or evolution of the system towards the mixture [Griffiths 1984, Omnés 1992].

As is well known, EPR correlations play a central role in the investigations aimed to establish the consistency —or inconsistency— between locality, realism and quantum mechanics. An outcome of the related experiments, performed up to now only with photons, is that such correlations are real, and that they survive over long macroscopic distances. It would be of immense theoretical value to verify, perhaps with an experiment such as that proposed by Fry [see, e.g., Fry and Li 1992], that atoms also do not decohere over macroscopic distances.

CHAPTER 2

QUANTUM MECHANICS AS A STOCHASTIC THEORY

From the discussion in chapter 1 it is clear that quite independently of the interpretation, chance enters into the quantum description as an essential ingredient. This strongly suggests interpreting the quantum behaviour of matter as result of an underlying stochastic phenomenon, a sort of Brownian process. However, any stochastic interpretation of quantum mechanics confronts immediately a fundamental difficulty, since, as was seen in section 1.2, quantum mechanics is not a genuine statistical theory. It follows that if a true stochastic description of quantum systems can be made, then it will not be identical to quantum theory, or else, stochastic models designed to just reproduce the quantum behaviour in detail will not correspond to a *bona fide* statistical description of a stochastic process. This book is devoted to study the first of these possibilities from the standpoint of SED. The present chapter, however, is much narrower in scope, most of it dealing with the study of a phenomenological model that goes along the second alternative; the rest of the chapter is dedicated to a short review of other theories within the same context, but of a clearly different nature.

2.1. Quantum mechanics and stochasticity

There are several brands of 'Brownian' models of quantum systems. Before entering into the subtleties of the problem, and as a prelude to it, let us convince ourselves that if a stochastic process is assumed to underlie quantum mechanics, this process cannot be Markovian.

For this purpose consider a quantum system with a time-independent Hamiltonian $\hat{H}$, which has been prepared in a pure state $|a\rangle$ at time t_0; at time t_1 the state has evolved to $|a1\rangle \equiv |a, t_1\rangle = [\exp -i\hat{H}(t_1 - t_0)/\hbar]\,|a, t_0\rangle \equiv \hat{U}(t_1 - t_0)\,|a, t_0\rangle$. Then the transition probability from state $|a\rangle$ to state $|b\rangle = |b, t_1\rangle \equiv |b1\rangle$ is given according to the rules of quantum mechanics by

$$P_{ab} = \left|\left\langle b, t_1 \left|\hat{U}(t_1 - t_0)\right| a, t_0\right\rangle\right|^2 = |\langle b1|a1\rangle|^2 . \qquad (2.1)$$

In terms of the projector $\hat{P}_a = |a1\rangle\langle a1|$ this expression reads

$$P_{ab} = \langle b1\,|a1\rangle\,\langle a1|\,b1\rangle = \left\langle b1\left|\hat{P}_a\right| b1\right\rangle . \qquad (2.2)$$

Let us consider now the more elaborate object

$$P_{abc} = \left| \left\langle c, t_2 \left| \hat{U}\left(t_2 - t_1\right) \right| b1 \right\rangle \left\langle b1 \left| \hat{U}\left(t_1 - t_0\right) \right| a, t_0 \right\rangle \right|^2$$

$$= \left| \langle c1 \left| b1 \right\rangle \langle b1 \right| a1 \rangle \right|^2, \tag{2.3}$$

which in usual quantum theory is interpreted as the combined probability that a measurement at t_1 gives $|b\rangle$, followed by a measurement at t_2 that gives $|c\rangle$. Then

$$\sum_b P_{abc} = \sum_b \left| \langle c1 \left| b1 \right\rangle \langle b1 \right| a1 \rangle \right|^2$$

$$= \sum_b \langle c1 \left| b1 \right\rangle \langle b1 \left| a1 \right\rangle \langle a1 | b1 \rangle \langle b1 | c1 \rangle = \sum_b \left\langle c1 \left| \hat{P}_b \hat{P}_a \hat{P}_b \right| c1 \right\rangle .$$

Writing $\hat{P}_a \hat{P}_b = \hat{P}_b \hat{P}_a + \left[\hat{P}_a, \hat{P}_b\right]$ and taking into account that $\sum_b \hat{P}_b = 1$ and $\hat{P}_b^2 = \hat{P}_b$, one gets

$$\sum_b P_{abc} = \sum_b \left\langle c1 \left| \hat{P}_b \hat{P}_a + \hat{P}_b \left[\hat{P}_a, \hat{P}_b\right] \right| c1 \right\rangle$$

$$= \left\langle c1 \left| \hat{P}_a \right| c1 \right\rangle + \sum_b \left\langle c1 \left| \hat{P}_b \left[\hat{P}_a, \hat{P}_b\right] \right| c1 \right\rangle$$

or finally, using equation (2.2),

$$\sum_b P_{abc} = P_{ac} + \sum_b \left\langle c1 \left| \hat{P}_b \left[\hat{P}_a, \hat{P}_b\right] \right| c1 \right\rangle . \tag{2.4}$$

This result shows that the quantity $\sum_b P_{abc}$ is not equal in general to the probability P_{ac} of getting the state $|c\rangle$ at time t_2 when the system was in state $|a\rangle$ at time t_0 independently of anything else: there are extra contributions generated by the projection of the state at the intermediate time t_1 that produced the result $|b\rangle$. Only when the projectors $\hat{P}_a$ and $\hat{P}_b$ commute do these additional contributions disappear; one may interpret this by saying that (2.4) describes a system with memory. This description cannot correspond to a Markov process, because for such processes, by definition, the state at time t_0 determines the probabilities of the states at any later time t_2 independently of the (possible or actual) path followed, and thus of the specific state attained at any intermediate time. This can be clearly seen to be a consequence of the superposition of amplitudes instead of probabilities. Thus we verify that any attempt to perform a direct reading of quantum results in terms of Markovian processes is doomed to failure, and more generally, that the addition of amplitudes is a feature bound to create problems with any stochastic interpretation of quantum theory.[1]

A closer analysis of equation (2.4) reveals the origin of the problem. In this expression any reference to times other than t_1 has been absorbed by the evolution operator, so that everything is described as if happening simultaneously, at time t_1. Thus one may forget the time altogether and say that the probability of getting $|c\rangle$ from state $|a\rangle$ depends on every possible intermediate state $|b\rangle$, however far away (in space or time) it happens to be from $|a\rangle$; this expression contains therefore a good deal of nonlocality. Note that the theory, being nonrelativistic, allows for instantaneous actions at a distance; to get rid of them here one would have to assume that the commutator of the projectors at equal times vanishes for any spatial separation different from zero, thus introducing a demand similar to that of microcausality in relativistic field theory. However, within the context of usual (nonrelativistic) quantum theory there is no room for this demand and the inequivalence between Markovian and quantum processes remains.

2.2. The stochastic description of quantum mechanics

To the arguments already given in (partial) support of the notion of a stochastic process underlying quantum mechanics, a couple more will be added here [see also Comisar 1965, Hall and Collins 1971]. The first one is superficial and merely formal, and is based on the analogy observed by Schrödinger (1931, 1932) [and later on by Fürth 1933] between his equation and the diffusion equation, which are related to one another by analytical continuation into imaginary times. Schrödinger specified both an initial boundary density at time $t_0 < t$ for the solution of the diffusion equation at time t, *and* a final density for a later time $t_1 > t$. The interpolating solution for the general problem is not known, but for a certain particularly simple case (when the backward solution at time t_1 coincides with the measure to which the natural evolution of the forward problem leads at that time), the distribution at the intermediate time is given by the product of the probabilities that solve each problem. Then the analogous quantum case leads to the standard product $\psi\psi^*$ for the probability at a given time. The seed planted by Schrödinger has been developed in recent times into a fuller theory in terms of Bernstein processes [Jamison 1974, Zambrini 1986; see also Cramer 1986 and Garbaczewski 1993].

A second argument is that if a quantum particle is considered to follow a stochastic process in configuration space, the resulting Hausdorff fractal dimension of such process is the same as that of a Brownian particle [Abbot and Wise 1981]. This fact is in itself interesting, although one can think with no less legitimacy of a phase-space description, and then the analogy breaks down. Yet various other reasons in support of a stochastic approach to the quantum problem have been given; for example, de Broglie (1967) needs to introduce it in order to make the particle change stochastically from one guiding wave to another.

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[1] A further example showing that quantum mechanics cannot be expressed as a Markov process is given in Gillespie (1994).

The first relatively accomplished stochastic theory of the quantum process was proposed by the Hungarian physicist Imre Féynes (1946, 1952), further developed by Kershaw (1964), and strongly criticized by Nicholson (1954). Féynes' theory is based on an ad hoc Lagrangian within a Brownian context; the author goes far enough as to recover a good part of the Hilbert-space formalism and concludes that quantum mechanics describes an inherently stochastic phenomenon.[2] Perhaps the most widely known theory of this kind, based on a nondissipative Markov process, is *stochastic mechanics*, initiated by Nelson (1966, 1985, 1986) and further developed by Davidson (1978, 1979a, b), Guerra, Ruggiero and other authors (see references below). A common characteristic of these works, at least during the earlier stages of development, was the identification of the underlying process as classical and of a Brownian nature. This led Jammer to the statement: "The main objective of the stochastic interpretation of quantum mechanics has been to show that quantum theory is fundamentally a classical theory of probabilities or stochastic processes, and as such conceptually of the same structure as, say, the Einstein-von Smoluchowski theory of Brownian motion..." [Jammer 1974, p. 418]. This sounds as astonishing and implausible as the complementary assertion would sound, namely, that Brownian motion is fundamentally a quantum theory conceptually of the same structure as the Schrödinger theory of the electron [de la Peña and Cetto 1982]. Indeed, the need for a clear conceptual distinction between the two stochastic processes gave rise to another branch of research, whose scope is also the development of a possible stochastic interpretation of quantum mechanics, but on the basis that the stochasticity is distinctly nonclassical, i.e., essentially different from the Brownian motion. This theory, which has been called *stochastic quantum mechanics*, is the one to be delineated here.

A general feature of these stochastic theories is their phenomenological nature; since they are aimed at reproducing quantum mechanics, whether the process is considered classical or not, they are in principle unable to go beyond quantum mechanics itself. In general, no specific assumption is made about the nature of the stochastic force, although one can find the most varied suggestions in this respect, ranging from collisions with vacuum particles or *zerons*, or interactions with a diversity of vacuum fields or even neutrinos, to fluctuations of the space-time metric. The description may even be made compatible with the idea of an indeterministic electron, which is far from the realistic persuasion that inspires the whole enterprise. Their phenomenological character is perhaps the strongest objection that can be made to these models, but taken at their face value they can be and indeed have been of some utility, as we will have opportunity to see in the discussion.

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[2] A detailed account of the first developments of the stochastic approach to quantum mechanics can be seen in Jammer (1974), chapter 9.

2.2.1. STOCHASTIC QUANTUM MECHANICS

Our first task is to construct a theory of stochastic processes in configuration space that is as general as possible, so that it can accommodate the quantum processes, assuming such a reduction is feasible. The (rather informal) exposition that follows, based on de la Peña (1969), de la Peña and Cetto (1975, 1982, 1991a) and Santos (1973), starts with the formulation of the appropriate kinematics. Complementary discussions can be seen in Guerra (1981), Nelson (1985), Blanchard et al. (1987) and Kyprianidis (1992).

Kinematics

Consider a particle undergoing a stochastic motion and moving under a given force field; construct a subensemble constituted by all those particles that happen to be within a small volume $d\mathrm{x}$ around a point x at a certain time t. At a slightly earlier time $t' = t-\Delta t$, owing to their stochastic motion these particles had different positions, which we denote generically by x'; also, at a slightly later time $t'' = t + \Delta t$ the same particles will occupy different positions, generically denoted by x''. For any smooth function g of the stochastic variable x one gets from a Taylor series expansion, with

$$\Delta_+\mathrm{x} = \mathrm{x}'' - \mathrm{x}, \qquad \Delta_-\mathrm{x} = \mathrm{x} - \mathrm{x}' \tag{2.5}$$

and omitting the summation signs for simplicity in the expressions,

$$\frac{g(\mathrm{x}'') - g(\mathrm{x}')}{2\Delta t}$$

$$= \frac{\partial g}{\partial x_i}\frac{\Delta_+x_i + \Delta_-x_i}{2\Delta t} + \frac{\partial^2 g}{\partial x_i \partial x_j}\frac{\Delta_+x_i\Delta_+x_j - \Delta_-x_i\Delta_-x_j}{4\Delta t} + \cdots \tag{2.6}$$

For a regular motion this reduces to $dg(\mathrm{x})/dt = (\boldsymbol{\nabla} g) \cdot (d\mathrm{x}/dt)$ in the limit $\Delta t \longrightarrow 0$. For a random process x it is possible to construct an approximate derivative, for small Δt, performing the appropriate changes. Since one would like the derivative to be a well-defined local function, whereas

$\Delta_\pm x_i(t)$ refer to whole families of functions, the first change consists in averaging over the subensemble defined above. This operation will be denoted by $\langle\cdot\rangle$, so that instead of $g(\mathrm{x}'') - g(\mathrm{x}')$ we will consider $\langle g(\mathrm{x}'') - g(\mathrm{x}')\rangle$.

A more delicate consideration is related to the usual limit $\Delta t \to 0$, since Δt cannot be taken arbitrarily small in the present case. For let δx be any of the Cartesian components of $\Delta_\pm x_i$ for a given member of the ensemble; at the time scale of the 'instantaneous' description (i.e., according to the available experimental time resolution) this δx may happen to be nondifferentiable, due for instance to abrupt kinks. We solve this problem by performing a moving averaging[3] of the function $\mathrm{x}(t)$ during a 'small' time Δt, much smaller than the smallest characteristic time T_o of the systematic motions (assuming this to be possible; otherwise the ensuing picture will be less useful), but much bigger than the correlation time t_c associated with the stochastic process, so as effectively to smooth out the most rapid changes in the instantaneous position. For example, in the case of Brownian motion the particle is so large compared with the solvent molecules that it receives a large number of molecular impacts during the time interval Δt, thus effectively averaging them into a (much) smoother function of time. These averaged quantities are the ones that obey the diffusion laws. Similarly, in the more general case we select Δt so as to embrace many of the most closely spaced violent changes in each particular 'instantaneous' $\mathrm{x}(t)$; thus,

$$T_o \gg \Delta t \gg t_c. \tag{2.7}$$

The resulting (coarse-time-scale) average time derivative or *systematic derivative* is denoted by the symbol $\mathcal{D}_c$; hence,

$$\mathcal{D}_c g(\mathrm{x}) = \frac{\langle g(\mathrm{x}'') - g(\mathrm{x}')\rangle}{2\Delta t}. \tag{2.8}$$

At this point it is necessary to make several assumptions about the properties of the stochastic motion. Considering the desired generality of the treatment and the lack of a specific model, these properties are unknown in principle; but for the cases of interest here it proves sufficient to assume that the stochasticity is due to a stationary, approximately isotropic and homogeneous source; then the second moments of $\Delta_\pm \mathrm{x}$ are independent of the sign for equal signs, $\langle\Delta_+ x_i \Delta_+ x_j\rangle = \langle\Delta_- x_i \Delta_- x_j\rangle$. Further, the fluctuations are assumed to be statistically independent, $\langle\Delta_+ x_i \Delta_- x_j\rangle = 0$ for all i and j. Each surviving second moment may have a contribution of order Δt due to the randomness of the motion, plus higher contributions,

$$\langle\Delta_+ x_i \Delta_+ x_j\rangle = \langle\Delta_- x_i \Delta_- x_j\rangle = 2D_{ij}(\mathrm{x}, t)\Delta t + \cdots. \tag{2.9}$$

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[3] The moving average $x_{\Delta t}(t)$ of $x(t)$ is defined as $x_{\Delta t}(t) = (1/\Delta t)\int_t^{t+\Delta t} d\tau x(\tau)$.

The difference $\left\langle (\Delta_+ x_i)^2 \right\rangle - \left\langle (\Delta_- x_i)^2 \right\rangle$ is therefore of order higher than the first in Δt, and from equations (2.6) and (2.8) we have to order zero in Δt (adding the contribution that may come from an explicit time dependence of g)

$$\mathcal{D}_c g(\mathrm{x}) = \frac{\partial g}{\partial t} + \mathrm{v} \cdot \nabla g, \tag{2.10}$$

with the *systematic velocity* $\mathrm{v}(\mathrm{x},t)$ given by

$$\mathrm{v}(\mathrm{x},t) = \frac{\langle \Delta_+ \mathrm{x} + \Delta_- \mathrm{x} \rangle}{2\Delta t} = \frac{\langle \mathrm{x}'' - \mathrm{x}' \rangle}{2\Delta t}. \tag{2.11}$$

Note that $\mathcal{D}_c$ coincides with the total time derivative of hydrodynamics. Further, application of equation (2.10) to x gives

$$\mathrm{v}(\mathrm{x},t) = \mathcal{D}_c \mathrm{x}. \tag{2.12}$$

The systematic derivative defined above is only one of an infinite number of possible derivatives; another one, equally important for what follows, is obtained by considering the symmetrical combination

$$g(\mathrm{x}'') + g(\mathrm{x}') = 2g(\mathrm{x}) + \frac{\partial g}{\partial x_i}(\Delta_+ x_i - \Delta_- x_i)$$

$$+ \frac{1}{2}\frac{\partial^2 g}{\partial x_i \partial x_j}(\Delta_+ x_i \Delta_+ x_j + \Delta_- x_i \Delta_- x_j) + \cdots \tag{2.13}$$

leading to the definition of the *stochastic derivative* of the function $g(\mathrm{x},t)$, namely,

$$\mathcal{D}_s g(\mathrm{x},t) = \frac{\langle g(\mathrm{x}'') + g(\mathrm{x}') - 2g(\mathrm{x}) \rangle}{2\Delta t} \tag{2.14}$$

or

$$\mathcal{D}_s g(\mathrm{x},t) =$$

$$= \frac{\partial g}{\partial x_i}\left\langle \frac{\Delta_+ x_i - \Delta_- x_i}{2\Delta t} \right\rangle + \frac{\partial^2 g}{\partial x_i \partial x_j}\left\langle \frac{\Delta_+ x_i \Delta_+ x_j + \Delta_- x_i \Delta_- x_j}{4\Delta t} \right\rangle + \cdots. \tag{2.15}$$

To zero order in Δt the components of the diffusion tensor are given according to equation (2.9) by

$$D_{ij}(\mathrm{x},t) = \frac{\langle \Delta x_i \Delta x_j \rangle}{2\Delta t}, \tag{2.16}$$

where the Δx refer to any of the two possibilities (++) or (−−). As to the first moments of the deviations of the coordinates entering into the expression for $\mathcal{D}_s g$, note that for a smooth motion the difference $(\Delta_+ \mathrm{x} - \Delta_- \mathrm{x})$

is $\mathcal{O}(\Delta t)^2$; however, if there is a 'diffusion pressure', i.e., if the distribution of particles is inhomogeneous, the average of this difference may have a contribution of order Δt. Therefore, we write

$$\mathbf{u} = \left\langle \frac{\Delta_+ \mathrm{x} - \Delta_- \mathrm{x}}{2\Delta t} \right\rangle = \left\langle \frac{\mathrm{x}'' + \mathrm{x}' - 2\mathrm{x}}{2\Delta t} \right\rangle \tag{2.17}$$

and call $\mathbf{u}$ the *stochastic velocity*. Collecting results, and neglecting again all higher-order terms, we get

$$\mathcal{D}_s g(\mathrm{x}, t) = u_i \frac{\partial g}{\partial x_i} + D_{ij} \frac{\partial^2 g}{\partial x_i \partial x_j}, \tag{2.18}$$

This Markovian approximation (retention of terms up to and including second-order moments) is by no means trivial and in each specific application its validity should be verified. It will prove sufficient and appropriate for the reproduction of the quantum description. Equation (2.18) applied to x gives now

$$\mathbf{u} = \mathcal{D}_s \mathrm{x}. \tag{2.19}$$

Note that neither $\mathbf{u}$ nor $\mathcal{D}_s$ exist in the Newtonian limit, i.e., when the motion is regular and the stochasticity ceases; this allows us to define the Newtonian limit through $\mathcal{D}_s \rightarrow 0$ and $\mathbf{u} \rightarrow 0$ simultaneously. As is now evident, by considering a sequence of time intervals previous to t' and following t'' it becomes possible to define as many different velocities as desired, and each additional one renders a more complete statistical description of the motion. However, for the present purposes the two velocities are sufficient. Further, for the simple case of a diagonal and isotropic diffusion tensor equation (2.18) reduces to

$$\mathcal{D}_s g(\mathrm{x}, t) = \left(\mathbf{u} \cdot \nabla + D\nabla^2 \right) g(\mathrm{x}, t). \tag{2.20}$$

Certain linear combinations of the velocities $\mathbf{v}$ and $\mathbf{u}$, as well as of the operators of time derivation $\mathcal{D}_c$ and $\mathcal{D}_s$, are particularly useful. Specifically, we have the following *exit* and *access* combinations, denoted by the indices e and a (also called *forward* and *backward*, with + and –, respectively), and with $\mathcal{D}_e = \mathcal{D}_c + \mathcal{D}_s$, $\mathcal{D}_a = \mathcal{D}_c - \mathcal{D}_s$,

$$\mathcal{D}_e = \frac{\partial}{\partial t} + \mathrm{v}_e \cdot \nabla + D\nabla^2, \quad \mathcal{D}_a = \frac{\partial}{\partial t} + \mathrm{v}_a \cdot \nabla - D\nabla^2, \tag{2.21}$$

$$\mathrm{v}_e = \frac{\langle \mathrm{x}'' - \mathrm{x} \rangle}{\Delta t} = \mathbf{v} + \mathbf{u} = \mathcal{D}_e \mathrm{x}, \quad \mathrm{v}_a = \frac{\langle \mathrm{x} - \mathrm{x}' \rangle}{\Delta t} = \mathbf{v} - \mathbf{u} = \mathcal{D}_a \mathrm{x}, \tag{2.22}$$

hence

$$\mathbf{v} = \frac{1}{2} \left(\mathrm{v}_e + \mathrm{v}_a \right), \quad \mathbf{u} = \frac{1}{2} \left(\mathrm{v}_e - \mathrm{v}_a \right). \tag{2.23}$$

The access velocity $\mathbf{v}_a$ is the local average velocity of the particles reaching the point x, whereas the exit velocity $\mathbf{v}_e$ is the local average velocity of the particles leaving this point. In the absence of diffusion both velocities are equal (to zero order in Δt); but if there is diffusion, there may be more (or less) particles leaving than entering the neighborhood of x in a given small time interval, the difference $2\mathbf{u}$ being a measure of the intensity of the diffusion (see equation (2.32) below).

An important feature of the velocities is their different behaviour with respect to time reversal. A time reversal operation interchanges t' and t'', and thus also the points x' and x''. If $\hat{T}$ denotes this operation, then $\hat{T}$ $\mathrm{x}'' = \mathrm{x}'$ and $\hat{T}$ $\mathrm{x}' = \mathrm{x}''$ and it follows from equations (2.22, 2.23) that

$$\begin{aligned} \hat{T}\mathbf{v}_e &= -\mathbf{v}_a \quad ; \quad \hat{T}\mathbf{v}_a = -\mathbf{v}_e \\ \hat{T}\mathbf{v} &= -\mathbf{v} \quad ; \quad \hat{T}\mathbf{u} = \mathbf{u}. \end{aligned} \tag{2.24}$$

Similarly one gets the following behaviour for the derivative operators:

$$\begin{aligned} \hat{T}\mathcal{D}_e &= -\mathcal{D}_a \quad ; \quad \hat{T}\mathcal{D}_a = -\mathcal{D}_e \\ \hat{T}\mathcal{D}_c &= -\mathcal{D}_c \quad ; \quad \hat{T}\mathcal{D}_s = \mathcal{D}_s \end{aligned} \tag{2.25}$$

The next step is to construct appropriate expressions for the acceleration; this can be readily achieved by applying a time derivation to the velocity. We have at our disposal two velocities and two time derivatives, whose different combinations afford four different accelerations; these accelerations and their corresponding behaviour under time reversal are

$$\begin{aligned} \mathbf{a}_{cc} &= \mathcal{D}_c\mathbf{v} = \mathcal{D}_c\mathcal{D}_c\mathrm{x}; & \hat{T}\mathbf{a}_{cc} &= +\mathbf{a}_{cc}, \\ \mathbf{a}_{ss} &= \mathcal{D}_s\mathbf{u} = \mathcal{D}_s\mathcal{D}_s\mathrm{x}; & \hat{T}\mathbf{a}_{ss} &= +\mathbf{a}_{ss}, \\ \mathbf{a}_{cs} &= \mathcal{D}_c\mathbf{u} = \mathcal{D}_c\mathcal{D}_s\mathrm{x}; & \hat{T}\mathbf{a}_{cs} &= -\mathbf{a}_{cs}, \\ \mathbf{a}_{sc} &= \mathcal{D}_s\mathbf{v} = \mathcal{D}_s\mathcal{D}_c\mathrm{x}; & \hat{T}\mathbf{a}_{sc} &= -\mathbf{a}_{sc}. \end{aligned} \tag{2.26}$$

The continuity and the Fokker-Planck equations

Consider any conserved quantity $f(\mathrm{x}, t)$; such a quantity must satisfy the conditions

$$\mathcal{D}_a f = 0; \quad \mathcal{D}_e f = 0. \tag{2.27}$$

Now let $\rho(\mathrm{x}, t)$ be the spatial density of particles; then one may write for the conserved quantity

$$\int d^3x\rho\mathcal{D}_a f = 0; \quad \int d^3x\rho\mathcal{D}_e f = 0. \tag{2.28}$$

Performing some integrations by parts and assuming that ρ vanishes rapidly enough at infinity so that the integrated terms vanish, the above equations can be recast in the alternative form

$$\int d^3xf\left(\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\mathbf{v}_e-D\nabla^2\rho\right)=0,$$

$$\int d^3xf\left(\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\mathbf{v}_a+D\nabla^2\rho\right)=0.$$

Since these conditions must be satisfied for any conserved quantity f, they are equivalent to

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\mathbf{v}_e-D\nabla^2\rho=0;\quad\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\mathbf{v}_a+D\nabla^2\rho=0. \tag{2.29}$$

These constitute the forward (exit) and backward (access) Fokker-Planck equations for the problem. By combining them and using (2.23) one is led to

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\mathbf{v}=0, \tag{2.30}$$

$$\nabla\cdot\rho\mathbf{u}=D\nabla^2\rho. \tag{2.31}$$

The first of these equations is the continuity equation expressing the local conservation of particles. The second one can be rewritten as $\nabla\cdot(\rho\mathbf{u}-D\nabla\rho)=0$, and its general integral is $\rho\mathbf{u}=D\nabla\rho+\nabla\times\mathbf{G}$, with $\mathbf{G}$ an arbitrary vector; however, by considering the balance of particles that go into and out of any small volume around a point $\mathbf{x}$ in space, it can be seen that one may take $\mathbf{G}=0$ without loss of generality, and thus one obtains the central formula

$$\mathbf{u}=D\frac{\nabla\rho}{\rho}. \tag{2.32}$$

Note that indeed the motions described by the stochastic velocity $\mathbf{u}$ are due to diffusion, as discussed in relation with equation (2.17); this verifies that the selection $\mathbf{G}=0$ is appropriate.

Dynamics

The lack of a specific model compels us to use a few very general arguments in the construction of the dynamical description. In the first place, one would expect the relationship between the (coarse-grained) 'accelerations' and the forces to be linear. For this to hold, the total acceleration $\mathbf{a}$ should be expressible as a linear combination of the previous four accelerations, equations (2.26), so that we start from the general expression:

$$\mathbf{a}=\lambda_1\mathbf{a}_{cc}+\lambda_2\mathbf{a}_{ss}+\lambda_3\mathbf{a}_{cs}+\lambda_4\mathbf{a}_{sc}, \tag{2.33}$$

where the λ's are constant parameters to be determined. Note that this expression is not time-reversal invariant, since upon time inversion the last two terms reverse their sign, whereas the first two remain unchanged.

Now the total force acting on the particles is the sum of the external force and a stochastic force that contributes both a dissipative term and a purely stochastic term. The dissipative force embodies the systematic effects of the stochastic force on the particle, since the purely random ones are taken into account separately; hence the expression for the effective, averaged force should reduce to the sum of the external force and a coarse-grained friction term which, in the specific quantum mechanical case, should be taken as zero in the absence of losses. Let us denote this total effective local force by $\mathbf{F}$; then we propose to write $\mathbf{F} = m\mathbf{a}$, with m the mass of the particle and $\mathbf{a}$ as expressed above.

However, a force that depends only on the position should remain invariant with respect to time reversal, while velocity-dependent forces may change their sign under such operation; thus, in general $\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_-$, where $\hat{T}\mathbf{F}_\pm = \pm\mathbf{F}_\pm$. By combining the equation of motion and its time-reversed version, one gets thus two separate equations, each one invariant with respect to time reversal

$$\begin{aligned} m\left(\lambda_1 \mathbf{a}_{cc} + \lambda_2 \mathbf{a}_{ss}\right) &= \mathbf{F}_+, \\ m\left(\lambda_3 \mathbf{a}_{cs} + \lambda_4 \mathbf{a}_{sc}\right) &= \mathbf{F}_-. \end{aligned} \tag{2.34}$$

The parameters can be selected so as to adjust the theory to different purposes [see, e.g., Davidson 1978, 1979b, Nassar 1985]; here we study the case that is most directly of interest for us. To make things as simple as possible, let us start by considering the conservative problem $\mathbf{F} = -\nabla V$, for which $\mathbf{F}_- = 0$ (the generalization to $\mathbf{F}_- \neq 0$ is straightforward, as can be seen in the example of the next section); the second equation (2.34) can then be written $m\left(\lambda_3 \mathbf{a}_{cs} + \lambda_4 \mathbf{a}_{sc}\right) = m\lambda_3\left(\mathcal{D}_c\mathbf{u} + \eta\mathcal{D}_s\mathbf{v}\right) = 0$, or

$$\mathcal{D}_c\mathbf{u} + \eta\mathcal{D}_s\mathbf{v} = \mathbf{0}, \tag{2.35}$$

with $\eta = \lambda_4/\lambda_3$. The meaning of this equation is most easily ascertained by taking the derivative of the continuity equation (2.29) with respect to x_j and combining with equations (2.10) and (2.20), to arrive after some simplifications at

$$\frac{\partial u_j}{\partial t} + v_i\frac{\partial u_i}{\partial x_j} + u_i\frac{\partial v_i}{\partial x_j} + D\frac{\partial^2 v_i}{\partial x_i \partial x_j} = 0. \tag{2.36}$$

From equation (2.32) it follows that $\boldsymbol{\nabla} \times \mathbf{u} = 0$, so $(\partial u_i/\partial x_j) = (\partial u_j/\partial x_i)$; further, in the potential case studied here it is possible to set also $\boldsymbol{\nabla} \times \mathbf{v} = 0$. Thus the above equation transforms into

$$\frac{\partial u_j}{\partial t} + v_i\frac{\partial u_j}{\partial x_i} + u_i\frac{\partial v_j}{\partial x_i} + D\nabla^2 v_j = 0, \tag{2.37}$$

which can be rewritten as

$$\mathcal{D}_c\mathbf{u} + \mathcal{D}_s\mathbf{v} = 0. \tag{2.38}$$

Comparison with (2.35) shows that $\eta = 1$ and allows us to identify the second of equations (2.34) as a form of the continuity equation; it represents therefore a constriction on the system rather than a dynamical relation. Hence only the first of these equations gives a true dynamical law, which can be recast in the form

$$m\lambda_1 \left(\mathbf{a}_{cc} - \lambda \mathbf{a}_{ss}\right) = \mathbf{F}_+, \tag{2.39}$$

with $\lambda = -\lambda_2/\lambda_1$. In the Newtonian limit, $\mathbf{a}_{ss} \rightarrow 0$ whereas $\mathcal{D}_c \rightarrow d/dt$, so that $\mathbf{a}_{cc} = \mathcal{D}_c^2\mathbf{x} \rightarrow d^2\mathbf{x}/dt^2$; hence to recover the correct classical limit one takes $\lambda_1 = 1$, and the equation of motion becomes finally

$$m\left(\mathcal{D}_c\mathbf{v} - \lambda\mathcal{D}_s\mathbf{u}\right) = \mathbf{F}_+, \tag{2.40}$$

or, using equations (2.10) and (2.18),

$$m\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} - \lambda\mathbf{u}\cdot\nabla\mathbf{u} - \lambda D\nabla^2\mathbf{u}\right) = \mathbf{F}_+, \tag{2.41}$$

with only one free parameter. This equation was first proposed for the specific value $\lambda = 1$ by Nelson (1966). More generally we have

$$\begin{aligned} m\left(\mathbf{a}_{cc} - \lambda\mathbf{a}_{ss}\right) &= \mathbf{F}_+, \\ m\left(\mathbf{a}_{cs} + \mathbf{a}_{sc}\right) &= \mathbf{F}_-. \end{aligned} \tag{2.42}$$

Integration of the equations of motion

Let us consider the system of equations (2.42) in the more general case when in addition to $\mathbf{F}_+ = -\nabla V$ there is a Lorentz force due to an electromagnetic potential $\mathbf{A}$. To integrate the equations in this case, it is convenient to write the systematic velocity in the form

$$\mathbf{v} = 2D\nabla S - \frac{e}{mc}\mathbf{A}. \tag{2.43}$$

The factor $2D$ is introduced for convenience (this makes S dimensionless); the function S may be understood as a velocity potential, or as a kind of statistical action function whose gradient gives the momentum associated with the systematic (local) velocity (not the velocity of a particular member of the ensemble). Note that S is specified up to an additive function of time.

Equations (2.42) with $\mathbf{u}$ and $\mathbf{v}$ given by (2.32, 2.43) look impressive: they form a system of coupled, nonlinear partial differential equations. However, this system has the remarkable property that it can be integrated (once), uncoupled and linearized if expressed in terms of appropriate functions. This is achieved as follows. First, note that in the second equation (2.42) there is now a contribution of the Lorentz force that changes sign under time reversal, $\mathbf{F}_- = (e/c)\mathbf{u} \times \mathbf{B} + (eD/c)\nabla^2\mathbf{A}$; with this and using $\boldsymbol{\nabla} \times \mathbf{v} = -(e/mc)\mathbf{B}$, the first integral of the equation still gives the continuity equation. Further, the first equation (2.42) can be brought after minor simplifications into

$$\boldsymbol{\nabla}\left[2D\frac{\partial S}{\partial t} + \frac{1}{2}\mathbf{v}^2 - \frac{\lambda}{2}\mathbf{u}^2 - \lambda D\boldsymbol{\nabla}\cdot\mathbf{u}\right] = -\boldsymbol{\nabla}V, \tag{2.44}$$

whence

$$-2mD\frac{\partial S}{\partial t} = +\frac{1}{2}m\left(\mathbf{v}^2 - \lambda\mathbf{u}^2\right) - m\lambda D\boldsymbol{\nabla}\cdot\mathbf{u} + V \tag{2.45}$$

except for an arbitrary function of time that comes from the integration and has been absorbed into the function S (this point has its subtleties, but we leave them aside).[4] A first integration of the equations is thus completed.

Now we write equation (2.32) for the stochastic velocity in the form

$$\mathbf{u} = D\boldsymbol{\nabla}\ln\rho = 2D\boldsymbol{\nabla}R \tag{2.46}$$

where R is again a real function of $\mathbf{x}$ and t; inverting,

$$\rho = e^{2R}. \tag{2.47}$$

With the help of these expressions the continuity equation transforms into

$$2D\frac{\partial R}{\partial t} + \mathbf{v}\cdot\mathbf{u} + D\boldsymbol{\nabla}\cdot\mathbf{v} = 0. \tag{2.48}$$

To separate equations (2.45) and (2.48), one introduces the linear combinations

$$w_+ = R + \frac{S}{\sqrt{-\lambda}}, \quad w_- = R - \frac{S}{\sqrt{-\lambda}}; \tag{2.49}$$

after some algebra one arrives at the pair of separate equations

$$\begin{aligned} -2m\zeta\frac{\partial w_+}{\partial t} &= -2m\zeta^2\left[(\boldsymbol{\nabla}w_+)^2 + \nabla^2 w_+\right] + V \\ &- \zeta\frac{e}{c}\left(2\mathbf{A}\cdot\boldsymbol{\nabla}w_+ + \boldsymbol{\nabla}\cdot\mathbf{A}\right) + \frac{e^2}{2mc^2}\mathbf{A}^2, \end{aligned} \tag{2.50}$$

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[4] Compare equation (2.45) with (2.32, 2.43) taken into account, with the so-called Hamilton-Jacobi equation in Bohm's theory, equation (1.12) [Bohm and Hiley 1987]. It is clear that the terms that give rise to the quantum potential come from the stochastic velocity **u**, and thus they have a kinematical (diffusive) meaning, as stated in section 1.3.

$$+2m\zeta\frac{\partial w_-}{\partial t} = -2m\zeta^2\left[(\nabla w_-)^2 + \nabla^2 w_-\right] + V$$

$$+\,\zeta\frac{e}{c}\left(2\mathbf{A}\cdot\nabla w_- + \nabla\cdot\mathbf{A}\right) + \frac{e^2}{2mc^2}\mathbf{A}^2, \qquad (2.51)$$

with $\zeta = D\sqrt{-\lambda}$. As a final step, this system of equations can be linearized by introducing the further change of functions

$$\psi_+ = e^{w_+}; \quad \psi_- = e^{w_-}, \qquad (2.52)$$

which leads to the equations

$$-2mD\sqrt{-\lambda}\frac{\partial\psi_+}{\partial t} = \frac{1}{2m}\left[+2mD\sqrt{-\lambda}\nabla - \frac{e}{c}\mathbf{A}\right]^2\psi_+ + V\psi_+,$$

$$2mD\sqrt{-\lambda}\frac{\partial\psi_-}{\partial t} = \frac{1}{2m}\left[-2mD\sqrt{-\lambda}\nabla - \frac{e}{c}\mathbf{A}\right]^2\psi_- + V\psi_-, \qquad (2.53)$$

where

$$\rho = e^{2R} = e^{w_+ + w_-} = \psi_+\psi_- \qquad (2.54)$$

and

$$\mathbf{v} = D\sqrt{-\lambda}\nabla\ln\frac{\psi_+}{\psi_-} - \frac{e}{mc}\mathbf{A} = D\sqrt{-\lambda}\left[\frac{\nabla\psi_+}{\psi_+} - \frac{\nabla\psi_-}{\psi_-}\right] - \frac{e}{mc}\mathbf{A}, \qquad (2.55)$$

$$\mathbf{u} = D\nabla\ln\psi_+\psi_- = D\left[\frac{\nabla\psi_+}{\psi_+} + \frac{\nabla\psi_-}{\psi_-}\right]. \qquad (2.56)$$

Choice of the parameters

Equations (2.53), each one having the structure of a Schrödinger equation, apply to *any* system that can be described by the present stochastic treatment. Here we perceive at once the strength and the weakness of the procedure; for on the one hand one indeed arrives at a quantum description of the stochastic system, with a mere appropriate choice of the parameters D and λ; but then, on the other hand, it turns out that this equation is quite unspecific and that the 'appropriate' selection of the parameters is arbitrary to a considerable extent. A complete theory should allow for an unambiguous derivation of the parameter λ and the coefficients D_{ij} (not necessarily constant in general) from first principles, which the present phenomenological approach is unable to do. Below in §2.2.3 a derivation is given that can help to understand why we have been led to such Schrödinger-like equations. For the moment, let us briefly explore the possible applications of the results just derived.

Observe that in the integrated equations (2.53) the important parameter is $\zeta = D\sqrt{-\lambda}$, and not each factor separately. One may therefore consider that D (which, for simplicity, we continue to treat as a constant) takes care of the scale, whereas the relevant property of λ is its sign. This means that qualitatively one has essentially two different theories, according to the sign of λ, namely:

a) $\lambda = -1$. In this case equations (2.53) are parabolic, the functions $\psi_+ = \exp(R+S)$ and $\psi_- = \exp(R-S)$ are both real and the process described by them is irreversible (it has been suggested to call it an *Einstein process*). This theory can be used to describe classical Markov processes, if due allowance is made for the friction force, which may be introduced via an expression such as $\mathbf{f} = -\beta \mathbf{v}_e$ with $\mathbf{v}_e = 2D(\nabla\psi_+/\psi_+)$. However, with such substitutions the theory ceases to be linear and it is simpler to go back to the linear Fokker-Planck equation [further elaborations can be seen in de la Peña and Cetto 1975, Skagerstam 1977, Nassar 1985, 1986, and references therein].

b) $\lambda = 1$. In this case equations (2.53) are hyperbolic, the functions $\psi_+ \equiv \psi = \exp(R+iS)$ and $\psi_- = \psi^* = \exp(R-iS)$ being each other's complex conjugate. The process is undulatory and reversible (it has been suggested to call it a *de Broglie process*). One arrives at the Schrödinger equation with the selection

$$D = \frac{\hbar}{2m} \tag{2.57}$$

for the diffusion coefficient. It should be stressed, however, that this strong selection is far from obvious; there is no a priori reason to assume that the tensor mD has a universal value, independent of the specific problem. Of course, arguments have been given in support of this formula, and there have been attempts to derive it from fundamental hypotheses [de Broglie 1967], to ground the theory on a more solid basis. But generally speaking, the problem of identifying the noise source behind the assumed stochastic process remains open in this kind of approach, and with it, that of deriving the detailed form of the diffusion tensor.

Discussion of the results

Before discussing more fundamental questions, let us combine equation (2.57) with the requirement $\Delta t > t_c$. Consider the standard deviation of the variable x, $\overline{\Delta x} =< (\Delta x)^2 >^{1/2} = \sqrt{2D\Delta t}$, from equation (2.16). In order that the theory makes sense the relation $\overline{\Delta x}/\Delta t = \sqrt{2D/\Delta t} < \bar{p}/m$ must hold, where $\bar{p}$ stands for a representative momentum along the x axis (indeed, for an accurate description the $<$ sign should be replaced by a $\ll$ sign). It follows that $\overline{\Delta x}^2 = 2D\Delta t > 4m^2D^2/\bar{p}^2$, or

$$\overline{\Delta x}^2\bar{p}^2 > \hbar^2, \quad \bar{K}\Delta t > \hbar/2, \tag{2.58}$$

where $\bar{K}$ represents the kinetic energy associated with the momentum $\bar{p}$ (the second inequality is immediate from the above considerations). Thus, the Heisenberg inequalities appear right from the beginning as indicating the limited descriptive capabilities of the theory: the description is too rough to go beyond them. From this point of view, the Heisenberg inequalities have nothing to do with the measurement process or with any assumed noncausality of the momentum or position variables.

Since for $\lambda = 1$ one has the Schrödinger equation and its complex conjugate, with the wave function as a probability amplitude, formally one can reproduce all results of quantum mechanics, including the operator formalism and results that follow from it. The theory can even be extended in several important directions. For example —and this is perhaps one of its most remarkable outcomes— conditional *probabilities* (not amplitudes) can be constructed describing interference phenomena and the like [Cufaro Petroni 1989]; this possibility arises because the kernel for such probabilities depends on the state of the system. In other instances the theory helps to give some intuition on quantum phenomena, as is the case with the tunnel effect [Jona-Lasinio et al. 1981, Yasue 1981], the (anti)symmetrization of the wave function [Nelson 1985, de la Peña and Cetto 1985a], numerical simulations to reconstruct the trajectories of single electrons [McClendon and Rabitz 1988], etc.

Stochastic quantum mechanics has also been extended to a wider range of problems, such as the electron spin [Dankel 1970, 1977, de la Peña 1971, Guerra and Marra 1984], the description of mixtures [Guerra 1981, Guerra and Loffredo 1981, Ruggiero and Zannetti 1982, Jackel and Pignon 1984, Vilela Mendes 1986], and the relativistic case [de la Peña 1971, Yasue 1977, Lehr and Park 1977, Bess 1979, Roy 1979, Vigier 1979]; systems with several particles have been considered, thereby exhibiting quite dramatically the characteristic nonlocalities of the description [de la Peña and Cetto 1969, Nelson 1985]. Variational methods [Santos 1969, Guerra and Marra 1983, Guerra 1985, Marra 1987, Loffredo and Morato 1989] and path-integral procedures have been introduced [Berrondo 1973, Weaver 1978, Yasue 1981, Zambrini 1984], and a field theory has been developed within the Nelson formulation, as well as the procedure of stochastic quantization [Moore 1979a, 1980, Guerra 1981, de Angelis et al. 1981], and so on.[5] An independent, very interesting development in a similar direction is the treatment of the Dirac equation in terms of a dichotomic (telegraph) stochastic process [Gaveau et al. 1984]. Stochastic quantum mechanics can also be of some value in the study of stochastic chaos in Brownian systems obeying a Fokker-Planck equation which is formally analogous to the Schrödinger equation [see, e.g., Alpatov and Reichl 1994].

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[5] Ample and complementary lists of references can be found in Jammer (1974), Guerra (1981, 1984), Blanchard et al. (1987) and de la Peña and Cetto (1991a). Recently Tiwari (1988) proposed a new and interesting phenomenological statistical description able to accomodate relativity and spin in a natural way.

As we have seen, a main point of this stochastic theory is its ability to distinguish classical from quantum stochastic processes, which become described by essentially different equations. One should therefore speak not of a Brownian analogue of quantum mechanics, but of a quantum stochastic process in itself; insufficient attention to this essential point is the cause of much confusion in the literature. More specifically, the accelerations $\mathbf{a}_B$ for the classical (Brownian) case and $\mathbf{a}_Q$ for the quantum system entering into the equation of motion (2.40) are, respectively,

$$\mathbf{a}_B = \mathcal{D}_c\mathbf{v}+\mathcal{D}_s\mathbf{u}, \quad \mathbf{a}_Q = \mathcal{D}_c\mathbf{v}-\mathcal{D}_s\mathbf{u}. \tag{2.59}$$

One can use these formulas to write the equations of motion in a more conventional way by observing, as was done in equation (2.45), that $\mathbf{a}_{ss}$ is a gradient, $\mathbf{a}_{ss} = \mathcal{D}_s\mathbf{u} = \boldsymbol{\nabla}[\frac{1}{2}\mathbf{u}^2 + D\boldsymbol{\nabla}\cdot\mathbf{u}]$; then one gets for the classical and quantum problems, respectively (in the potential case),

$$m\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}\right) = -\boldsymbol{\nabla}\left(V + \frac{1}{2}m\mathbf{u}^2 + mD\boldsymbol{\nabla}\cdot\mathbf{u}\right), \tag{2.60}$$

$$m\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}\right) = -\boldsymbol{\nabla}\left(V - \frac{1}{2}m\mathbf{u}^2 - mD\boldsymbol{\nabla}\cdot\mathbf{u}\right). \tag{2.61}$$

In these equations the difference in dynamical behaviour is attributed solely to different effective forces acting on the system. This brings us a bit closer to the point of view adopted in the approach to the stochastic theory of quantum mechanics in terms of Markov-Bernstein processes [see, e.g., Garbaczewski 1993].

The stochastic approach to quantum mechanics has been criticized from several points of view [see, e.g., Kracklauer 1974, Claverie and Diner 1976a, Ghirardi et al. 1978]. As stated above, the theory possesses several congenital shortcomings, so that we comment only on a couple of the other criticisms. A first one is that not a single stochastic process but an infinity of them can be associated with a quantum state [Davidson 1979b]; this seems to be a peculiarity of the description in terms of a Schrödinger equation involving only the product $D\sqrt{-\lambda}$, as discussed above, rather than a problem for its stochastic interpretation. Further, it is argued that contrary to what happens with classical diffusions, the quantum stochastic process cannot be separated into 'subprocesses' satisfying a given set of initial conditions [Grabert et al. 1979]; this means that the trajectory of a given particle depends nonlocally on all other trajectories that it *could* have followed, which is of course quite unrealistic and unacceptable. The nonlocality problem in stochastic quantum mechanics, which may be even worse than what has just been said, has been strongly—and rightly—stressed by Nelson (1985, section 23), on the grounds that any physical theory that violates locality is untenable. Thus, some comments on this important question seem in place.

If not the single one, an important cause for the nonlocal behaviour just described is to be found in the arbitrary reduction of the description of the stochastic process to configuration space, which means that partial (local) averagings have been performed on the real physical variables. A clear indication of this is just the expression for the acceleration $\mathbf{a}_Q$, equation (2.59), which is a function of the stochastic velocity and thus of the density of particles ρ (see equation (2.19)): the essentially nonlocal nature of the stochastic velocity $\mathbf{u}$ is transmitted to the acceleration through $\lambda \neq 0$. A similar property of the quantum potential makes Bohm's theory nonlocal; but a fundamental expression for the acceleration *must* be local.

This confirms what we already know: the stochastic formulation of quantum mechanics does not constitute a fundamental theory of the quantum domain, but a phenomenological description of it. As such, it is not to be doomed for its shortcomings; after all, the formula for the acceleration $\mathbf{a}_B$ for classical particles is as nonlocal as the quantum acceleration, but nobody denies the usefulness of the Brownian-motion theory of Einstein and Smoluchowski within its domain of applicability. It even played a most important and historic role in the empirical demonstration of the reality of molecules at the beginning of the century! The error would lie in taking such a partial description as the complete theory. The parameters of the stochastic description of quantum mechanics are selected so as to reproduce the latter, with all its virtues and limitations. Contradictions appear due to the poorness of the configuration-space description: it is too restricted to hold the richness of real phenomena. Moreover, only terms up to second order were considered in the derivative operators, as corresponds to a Markov approximation; but a nonlocal process cannot be Markovian, therefore all higher-order derivatives should have been kept in principle. These basic shortcomings apply to both the classical and the quantum domains, but in the latter case they become much more important due to the relatively high value of the diffusion constant and to the linearity of the Schrödinger equation, which leads to the superposition of amplitudes (cf. the discussion in section 2.1).

Summarizing, one is led to conclude that a stochastic description of quantum systems seems to be feasible, but a correct picture of the microworld in terms of a space-time description would require a more detailed treatment of the underlying stochastic process than the one afforded by the configuration-space theory.

2.2.2. THE MEANING OF OPERATOR ORDERING

Since an appropriate choice of the parameters leads to the usual quantum mechanics, the operator formalism (as a formalism) is easily arrived at from the above formulation. Let us use this possibility to discuss from the standpoint of the above theory the meaning that can be ascribed to the ordering of operators. In order to keep the discussion as simple as possible, we shall consider only products of the fundamental operators x and p. First note that an integration by parts allows us to write

$$\int dx\psi^*(\hat{x}\hat{p}+\hat{p}\hat{x})\psi = -i\hbar\int dx\psi^*\left(x\frac{\partial\psi}{\partial x}+\frac{\partial}{\partial x}x\right)\psi$$

$$= -i\hbar\int dxx\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right) = 2m\int dxxv\rho = 2m\left\langle xv\right\rangle, \tag{2.62}$$

with the systematic velocity

$$v = -\frac{i\hbar}{2m}\left(\frac{1}{\psi}\frac{\partial\psi}{\partial x}-\frac{1}{\psi^*}\frac{\partial\psi^*}{\partial x}\right), \tag{2.63}$$

as given by equations (2.55) and (2.57). Hence,

$$\tfrac{1}{2}\left\langle\hat{x}\hat{p}+\hat{p}\hat{x}\right\rangle = m\left\langle xv\right\rangle, \tag{2.64}$$

showing that the usual quantum correlation $\frac{1}{2}\left\langle\hat{x}\hat{p}+\hat{p}\hat{x}\right\rangle$ gives indeed the correlation between the position coordinate and the systematic velocity v. A similar calculation gives for the commutator

$$\int dx\psi^*(\hat{x}\hat{p}-\hat{p}\hat{x})\psi = -i\hbar\int dx\psi^*\left(x\frac{\partial\psi}{\partial x}-\frac{\partial}{\partial x}x\right)\psi$$

$$= i\hbar\int dx\psi^*\frac{\partial x}{\partial x}\psi = -i\hbar\int dxx\frac{\partial\rho}{\partial x} = -2im\int dxxu\rho = -2im\left\langle xu\right\rangle, \tag{2.65}$$

where equations (2.56) and (2.57) were used for the stochastic velocity,

$$u = \frac{\hbar}{2m}\left(\frac{1}{\psi}\frac{\partial\psi}{\partial x}+\frac{1}{\psi^*}\frac{\partial\psi^*}{\partial x}\right) = \frac{\hbar}{2m}\frac{1}{\rho}\frac{\partial\rho}{\partial x}. \tag{2.66}$$

Hence,

$$\tfrac{1}{2}\left\langle\hat{x}\hat{p}-\hat{p}\hat{x}\right\rangle = -im\left\langle xu\right\rangle, \tag{2.67}$$

which shows that the expectation of the commutator $[\hat{x}, \hat{p}]$ is proportional to the correlation between the position coordinate and the stochastic velocity u. Adding and subtracting the above results, it follows that

$$\frac{1}{m}\langle\hat{x}\hat{p}\rangle = \langle xv\rangle - i\langle xu\rangle = \langle x(v-iu)\rangle, \tag{2.68}$$

$$\frac{1}{m}\langle\hat{p}\hat{x}\rangle = \langle xv\rangle + i\langle xu\rangle = \langle x(v+iu)\rangle. \tag{2.69}$$

These expressions show that the order of the (time-independent) operators in the calculation of the expectation of their product has nothing to do with a temporal order of observations or measurements, but with the sign of some imaginary contributions that have no classical analog, although they are directly related to the effect of the stochastic motions on the respective correlations.

2.2.3. STATISTICAL MEANING OF SCHRÖDINGER-LIKE EQUATIONS

Within the framework of the stochastic theory discussed above the Schrödinger-like equation turns out to be quite unspecific, as was pointed out in relation with equations (2.53). With the aim of making this point clearer, and also for later purposes, we recall here an alternative way of 'deriving' the Schrödinger equation, based merely on general arguments of a statistical nature. Although we have argued earlier on for the need to perform the most complete possible statistical description to arrive at a healthy theory, here the sole intention is to elaborate the relationship between the Schrödinger equation and a naive (configuration-space) stochastic description.

For this purpose it suffices to use as starting point the continuity equation $(\partial\rho/\partial t) + \nabla\cdot\rho\mathbf{v} = 0$ for the density of particles; we assume the flux to be laminar and write the drift (systematic) velocity $\mathbf{v}$ as before in terms of a velocity potential

$$\mathbf{v} = \frac{b}{m}\nabla S, \tag{2.70}$$

with the parameter b so selected as to make the 'action' S a dimensionless function of $\mathbf{x}$ and t. A change of variables from ρ, S to a new complex pair ψ, ψ^*, given by

$$\psi = Re^{iS}, \quad \psi^* = Re^{-iS}, \tag{2.71}$$

whence

$$\rho = \psi^*\psi, \quad R = \sqrt{\rho} \tag{2.72}$$

and

$$\mathbf{v} = \frac{ib}{2m}\left(\frac{\nabla\psi^*}{\psi^*} - \frac{\nabla\psi}{\psi}\right), \tag{2.73}$$

transforms the continuity equation into

$$\psi^*\left(i\frac{\partial\psi}{\partial t}+\frac{b}{2m}\nabla^2\psi\right)-\psi\left(-i\frac{\partial\psi^*}{\partial t}+\frac{b}{2m}\nabla^2\psi^*\right)=0.$$

This expression can be separated into

$$i\frac{\partial\psi}{\partial t}=-\frac{b}{2m}\nabla^2\psi+U\psi \tag{2.74}$$

and its complex conjugate, with U a real separating function that may depend on x, t *and even* on ψ and ψ^*. Note that with small changes (to allow S to be real or imaginary) the procedure can be applied to *any* mechanical system obeying the continuity equation (with a laminar flow), and hence to classical or quantum particles alike [de la Peña 1967]. However, two problems remain, namely, the determination of the constant b and of the function U for a given specific problem. The latter is much simpler, and can be approximately solved for a broad enough class of situations, as follows [Kracklauer 1992, de la Peña and Cetto 1993b, 1994a].

Consider a particle subject to a conservative (external) force $\mathbf{F}(\mathrm{x})=-\boldsymbol{\nabla}V$, a purely stochastic force $\mathbf{F}_{\mathrm{st}}(\mathrm{x})$ and a dissipative force; for example, in the case of an electron the latter would be the radiation reaction $\mathbf{F}_{\mathrm{rr}}(\mathrm{x})=m\tau\,\dddot{\mathrm{x}}$, where $\tau=2e^2/3mc^3$ (this force is discussed at length in section 3.2). For any particle of the ensemble such a situation is described by the Langevin-type equation of motion

$$m\ddot{\mathrm{x}}=\mathbf{F}(\mathrm{x})+\mathbf{F}_{\mathrm{rr}}(\mathrm{x})+\mathbf{F}_{\mathrm{st}}(\mathrm{x}). \tag{2.75}$$

The mere existence of the random force (and possibly of other sources of randomness, such as distributed initial conditions) demands a statistical treatment of the system. Let us therefore consider the ensemble average of equation (2.75)

$$m\left\langle\ddot{\mathrm{x}}\right\rangle=\left\langle\mathbf{F}(\mathrm{x})\right\rangle+\left\langle\mathbf{F}_{\mathrm{rr}}(\mathrm{x})\right\rangle+\left\langle\mathbf{F}_{\mathrm{st}}(\mathrm{x})\right\rangle \tag{2.76}$$

and introduce the following two approximations. Firstly, since $\mathbf{F}_{\mathrm{st}}(\mathrm{x})$ is originally a purely stochastic force, assuming it is negligibly affected by the interaction one can write $<\mathbf{F}_{\mathrm{st}}(\mathrm{x})>=0$. Secondly, in stationary or nearly stationary situations the radiation reaction is a small force and in a first (radiationless) approximation its average value can be neglected; one is thus left with the approximate average expression

$$m\left\langle\ddot{\mathrm{x}}\right\rangle=\left\langle\mathbf{F}(\mathrm{x})\right\rangle. \tag{2.77}$$

(This result can be also used in those classical situations where the mean force $\langle\mathbf{F}\rangle$ can be made to contain the average effect of the frictional forces, although then some modifications in the treatment that follows are required.) Since from the statistical description it follows that $\langle \mathbf{F}(\mathbf{x})\rangle = -\int d^3x\rho\nabla V$ and $\langle \dot{\mathbf{x}}\rangle = \langle \mathbf{v}(\mathbf{x})\rangle = \int d^3x\rho\mathbf{v}(\mathbf{x})$, by combining equations (2.74) and (2.77) one is led to

$$\begin{aligned} m\frac{d\langle \dot{\mathbf{x}}\rangle}{dt} &= \frac{1}{2}b\int d^3x\left(U\psi\nabla\psi^* - \psi\nabla U\psi^* + U\psi^*\nabla\psi - \psi^*\nabla U\psi\right) \\ &= -b\int d^3x\rho\nabla U = \langle \mathbf{F}(\mathbf{x})\rangle = -\int d^3x\rho\nabla V. \end{aligned} \tag{2.78}$$

From this result it follows immediately that $bU = V$, and equation (2.74) becomes

$$ib\frac{\partial\psi}{\partial t} = -\frac{b^2}{2m}\nabla^2\psi + V\psi. \tag{2.79}$$

One thus arrives at a Schrödinger-like equation. The unspecific nature of this equation is clearly revealed by the derivation: (2.79) is one thing and the true Schrödinger equation is another. Here the parameter b is still free, and its value is fixed in general by the specific problem at hand (it may even be imaginary), as follows from the fact that the 'action' has been written as bS, with S a dimensionless function, which can be assumed to be of order 1; the constant b is therefore, in principle, problem-dependent. As a proof of this let us perform the following calculation, that conforms to the above formalism: we write the average energy in the usual form

$$E = \int d^3x\psi^*\left(-\frac{b^2}{2m}\nabla^2\psi + V\psi\right), \tag{2.80}$$

and differentiate this expression with respect to the parameter b, to get

$$\frac{\partial E}{\partial b} = \int d^3x\psi^*\left(-\frac{b}{m}\nabla^2\right)\psi + E\int d^3x\left(\frac{\partial\psi^*}{\partial b}\psi + \psi^*\frac{\partial\psi}{\partial b}\right). \tag{2.81}$$

The second integral vanishes due to the preservation of the normalization, and one is left with

$$b\frac{\partial E}{\partial b} = \int d^3x\psi^*\left(-\frac{b^2}{m}\nabla^2\right)\psi = 2\langle K\rangle, \tag{2.82}$$

where K stands for the kinetic energy.[6] For a one-dimensional system in a stationary situation described by an action integral J we write, assuming the time average and the ensemble average to be equivalent,

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[6] This result can be obtained by direct application of the Feynman-Hellman formula [Feynman 1939], which states that if $\hat{F}(\lambda)$ is a Hermitian operator depending on a parameter λ, and $|\psi(\lambda)\rangle$ and $f(\lambda)$ are its (normalized) eigenvectors and eigenvalues, then $\partial f/\partial\lambda = \langle\psi|\,\partial\hat{F}/\partial\lambda\,|\psi\rangle$.

$$J = \oint p dq = \frac{1}{m}\int_0^{1/\nu} p^2 dt = \frac{2}{\nu}\langle K\rangle = \frac{1}{\nu} b\frac{\partial E}{\partial b}, \tag{2.83}$$

or, since the frequency associated with this action is given by $\nu = \partial H/\partial J$ [see, e.g., Goldstein 1980],

$$b\frac{\partial E}{\partial b} = J\frac{\partial E}{\partial J}. \tag{2.84}$$

This equation has the solution $b = \text{const}\times J$, so that the scale of J is in general fixed by b. In the multidimensional, periodic or quasiperiodic, case the result is qualitatively the same, and one concludes that the parameter b in equation (2.79) should be considered in general problem-dependent. Hence the a priori identification of equation (2.79) with the Schrödinger equation is not justified. Quite on the contrary, a most important task remains open, namely, to demonstrate that in the quantum case there exists a specific force field $\mathbf{F}_{\text{st}}(\mathrm{x})$ leading to a universal value for b (and moreover, equal to Planck's constant), which is of course a highly nontrivial problem (and central to stochastic electrodynamics).

As long as nothing is said about b, equation (2.79) is general and can be applied equally well to classical and quantum systems; but in the former the principle of universality simply does not hold, and the equation becomes useless. This important difference explains the singular role played by Schrödinger's equation in (and only in) quantum theory, and therein resides a specific feature of quantum systems: whereas in the classical case the value of the action integrals is determined by the initial conditions (and thus b becomes highly arbitrary), in the quantum case this parameter should become fixed in a more fundamental way.

Despite its essential character, this point is normally overlooked in the literature, characteristically in the many published attempts to present variants of the above procedure as derivations of the Schrödinger equation from classical arguments. We elaborate further on this question in §10.4.3.

2.3. Semiclassical theories

The feeling of dissatisfaction with some very basic quantum precepts is not confined to the domain of quantum mechanics, but penetrates to some extent into the provinces of the quantum theory of radiation. Despite the fact that QED continues to be the most accurate theoretical instrument available, and is used even as model for the construction of other field theories, a coherent quantum-field-theoretical formulation of it does not exist hitherto. We have not only its ubiquitous divergencies in mind, but also its pragmatic conception leading to a merely formal recipe to quantize the field in total independence of matter (which is its ultimate source, and from which the quantum rules are nevertheless borrowed) and making the renormalization program admissible, despite the fact that it clashes in principle with the canonical formulation, according to which the Lagrangian and the commutation relations should suffice to develop the full theory. For the rest, there is ample acknowledgment of the severe formal problems met by QED, for example with regard to the definition of the product of field operators at equal times, or even the very existence of the fields. The astonishing power of QED very frequently leads people to forget its weaknesses; but they *are* important, precisely in view of the successes of the theory.[7]

Among the problematic concepts of QED, particularly obscure is the notion of the photon. So much so, that a critical attitude towards it was already championed by Einstein himself; recall, for instance, that he never used this term, sticking faithfully to the expression *quantum of radiation*, and near the end of his life he expressed his deep dissatisfaction in a well-known confession [Einstein 1951]:

> All the fifty years of conscious brooding have brought me no closer to the answer to the question, 'What are light quanta?' Of course today anyone thinks he knows the answer, but he is deluding himself.

It is therefore not surprising that the quantum theory of radiation in the optical domain, quantum optics, has also been challenged on numerous occasions, both on intuitive grounds and with more formal arguments related to its mathematical difficulties. Not less than four competing pictorial descriptions of the photon exist at present [Kidd et al. 1989; see also Scully and Sargent 1972], namely: the photon as a fuzzy ball, the photon as a singularity in the electromagnetic field, the photon as a wave packet [recalling the directed needles of radiation of Einstein 1917], and, finally, the formal description used in modern quantum electrodynamics for the propagating photon. The image of the photon as a localized corpuscle, as apparently suggested by the everyday detection processes by localized atoms at fixed times, is inconsistent with quantum electrodynamics [see, e.g., Pryce 1948, Newton and Wigner 1949, Kramers 1958, Pike and Sarkar 1986], since the theory does not contain any dynamical variable that can be put into correspondence with a position operator. In daily practice, the field-theoretic photons are always described in momentum space, as opposed to the fuzzy-ball pictures. In its turn, any attempt to express the electromagnetic field in terms of wave packets with a sharp discrete value of energy and propagating with unlimited stability, seems to be a no less tough enterprise. Even accepting that the photons are shaped during their emission by matter, which apparently was the point of view dominant among German physicists during the heroic quantum period, their apparent stability remains a mystery.

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[7] Detailed discussions can be found in the specialized literature; short discussions are, e.g., Zimmermann (1970) and Steinmann (1981). For comments on these problems from one of the founders of the semiclassical approach, see Jaynes (1990). We are grateful to U. Studer for calling our attention to this work. See also Pais (1986).

During the 19th century —from Dalton to Einstein, the two great atomists, to be more precise— a great many physicists considered the molecules merely as a heuristic concept, devoid of real ontological significance but useful for the concise expression of some properties of matter. Against both the common sense of the day and the continuum description afforded by the classical theories, the notion of discrete molecules emerged victorious at the end. Today, when nobody would question the reality of molecules —we even can 'look' at them under the electron microscope—, it seems that a similar process is taking place with the photon: the great majority of physicists assign to it an ontological status, and only a small, though not negligible, part of them consider it as mere heuristics invented to express efficiently what the formalism says in its own efficient language, and persist in their battle in favour of the continuum. Any prognosis of the outcome is still uncertain; so, it seems better to allow the search to go on.

Among the alternatives to QED that have been developed in the course of time as attempts to solve its problems by challenging the concept of the photon, there is a class of theories known under the generic name of *semiclassical theories.* Their distinctive feature is that they treat matter quantum mechanically, but the electromagnetic field is treated classically. In some sense these theories can be considered as intermediate between classical physics, and a full quantum theory with both matter and field intrinsically quantum. There exist actually different versions of the semiclassical program, with a declared aim that varies widely from being an eventual substitute to QED [Jaynes 1973], to being just an approximate form of the latter [Senitzky 1978]. For reasons of completeness we give here an overview of the semiclassical theories, though very informal and brief, since they are only of collateral interest to us; we refer the interested reader to the specialized literature for more detailed accounts.[8]

The simplest semiclassical theory is constructed by adding the classical electromagnetic field to the quantum description of the atom; however, under such a scheme it becomes impossible to consider consistently the back reaction of the (quantum) atom on the field, and one ends up with a combination of a classical and a quantum (radiated) field. To solve this difficulty, the dynamical variables referring to the matter-field interaction are replaced by their expectation values and are treated not as expectations, but as actual values of the variables. Then the radiated field can be expressed directly in terms of the source variables (c-numbers). Such substitution amounts to decorrelating atomic and field variables, but it is unclear at which moment along the calculation this decorrelation should be performed. For example, for $\hat{A}$ and $\hat{B}$ two atomic dynamic variables the quantum expression $<\hat{A}\hat{B}>$ is replaced by $<\hat{A}><\hat{B}>$; there is always the doubt of whether this substitution was not performed too early, thereby loosing possibly important contributions coming from the correlation $<\hat{A}\hat{B}>$-$<\hat{A}><\hat{B}>$.

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[8] Excellent reviews on the subject are Mandel (1976), Milonni (1976), Jaynes (1978) and Senitzky (1978). The present discussion relies heavily on the last cited reference.

A variant of the theory which pays attention to the above problem is known as *neoclassical theory* [Crisp and Jaynes 1969, Stroud and Jaynes 1970, Jaynes 1973, 1978]. In this approach all variables (including those that refer to matter) are c-numbers, but the laws they satisfy are so postulated that the theory reproduces as much of the quantum atomic behaviour as possible. For example, the atomic Hamiltonian is written right from the beginning in the form of a collection of independent harmonic oscillators, $H = \sum \hbar\omega_n a_n^* a_n$, where the amplitudes $a_n^*(t)$, $a_n(t)$ are c-numbers. The radiation field is deprived of any stochastic element and there is no vacuum electromagnetic field. However, the theory is not free of some of the conceptual problems it tries to avoid. For instance, since the quantum description of the atomic dynamical variables is interpreted as actual, in a decay process the intermediate states with elements from both the decaying and the decay states acquire the status of actual states, leading to a situation conceptually akin to the collapse of the wave function [Milonni 76].

To escape from some of these drawbacks, a different form of the theory has been developed as the semiclassical limit of QED [Senitzky 1978]. Consider for instance the expectation of the number operator $\left\langle \hat{a}_n^\dagger \hat{a}_n \right\rangle = N$, with $\left[\hat{a}_n, \hat{a}_n^\dagger\right] = 1$, and assume that $N \gg 1$; under such conditions the physical effects of the commutator are normally negligible and the treatment of the $\hat{a}_n^\dagger$, $\hat{a}_n$ variables as c-numbers $a_n^*(t)$, $a_n(t)$ becomes possible. In this case $<\hat{a}_n^\dagger \hat{a}_n> \simeq <\hat{a}_n \hat{a}_n^\dagger>$, and thus $<|a_n|^2>= N$; however, since also $<a_n>= 0$, the amplitudes $a_n^*(t)$, $a_n(t)$ should be treated as stochastic variables to account for their dispersion. In other words, it is recognized that if a semiclassical approximation to QED is to be consistent with the quantum behaviour of matter, its electromagnetic field *must* be random.

The semiclassical theories have thus been enriched by allowing the vacuum field to fluctuate. This gives them indeed the possibility of explaining a wide range of phenomena pertaining to the domain of QED, such as spontaneous emission from excited atoms, photoelectric correlations, resonance fluorescence, interference experiments of the Pfleegor-Mandel type[9] [Allen and Eberly 1975], and so on; they have also been applied extensively to laser theory [Haken 1970, Sargent et al. 1974]. However, there are phenomena for which QED and the semiclassical theories give substantially different predictions, as is the case with photon polarization correlations found in a three-level cascade [Clauser 1974] and quantum beats [Chow et al. 1975, Herman et al. 1975]. A central point is that the discrepancies are due not to an approximation of the type 'neglecting the commutator' (treating q-numbers as c-numbers), but to the decorrelation between the variables of the radiated field and those of the radiating atomic dipole; this decorrelation can, for example, bring about a failure to distinguish between pure states and mixtures.[10] From these and other observations, as well as from the results of experiments on photon-counting rates of the kind performed by Aspect et al. (1982b) to test for the violation of certain Bell inequalities, it has been widely concluded that only a consistently quantum calculation can reproduce in detail all the observed results. We defer this discussion until chapter 13, where it will be shown that a consistent consideration of the stochastic zeropoint radiation field may lead to a qualitatively different situation in which the experimental results could find due place and explanation.

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[9] The Pfleegor-Mandel experiment shows interference between two light rays produced

2.4. A semiquantum theory

Fourty years ago Sokolov and Ternov (1955) noticed that the fluctuations in the motions of a quantum electron can be reproduced by considering a classical charge subject to an external stochastic force. In particular, their treatment of synchrotron radiation by means of the notion of 'macromolecules' subject to a random force, constituted one of the earliest attempts to get quantum results from the theory of Markov processes. In a somewhat more sophisticated approach, Sokolov and Tumanov (1956) proposed that the source of the fluctuations resides in the electromagnetic vacuum of QED, or, as is usually stated, in the fluctuations of virtual photons, coupled to the (otherwise classical) electrons.

In this book we will study in detail a theory constructed around a stochastic (c-number) electromagnetic field (the zeropoint radiation field discussed in chapter 4), rather than a quantized field. It is therefore pertinent to give an account of the little known approach to the problem of the origin of the quantum rules by Sokolov and Tumanov, some of whose

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by different coherent sources of low intensity. Interference phenomena of this sort are easy to understand within the wave theory, but they become hard to explain in terms of photons.

[10] Detailed accounts may be seen in Scully (1980) and Milonni (1984b). An analysis of the semiclassical theory with a random zeropoint field from a different viewpoint is given in Theimer and Peterson (1977).

results will be of subsequent interest. The theory as first proposed was restricted to the study of the harmonic oscillator in its ground state, but several years later it was extended by Schiller and Tesser (1971)[11] to the study of mixtures of oscillators and to some other simple problems.

Let us consider a classical charged oscillator in interaction with an electromagnetic radiation field; the calculation will be nonrelativistic and the magnetic term of the Lorentz force will be neglected. Also, to further simplify matters we will consider the one-dimensional problem, the extension to three dimensions being immediate. With the force of radiation reaction taken into account, the equation of motion of the oscillating particle reads

$$m\ddot{x} = -m\omega_0^2 x + e\left(E_x^s + E_x\right), \tag{2.85}$$

where the radiation reaction self-force is given by

$$eE_x^s = m\tau\,\dddot{x}, \quad \tau = \frac{2e^2}{3mc^3}. \tag{2.86}$$

The Coulomb gauge is used, so that the scalar potential is equal to zero. As said above, the electromagnetic field to which the particle is coupled is to be identified with the vacuum field of QED, so we write (see section 3.1)

$$E_x = i\int d^3k\left(\frac{\hbar\omega}{8\pi^2}\right)^{1/2}\left[\hat{a}_x(\mathbf{k})\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} - \hat{a}_x^\dagger(\mathbf{k})\,e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right], \tag{2.87}$$

with the creation and annihilation operators satisfying the transverse field commutation rules

$$\left[\hat{a}_i(\mathbf{k}), \hat{a}_j^\dagger(\mathbf{k}')\right] = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\delta(\mathbf{k}-\mathbf{k}'). \tag{2.88}$$

The problem is further simplified by introducing the long-wavelength approximation, which means neglecting the spatial dependence of the field in (2.87), consistently with the nonrelativistic treatment made here. Then it is possible to solve the resulting equation of motion by a Fourier expansion of $x(t)$, which gives for the stationary solution

$$x = \frac{ie}{2\pi m}\int d^3k\sqrt{\frac{1}{2}\hbar\omega}\,\frac{\hat{a}_x e^{-i\omega t}}{\Delta(\omega)} + \text{h.c.}, \tag{2.89}$$

with

$$\Delta(\omega) = \omega_0^2 - \omega^2 - im\tau\omega^3. \tag{2.90}$$

The authors argue that in this problem the momentum of the particle must be defined as the canonical momentum with the full field taken into account

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[11] The same theory is studied later in Milonni (1984a, 1988) and Eckardt (1986).

(vacuum plus radiated field), and not simply as the mechanical part $m\dot{x}$. The correct definitions can reliably be obtained only through the development of the complete theory, the reason for the proposed selection being that it helps to reduce spurious contributions from the field components of very high frequency, which give rise to infinities and other problems. A similar problem arises when defining the energy of the particle, since it is not clear which part of the total energy is to be ascribed to the particle and which to the field. After some simplifications one arrives at

$$p = m\dot{x} + \frac{e}{c}\left(A_x + A_x^s\right) = \frac{e\omega_0^2}{2\pi}\int d^3k \sqrt{\frac{\hbar}{2\omega}}\frac{\hat{a}_x e^{-i\omega t}}{\Delta(\omega)} + \text{h.c.} \tag{2.91}$$

Since x and p are functions of the field operators $\hat{a}_x$ and $\hat{a}_x^\dagger$, they have become (noncommuting) operators. Their commutator follows from equation (2.88) and one finds, after performing the angular integration,

$$[x,p] = i\frac{e^2\omega_0^2\hbar}{8\pi^2 m}\frac{8}{3}\int\frac{d^3k}{|\Delta(\omega)|^2} = i\frac{4e^2\omega_0^2\hbar}{3\pi mc^3}\int_0^\infty\frac{\omega^2 d\omega}{|\Delta(\omega)|^2}. \tag{2.92}$$

It is not necessary to evaluate the integral in detail; since it is convergent one may simplify it by noting that the main contribution comes from the values of ω near the resonance. With $\omega = \omega_0 + \frac{1}{2}\tau\omega_0^2 z$ and approximating $\left(\omega^2 - \omega_0^2\right)^2 \simeq 4\omega_0^2\left(\frac{1}{2}\tau\omega_0^2 z\right)^2$, one gets

$$\int_0^\infty\frac{\omega^2 d\omega}{|\Delta(\omega)|^2} = \frac{1}{2\tau\omega_0^2}\int_{-2/\tau\omega_0}^\infty\frac{dz}{z^2+1} \simeq \frac{\pi}{2\tau\omega_0^2},$$

where the last integral could be extended to $-\infty$ without too much affecting its value, since $\tau\omega_0 \ll 1$ for optical frequencies (this point is discussed in detail in section 3.2). Finally, one obtains

$$[x,p] = i\frac{2\tau\omega_0^2\hbar}{\pi}\cdot\frac{\pi}{2\tau\omega_0^2} = i\hbar. \tag{2.93}$$

A more accurate approximation would add to this canonical result a small radiative correction, which is here left aside.

Observe that the classical oscillator has been transformed into a quantum oscillator by its contact with the quantum vacuum; its commutator is determined by that of the vacuum field, as is also its average energy at equilibrium. To calculate the energy we use the following formulas (a summation over the polarizations is performed)

$$\langle 0|\,\hat{a}_i(\mathbf{k})\,\hat{a}_j^\dagger(\mathbf{k}) + \hat{a}_j^\dagger(\mathbf{k})\,\hat{a}_i(\mathbf{k})\,|0\rangle = 2\delta_{ij}, \tag{2.94}$$

$$\langle 0|\,\hat{a}_i(\mathbf{k})\,\hat{a}_j(\mathbf{k})\,|0\rangle = 0; \quad \langle 0|\,\hat{a}_j^\dagger(\mathbf{k})\,\hat{a}_i^\dagger(\mathbf{k})\,|0\rangle = 0, \tag{2.95}$$

in combination with equations (2.89, 2.91), giving

$$\mathcal{E}_0 \equiv \langle 0|\,\frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2\,|0\rangle = \frac{4\pi}{3}\frac{\tau\hbar\omega_0^2}{8\pi}\int_0^\infty d\omega\frac{\omega\left(\omega^2+\omega_0^2\right)}{|\Delta(\omega)|^2}. \tag{2.96}$$

Here it is convenient to perform a more exact integration in order to compare with the results of QED; one thus gets

$$\mathcal{E}_0 = \frac{1}{2}\hbar\omega_0 + \frac{1}{2}\hbar\omega_0\cdot\frac{\tau\omega_0}{\pi}\left(\ln\frac{1}{\tau\omega_0} - 1\right). \tag{2.97}$$

The first term is recognized as the energy of the ground state of the oscillator, and coincides with the energy of the field mode of the same frequency. The second term is a radiative correction, similar to the Lamb shift of QED. This correction has been obtained without the need of any regularization, which is apparently a most attractive outcome; however, its numerical value depends strongly on the arbitrary definitions used above.

Although the correction predicted is of the right order of magnitude, the argument of the logarithm is too large. Since this is due to the substantial contributions of the very high frequencies to the integral, the problem can be alleviated by introducing a cutoff at an appropriate high frequency ω_c in the integral of equation (2.96). The correction to the energy thus becomes

$$\delta\mathcal{E}_0 = \frac{\hbar\tau\omega_0^2}{2\pi}\left(\ln\frac{\omega_c}{\omega_0} - 1\right). \tag{2.98}$$

With the usual choice $\omega_c = mc^2/\hbar$ this result virtually coincides with that of QED (see equation (7.102)), where a similar cutoff is required in its nonrelativistic version. In Schiller and Tesser (1971) an even better agreement with QED is secured by introducing a spin term into the equation of motion, but retaining the original definition of the mean energy. Such a procedure reproduces once more equation (2.98) with the ω_c given above, but without the need of any cutoff and with the plus of an extra Darwin term. This is a significant result, which shows an intimate relationship between the zitterbewegung (source of the Darwin term) and the spin, even in a nonrelativistic context and neglecting the dynamics of the latter. We will have opportunity to come back to these matters in section 8.3.

To explore the possibilities of the theory in regard to excited states, Schiller and Tesser (1971) modify the properties of the operators $\hat{a}_i(\mathbf{k})$, $\hat{a}_j^\dagger(\mathbf{k})$ describing the excited field; for a field that is still homogeneous and isotropic, but has a modified energy spectrum, they simply write

$$\left\langle \hat{a}_i(\mathbf{k})\,\hat{a}_j^\dagger(\mathbf{k}) + \hat{a}_j^\dagger(\mathbf{k})\,\hat{a}_i(\mathbf{k})\right\rangle = 2\delta_{ij}S(\omega) \tag{2.99}$$

instead of equation (2.94). Note that this proposal departs from the rules of QED, where the excitations are introduced through the state vectors rather than the field operators; note also that the commutator (2.88) must be left unchanged for the commutator $[x, p]$ to remain unaltered. The mean energy of the oscillator, however, is affected by the change; to lowest order in e it becomes

$$\mathcal{E}_S = \tfrac{1}{2}\hbar\omega_0 S\left(\omega_0\right). \tag{2.100}$$

Since the formalism describes a stationary situation, $S\left(\omega\right)$ must correspond to the equilibrium field, and hence is given by the Langevin function, $S\left(\omega\right) = \coth\left(\hbar\omega\beta/2\right)$, with β the inverse temperature $\beta = 1/k_BT$ and k_B the Boltzmann constant. With this selection, equation (2.100) gives the average energy of a mixture of bosonic oscillators of natural frequency ω_0 in equilibrium with the radiation field at temperature T, in correspondence with Planck's law and the quantum results. Of course, in the limit of high temperatures $S\left(\omega\right) \to 2/(\hbar\omega\beta) = \left(\hbar\omega/2\right)^{-1} k_BT$, and the average energy becomes that of classical oscillators, k_BT.

It is difficult to consider the Sokolov-Tumanov theory as a fundamental description of the electron, since it rests on the highly abstract quantum description of the radiation field, which was constructed after the quantization rules of matter. From the above considerations, however, one may nevertheless draw the conclusion, indulging in some extrapolation, that classical particles and quantum fields cannot coexist if they interact: through the interaction the whole system becomes consistently quantum. Recall that a complementary conclusion was reached with regard to semiclassical theories, where the idea of a purely classical field in interaction with quantum particles was observed to be inconsistent. We thus arrive at a firmly established dictum of modern physics, namely, that no peaceful coexistence among classical and quantum subsystems in interaction seems to be possible; consistent theories require all their interacting subsystems to be either classical or quantum.

2.5. The source of stochasticity

2.5.1. FLUCTUATING METRIC

Once the electron is visualized as an object performing a perpetual random movement, the question arises about the possible causes for such behaviour. The Sokolov-Tumanov theory affords of course a plausible answer; but other, less formal answers can be and have been conceived. In particular, the possibility of considering space-time metric fluctuations as being responsible for such fundamental stochasticity seems very attractive, if only because of the universality of the phenomenon they would produce. Despite its phenomenological description, the idea fits the general rules of modern physics. Indeed, according to quantum theory and general relativity, the energy of the vacuum itself should fluctuate and such fluctuations give rise to fluctuations of the metric at small distances. Moreover, if one believes in quantum gravity and the general theory of relativity, metric fluctuations at short distances are inescapable, and analogous conclusions are suggested by the recent developments in string theory or, more generally, in high energy physics.

The idea of metric fluctuations has indeed received some attention, in different contexts; for instance, in a pioneering work Frederick (1976) studied the effects of stochastic perturbations of the geodesic equation and observed that the random motions induced by the fluctuations of the metric occur with the speed of light. Later on, random transitions with the speed of light were considered in the relativistic versions of stochastic quantum mechanics [Lehr and Park 1977, Vigier 1979, Gaveau et al. 1984].

A complementary and more fundamental point of view is also possible: it goes back to a work by Einstein (1924), where stochastic fluctuations of the metric tensor are considered as a possible representation of a real, all-pervading material field. Then even a small piece of matter uncoupled from that field would be indirectly affected by the fluctuations of the former, through the fluctuations induced on the metric. Little attention has been paid to this idea, but it is still alive; for example, in a recent work [Roy 1992] the background field is identified with the zeropoint radiation field, and this identification is shown to lead to a finite energy density for the vacuum (due to the appearance of a natural cutoff).

Both the mathematical and the physical sides of the fluctuating geometry problem have received some attention over the years. The mathematicians have worked on the construction of different types of statistical geometries and their probabilistic topologies [see, e.g., Schweizer and Sklar 1983], whereas de Broglie (1932-34), Rosen (1949-66) and Blokhintsev (1960-75), among others, have studied the metric fluctuations as a source of the quantum fluctuations; but the results appear scattered in the literature, without having led to an integrated and accomplished theory.[12] A renewed interest seems to be flourishing more recently, with different ideas and viewpoints being explored [we refer for example to Carlton 1976, Namsrai 1986, Cufaro Petroni and Vigier 1984, Bergia et al. 1989, Bergia 1991 and Roy 1986, 1992]. The connections between a phenomenological description of the fluctuations of the metric and the stochastic theory of quantum mechanics, among other things, have been explored by Roy, whereas in their formulation of geometrodynamics, Bergia et al. have considered a model within the Kaluza-Klein framework, showing however that some fundamental problems arise in connection with the conservation of energy. The subject appears worth further efforts, and it would be premature to extract definitive conclusions about its merits and possibilities.

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[12] A list of references until ca. 1980 is given in Vigier (1982).

2.5.2. STOCHASTIC ELECTRODYNAMICS

We have just mentioned the fluctuations of the metric as possible cause and effect of matter fluctuations. Let us now avoid the problem of the chicken and the egg, and simply consider that every physical field fluctuates on a microscopic scale, including the vacuum fields. In modern quantum theory the vacuum fluctuations are commonplace, but their consequences seem not to be fully appreciated. It is true that the fluctuations of the best known vacuum field, the electromagnetic radiation field, are commonly considered to be responsible for several real physical phenomena, such as spontaneous radiation from excited systems or the Casimir forces [see, e.g., Davydov 1965, p. 576; Ballentine 1989, chapter 19; Milonni 1994, chapter 3]. But apart from serving to explain these quantum *corrections*, the vacuum field is usually viewed at as a sort of nuisance —almost as a pest— because it is responsible for not few of the infinities that spoil the otherwise smooth quantum calculations. Thus it is swept away as soon as possible (only to reappear by the back door), with the argument that it is merely a virtual (non-energy conserving) field.

In this volume the point of view adopted is quite different. The vacuum state of the radiation field is seen as a *real* (as opposed to virtual), all pervading stochastic field, as real as any other electromagnetic field. Moreover, far from being considered as merely the source of some small corrections to be put on top of the quantum behaviour of matter, this field is seen as the *source* itself of the stochastic behaviour of matter on the microscopic scale, and therefore also, in principle, as the source of its quantum behaviour. This is the central premise of stochastic electrodynamics. Of course one must admit that all vacuum fields can contribute in principle to the universal background noise and with it, to the fundamentally stochastic behaviour of matter on the microscopic level. However, at least at the scale of atomic and molecular phenomena, which are basically of an electromagnetic nature, it seems reasonable to assume that it is the electromagnetic vacuum that bears the main responsibility for the quantum behaviour of matter. Certainly at deeper levels other vacua may become relevant; one can even play with the idea that all vacuum fields have similar statistical properties, including the same energy per normal mode, so that there appears a kind of *universality*, in the sense that the essential stochasticity of matter is basically independent of the nature of the dominant background field. This point is touched upon again in chapter 4.

In support of the proposal to consider the background field as the source not only of stochasticity, but of the quantum behaviour of atomic systems, we advance the following argument, which will be repeated in chapter 9. Consider, for example, a hydrogenlike atom; such a simple system is characterized by an infinite number of transition frequencies, which, as we know, are not harmonically related for excitation levels that are not very high (for Rydberg or highly excited atoms, the spectrum may become approximately harmonic; but then we are in the semi-classical limit). A spectrum with an infinite number of nonharmonic terms is characteristic of a continuum, i.e., of a system with an infinite number of degrees of freedom, rather than of an isolated system consisting of a single particle (or a limited number of them) and moving in three-dimensional space, which possesses at most three fundamental frequencies (and an infinite number of overtones linearly related to them, of course). It is therefore more reasonable to look at the atom as a mechanical system in intimate contact with a continuum, than to consider it merely as a closed, few-degrees-of-freedom mechanical system.

Taking this argument even further, one realizes that this same intimate contact of the atom with the background field provides in principle a possible mechanism for the appearance of wavelike properties and other similar phenomena that are not understandable in terms of a simple mechanical model for the atom. Far from being a nuisance, the zeropoint radiation field can thus become central for the understanding of the behaviour of atomic matter. This is the hypothesis that is put to test in SED.

CHAPTER 3

ELEMENTS OF ELECTRODYNAMICS

The first part of this chapter has mainly a reference purpose; it is designed to gather some basic tools of electrodynamic theory which most probably the reader knows well, and to present them in a form that will be useful in the following chapters. The last sections are devoted to the electrodynamics of the point charge, with emphasis on the radiation self-effects; a causal theory of the radiating nonrelativistic electron is presented in some detail, for both a point-like electron and an extended structure.

3.1. The free radiation field

3.1.1. ELECTROMAGNETIC POTENTIALS

In the absence of matter, when all source terms (charges and currents) vanish, the electromagnetic field that is left is a pure radiation field, described by the homogeneous Maxwell equations

$$\nabla \cdot \mathbf{D} = 0, \tag{3.1}$$

$$-\frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \nabla \times \mathbf{H} = 0, \tag{3.2}$$

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{3.3}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{3.4}$$

The presence of sources modifies only the first two of these equations. From the last one (which expresses the conjectured absence of magnetic monopoles) it follows that the magnetic field **B** can be written always as deriving from a vector potential **A**,

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{3.5}$$

By combining this with the third Maxwell equation (Faraday's law) one gets $\nabla \times (\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \mathbf{E}) = 0$, which means that one can introduce a scalar potential Φ and thus write (the minus sign in front of Φ is conventional)

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{3.6}$$

With these definitions of the potentials, the third and fourth Maxwell equations are identically satisfied. The first two ones, in combination with the constitutive equations $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, then determine the evolution of the potentials.[1] Since both permittivities are equal to one in empty space (in the cgs units system used here), one gets with the aid of the vector identity $\nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$,

$$\nabla^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = 0, \tag{3.7}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) = 0. \tag{3.8}$$

However, equations (3.5) and (3.6) above do not specify the potentials uniquely; since $\nabla \times (\nabla\Lambda) \equiv 0$ for any Λ, the gradient of a scalar function can be added to the vector potential,

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\Lambda. \tag{3.9}$$

This leaves the electric field given by (3.6) unaffected if the scalar potential is simultaneously transformed into

$$\Phi \longrightarrow \Phi' = \Phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}. \tag{3.10}$$

It is easy to verify that equations (3.7) and (3.8) are invariant under this transformation. The invariance of the field equations with respect to the gauge transformation defined by equations (3.9) and (3.10) reveals that the potentials contain some redundancy, since an infinity of different potentials $(\mathbf{A}, \Phi)$ correspond to given fields $\mathbf{E}, \mathbf{B}$. This arbitrariness is frequently used to simplify the field equations with appropriate gauge selections. Since these matters are discussed in detail in the textbooks, we restrict ourselves here to the two most commonly used gauges.

Let us first choose the potentials so as to uncouple equations (3.7) and (3.8). An inspection reveals that the separation can be achieved if

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0. \tag{3.11}$$

Indeed, substituting in the aforementioned equations one gets

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0, \tag{3.12}$$

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[1] As is usual at the present level of description, we are treating the permittivities ε and μ as constant (in the static approximation), scalar quantities; in the presence of matter they may become (dispersive) tensors.

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \tag{3.13}$$

The set of equations (3.11)-(3.13) is equivalent in all respects to the Maxwell equations in vacuum; equation (3.11) is the Lorentz condition for the potentials, known as the *Lorentz gauge.* From (3.9) and (3.10) it follows that the Lorentz condition can always be satisfied by selecting Λ such that

$$\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0. \tag{3.14}$$

The Lorentz gauge is much used in relativistic quantum electrodynamics because it leads to an explicitly Lorentz invariant theory.

Another useful selection —and the one adopted in this book— is the *Coulomb gauge*, also called *transverse* or *radiation gauge*, defined by the condition

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0. \tag{3.15}$$

Under this condition the Maxwell equations in vacuum transform into

$$\nabla^2 \Phi = 0, \tag{3.16}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{1}{c}\boldsymbol{\nabla}\frac{\partial \Phi}{\partial t}. \tag{3.17}$$

Since there is no further restriction on Φ, one can take $\Phi = 0$; equation (3.17) reduces then to (3.13). Of course this reduction does not occur in the presence of sources. Although the equations arrived at are apparently more complicated, the Coulomb gauge is of great practical value. The reason for this is that the Φ term in equation (3.17) combines with other (longitudinal) contributions, leading to simpler final expressions. The transversality condition (3.15) states that for a free field $\mathbf{A}$ is purely transverse; the precise meaning of this will become apparent below.[2]

Irrespectively of the gauge, in vacuum the fields $\mathbf{E}$ and $\mathbf{B}$ always satisfy a wave equation without sources; indeed, taking the time derivative of (3.2) and the curl of (3.3), and noting from equation (3.1) that $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{E} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$, one gets

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0. \tag{3.18}$$

From the symmetrical position of the electric and magnetic fields in the Maxwell equations it follows that $\mathbf{B}$ satisfies the same wave equation. Summarizing, in the Coulomb gauge the source-free electromagnetic field vectors are solutions of the homogeneous equations

$$\Box^2 \mathbf{A} = 0, \quad \Box^2 \mathbf{B} = 0, \quad \Box^2 \mathbf{E} = 0, \tag{3.19}$$

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[2] A careful discussion of these matters can be found in Cohen-Tannoudji et al. (1989).

where the d'Alambertian operator is defined as $\Box^2 = (1/c^2)\partial^2/\partial t^2 - \nabla^2$.

3.1.2. NORMAL MODES OF THE FIELD

We shall most often be interested in the stationary solutions of equations (3.19). A solution procedure that is most suited for our purpose involves a time Fourier analysis, which implies a spectral decomposition of the fields, with components depending on the position variable x and the frequency ω. In free space it is normal to consider a finite volume of rectangular shape and perform a discrete development of these field components in terms of plane waves. However, it is convenient to write first a more general development that allows for other possible geometries of interest, and specialize afterwards, if desired, to the plane–wave case. Assume the space of interest to be delimited by boundaries of arbitrary geometry, on which the fields satisfy appropriate boundary conditions (typically, those of perfectly conducting walls); the field inside this volume can be written in the form

$$\mathbf{A}(\mathrm{x},t) = \sum_{\alpha}\left(\tilde{\mathbf{A}}_{\alpha}(\mathrm{x},\omega_{\alpha})e^{-i\omega_{\alpha}t} + \tilde{\mathbf{A}}^{*}_{\alpha}(\mathrm{x},\omega_{\alpha})e^{i\omega_{\alpha}t}\right). \tag{3.20}$$

The subscript α represents the set of required indices for the specific development, including the polarization. Substituting in the first of equations (3.19) and the transversality condition, one gets

$$\nabla^2\tilde{\mathbf{A}}_{\alpha} + k_{\alpha}^2\tilde{\mathbf{A}}_{\alpha} = 0, \quad c^2k_{\alpha}^2 = \omega_{\alpha}^2, \tag{3.21}$$

$$\boldsymbol{\nabla}\cdot\tilde{\mathbf{A}}_{\alpha} = 0. \tag{3.22}$$

The remaining equations (3.19) for **E** and **B** are of course automatically satistifed if **A** is a solution of (3.20–3.22). The Helmholtz equation (3.21) together with equation (3.22) and the boundary conditions jointly define a Hermitian eigenvalue problem; hence the modes corresponding to different eigenvalues are orthogonal. This allows us to express the solutions in terms of a family of orthonormal functions, which is achieved by writing

$$\tilde{\mathbf{A}}_{\alpha}(\mathrm{x},\omega_{\alpha}) = b_{\alpha}\mathbf{G}_{\alpha}(\mathrm{x},\omega_{\alpha}), \tag{3.23}$$

so that

$$\mathbf{A}(\mathrm{x},t) = \sum_{\alpha}\left(\mathbf{G}_{\alpha}(\mathrm{x},\omega_{\alpha})b_{\alpha}e^{-i\omega_{\alpha}t} + \mathbf{G}^{*}_{\alpha}(\mathrm{x},\omega_{\alpha})b^{*}_{\alpha}e^{i\omega_{\alpha}t}\right), \tag{3.24}$$

and selecting the functions $\mathbf{G}_{\alpha}$ so as to satisfy the equations

$$\nabla^2\mathbf{G}_{\alpha} + k_{\alpha}^2\mathbf{G}_{\alpha} = 0, \tag{3.25}$$

$$\nabla \cdot \mathbf{G}_\alpha = 0, \tag{3.26}$$

$$\int d^3x \mathbf{G}_\alpha^* \cdot \mathbf{G}_\beta = \delta_{\alpha\beta}. \tag{3.27}$$

In addition, products of any two $\mathbf{G}_\alpha$'s or any two $\mathbf{G}_\alpha^*$'s integrate to zero. The values of the amplitude coefficients b_α are fixed by the specific application. From equations (3.5, 3.6) one obtains for the electric and magnetic fields

$$\mathbf{E}(\mathrm{x}, t) = i \sum_\alpha \frac{\omega_\alpha}{c} \left(\mathbf{G}_\alpha(\mathrm{x}, \omega_\alpha) b_\alpha e^{-i\omega_\alpha t} - \mathbf{G}_\alpha^*(\mathrm{x}, \omega_\alpha) b_\alpha^* e^{i\omega_\alpha t} \right) \tag{3.28}$$

$$\mathbf{B}(\mathrm{x}, t) = \sum_\alpha \left(\nabla \times \mathbf{G}_\alpha(\mathrm{x}, \omega_\alpha) b_\alpha e^{-i\omega_\alpha t} + \nabla \times \mathbf{G}_\alpha^*(\mathrm{x}, \omega_\alpha) b_\alpha^* e^{i\omega_\alpha t} \right). \tag{3.29}$$

3.1.3. HAMILTONIAN OF THE RADIATION FIELD

To evaluate the Hamiltonian of the field we start from the usual expression

$$H = \frac{1}{8\pi} \int d^3x \left(\mathbf{E}^2 + \mathbf{B}^2 \right). \tag{3.30}$$

The electric contribution can be easily calculated with the help of the orthonormality condition (3.27), to get

$$\frac{1}{8\pi} \int d^3x \mathbf{E}^2 =$$

$$\sum_{\alpha,\beta} \frac{\omega_\alpha \omega_\beta}{8\pi c^2} \int d^3x \left(\mathbf{G}_\alpha b_\alpha e^{-i\omega_\alpha t} - \mathbf{G}_\alpha^* b_\alpha^* e^{i\omega_\alpha t} \right) \left(-\mathbf{G}_\beta b_\beta e^{-i\omega_\beta t} + \mathbf{G}_\beta^* b_\beta^* e^{i\omega_\beta t} \right)$$

$$= \sum_{\alpha,\beta} \frac{\omega_\alpha \omega_\beta}{8\pi c^2} \int d^3x \left(\mathbf{G}_\alpha \cdot \mathbf{G}_\beta^* b_\alpha b_\beta^* e^{-i(\omega_\alpha - \omega_\beta)t} + \mathbf{G}_\alpha^* \cdot \mathbf{G}_\beta b_\alpha^* b_\beta e^{i(\omega_\alpha - \omega_\beta)t} \right)$$

$$= \frac{1}{4\pi c^2} \sum_\alpha \omega_\alpha^2 \left| b_\alpha \right|^2. \tag{3.31}$$

To obtain the contribution of the magnetic field we calculate

$$(\nabla \times \mathbf{G}_\alpha) \cdot \left(\nabla \times \mathbf{G}_\beta^* \right) = \nabla \cdot \left(\mathbf{G}_\beta^* \times \nabla \times \mathbf{G}_\alpha \right) + \mathbf{G}_\beta^* \cdot \nabla \times (\nabla \times \mathbf{G}_\alpha)$$

$$= \nabla \cdot \left(\mathbf{G}_\beta^* \times \nabla \times \mathbf{G}_\alpha \right) + k_\alpha^2 \mathbf{G}_\beta^* \cdot \mathbf{G}_\alpha, \tag{3.32}$$

where we used (3.25) and (3.26). Thus, with the help of the divergence theorem and equation (3.27) we get

$$\int d^3x \, (\nabla \times \mathbf{G}_\alpha) \cdot \left(\nabla \times \mathbf{G}_\beta^* \right)$$

$$= \oint_S d\mathbf{S} \cdot \left(\mathbf{G}_\beta^* \times \nabla \times \mathbf{G}_\alpha\right) + k_\alpha^2 \int d^3x \mathbf{G}_\beta^* \cdot \mathbf{G}_\alpha. \tag{3.33}$$

The surface integral represents a contribution to the total power $\oint_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{B})$ flowing out of the volume V; for a source-free field this integral vanishes and we are left with

$$\int d^3x \left(\nabla \times \mathbf{G}_\alpha\right) \cdot \left(\nabla \times \mathbf{G}_\beta^*\right) = k_\alpha^2 \delta_{\alpha\beta}, \tag{3.34}$$

so that

$$\frac{1}{8\pi} \int d^3x \mathbf{B}^2 = \frac{1}{4\pi c^2} \sum_\alpha \omega_\alpha^2 \left|b_\alpha\right|^2. \tag{3.35}$$

Thus, independently of the configuration the electric and magnetic fields contribute equally to the Hamiltonian. Their sum gives

$$H = \sum_\alpha H_\alpha, \tag{3.36}$$

with the contribution of each independent mode α given by

$$H_\alpha = \frac{1}{2\pi c^2} \omega_\alpha^2 \left|b_\alpha\right|^2 = \frac{1}{2\pi} k_\alpha^2 \left|b_\alpha\right|^2. \tag{3.37}$$

Since the fields are stationary, every mode contributes with a constant term to the energy. The result obtained suggests replacing the set of coefficients $\{b_\alpha\}$ by another set $\{a_\alpha\}$ related to the first by

$$b_\alpha = \sqrt{\frac{2\pi c^2 \mathcal{E}_\alpha}{\omega_\alpha^2}} a_\alpha, \tag{3.38}$$

so that the contribution of each mode becomes

$$H_\alpha = \mathcal{E}_\alpha \left|a_\alpha\right|^2, \tag{3.39}$$

where $\mathcal{E}_\alpha = \mathcal{E}(\omega_\alpha)$ is the energy associated with mode α when $\left|a_\alpha\right|^2 = 1$. The vector potential reads now:

$$\mathbf{A}(\mathbf{x}, t) = \sum_\alpha \sqrt{\frac{2\pi c^2 \mathcal{E}_\alpha}{\omega^2}} \left(\mathbf{G}_\alpha(\mathbf{x}, \omega_\alpha) a_\alpha e^{-i\omega_\alpha t} + \mathbf{G}_\alpha^*(\mathbf{x}, \omega_\alpha) a_\alpha^* e^{i\omega_\alpha t}\right). \tag{3.40}$$

Development in terms of plane waves

This is an appropriate moment to go over to the Cartesian description. In the absence of natural boundaries defined by macroscopic bodies, the simplest description is attained by taking the reference volume as a parallelopiped of sides L_1, L_2, L_3 and volume $V = L_1 L_2 L_3$, and imposing periodicity conditions (not boundary conditions) on the fields over the reference volume. In this case the Helmholtz equation (3.25) is satisfied by the normalized harmonic functions

$$\mathbf{G}_{\mathbf{n}\lambda}(\mathbf{x},\mathbf{k}_{\mathbf{n}}) = \sqrt{\frac{1}{V}}\mathbf{e}_{\mathbf{n}}^{\lambda}e^{i\mathbf{k}_n\cdot\mathbf{x}}. \tag{3.41}$$

The vector $\mathbf{k}_{\mathbf{n}}$ has components

$$k_i = \frac{2\pi}{L}n_i, \quad i = x, y, z \tag{3.42}$$

and each n_i can assume any integer value, including zero, so that the potential satisfies the periodicity condition $\mathbf{A}(x_i) = \mathbf{A}(x_i + L_i)$ along each coordinate axis; the mode frequencies are then given by

$$\omega_{\mathbf{n}} = ck_{\mathbf{n}} = 2\pi\sqrt{\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2}}. \tag{3.43}$$

From the transversality condition (3.26) it follows that $\mathbf{k}_{\mathbf{n}} \cdot \mathbf{G}_{\mathbf{n}\lambda} = 0$, and thus for each wave vector $\mathbf{k}_{\mathbf{n}}$ there are two orthogonal directions of the vector $\mathbf{G}_{\mathbf{n}}$, which we have distinguished in (3.41) by means of the polarization index $\lambda = 1, 2$. In other words, for every $\mathbf{n} = (n_1, n_2, n_3)$ the $\mathbf{e}_{\mathbf{n}}^{\lambda}$ represent two polarization vectors which along with $\hat{\mathbf{k}}_{\mathbf{n}} = \mathbf{k}_{\mathbf{n}}/k_{\mathbf{n}}$ form a triplet of orthogonal unit vectors, so that

$$\mathbf{k}_{\mathbf{n}} \cdot \mathbf{e}_{\mathbf{n}}^{\lambda} = 0, \quad \mathbf{e}_{\mathbf{n}}^{\lambda} \cdot \mathbf{e}_{\mathbf{n}}^{\lambda'} = \delta_{\lambda\lambda'}, \tag{3.44}$$

$$\hat{\mathbf{k}}_{\mathbf{n}} = \mathbf{e}_{\mathbf{n}}^{1} \times \mathbf{e}_{\mathbf{n}}^{2}, \quad \mathbf{e}_{\mathbf{n}}^{1} = -\hat{\mathbf{k}}_{\mathbf{n}} \times \mathbf{e}_{\mathbf{n}}^{2}, \quad \mathbf{e}_{\mathbf{n}}^{2} = \hat{\mathbf{k}}_{\mathbf{n}} \times \mathbf{e}_{\mathbf{n}}^{1}. \tag{3.45}$$

The subscript n will sometimes be omitted when there is no risk of confusion; in particular, we shall write $\mathbf{k}$ instead of $\mathbf{k}_{\mathbf{n}}$. Introducing equation (3.41) into (3.40) we have

$$\mathbf{A}(\mathbf{x},t) = \sum_{\mathbf{n},\lambda}\sqrt{\frac{2\pi c^2\mathcal{E}_n}{\omega_n^2 V}}\mathbf{e}_{\mathbf{n}}^{\lambda}\left(a_{\mathbf{n}\lambda}e^{-i\omega_n t+i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{n}\lambda}^{*}e^{i\omega_n t-i\mathbf{k}\cdot\mathbf{x}}\right). \tag{3.46}$$

The contribution from each polarization has been explicitly separated, so that $\mathcal{E}_n$ refers to the energy of the mode with wave vector $\mathbf{k}$ and polarization λ. The electric and magnetic fields are then

$$\mathbf{E}(\mathbf{x},t) = i\sum_{\mathbf{n},\lambda}\sqrt{\frac{2\pi\mathcal{E}_n}{V}}\mathbf{e}_{\mathbf{n}}^{\lambda}\left(a_{\mathbf{n}\lambda}e^{-i\omega_n t+i\mathbf{k}\cdot\mathbf{x}} - a_{\mathbf{n}\lambda}^{*}e^{i\omega_n t-i\mathbf{k}\cdot\mathbf{x}}\right), \tag{3.47}$$

$$\mathbf{B}(\mathbf{x},t) = i\sum_{\mathbf{n},\lambda}\sqrt{\frac{2\pi c^2\mathcal{E}_n}{\omega_n^2 V}}\left(\mathbf{k}\times\mathbf{e}_{\mathbf{n}}^{\lambda}\right)\left(a_{\mathbf{n}\lambda}e^{-i\omega_n t+i\mathbf{k}\cdot\mathbf{x}} - a_{\mathbf{n}\lambda}^{*}e^{i\omega_n t-i\mathbf{k}\cdot\mathbf{x}}\right). \tag{3.48}$$

The complete set of integrals of motion of the free field can be calculated by following a procedure similar to that used above to calculate H. One gets, using the conventional definitions for the Hamiltonian, momentum and intrinsic angular momentum,[3]

$$H = \frac{1}{8\pi}\int d^3x\left(\mathbf{E}^2+\mathbf{B}^2\right) = \sum_{\mathbf{n},\lambda}\mathcal{E}_n a_{\mathbf{n}\lambda}^{*}a_{\mathbf{n}\lambda}, \tag{3.49}$$

$$\mathbf{P} = \frac{1}{4\pi c}\int d^3x\mathbf{E}\times\mathbf{B} = \sum_{\mathbf{n},\lambda}\frac{\mathcal{E}_n}{c}\hat{\mathbf{k}}a_{\mathbf{n}\lambda}^{*}a_{\mathbf{n}\lambda}, \tag{3.50}$$

$$\mathbf{M} = \frac{1}{4\pi c}\int d^3x\mathbf{E}\times\mathbf{A} = i\sum_{\mathbf{n}}\frac{\mathcal{E}_n}{\omega_n}\hat{\mathbf{k}}\left(a_{\mathbf{n}2}^{*}a_{\mathbf{n}1} - a_{\mathbf{n}1}^{*}a_{\mathbf{n}2}\right). \tag{3.51}$$

Note that for every mode, the components of $\mathbf{P}$ and $\mathbf{M}$ are parallel or antiparallel to $\mathbf{k}$ and hence orthogonal to the corresponding components of $\mathbf{E}$ and $\mathbf{B}$. The gauge invariance of H and $\mathbf{P}$ is evident; that of $\mathbf{M}$ follows from the fact that the vector potential is transverse in the Coulomb gauge and the transverse part of $\mathbf{A}$ is gauge invariant. The latter is true because under the gauge transformation (3.9), $\mathbf{A}_{\mathbf{n}}$ transforms as

$$\mathbf{A}_{\mathbf{n}} \to \mathbf{A}'_{\mathbf{n}} = \mathbf{A}_{\mathbf{n}} + i\mathbf{k}\tilde{\Lambda}_{\mathbf{n}}, \tag{3.52}$$

i.e., only the longitudinal part of $\mathbf{A}_{\mathbf{n}}$ can be affected, and this is zero here.

States of circular polarization

Equation (3.51) shows that the intrinsic angular momentum of the field modes is not diagonal in the basis of the linear polarization vectors $\{\mathbf{e}_{\mathbf{n}}^{\lambda}\}$. For certain purposes it is convenient to use a diagonal form, which can be obtained by a change of basis:

$$\mathbf{e}_{\mathbf{n}}^{+} = -\sqrt{\frac{1}{2}}\left(\mathbf{e}_{\mathbf{n}}^{1}+i\mathbf{e}_{\mathbf{n}}^{2}\right), \quad \mathbf{e}_{\mathbf{n}}^{-} = \sqrt{\frac{1}{2}}\left(\mathbf{e}_{\mathbf{n}}^{1}-i\mathbf{e}_{\mathbf{n}}^{2}\right). \tag{3.53}$$

The new unit vectors are again orthogonal,

$$\mathbf{e}_{\mathbf{n}}^{A*}\cdot\mathbf{e}_{\mathbf{n}}^{B} = \delta_{AB}, \quad \mathbf{e}_{\mathbf{n}}^{A*}\times\mathbf{e}_{\mathbf{n}}^{B} = iA\hat{\mathbf{k}}\delta_{AB}, \quad A,B=\pm \tag{3.54}$$

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[3] See, e.g., Cohen-Tannoudji et al. (1989). In particular, a detailed presentation of the angular momentum of the electromagnetic field is given in Compl. B.I; see also Heitler (1966), Appendix.

and the new indices A, B denote right (+) or left (−) circular polarization. The corresponding amplitude coefficients are

$$a_{\mathbf{n}+} = -\sqrt{\frac{1}{2}}\left(a_{\mathbf{n}1} + ia_{\mathbf{n}2}\right), \quad a_{\mathbf{n}-} = \sqrt{\frac{1}{2}}\left(a_{\mathbf{n}1} - ia_{\mathbf{n}2}\right) \tag{3.55}$$

and the Hamiltonian, momentum and intrinsic angular momentum become

$$H = \sum_{\mathbf{n},A} \mathcal{E}_n a^*_{\mathbf{n}A} a_{\mathbf{n}A}, \tag{3.56}$$

$$\mathbf{P} = \sum_{\mathbf{n},A} \frac{\mathcal{E}_n}{c} \hat{\mathbf{k}} a^*_{\mathbf{n}A} a_{\mathbf{n}A}, \tag{3.57}$$

$$\mathbf{M} = \sum_{\mathbf{n},A} A \frac{\mathcal{E}_n}{\omega_n} \hat{\mathbf{k}} a^*_{\mathbf{n}A} a_{\mathbf{n}A}, \quad A = \pm. \tag{3.58}$$

Note that for every single mode,

$$H_{\mathbf{n}A} = \mathcal{E}_n; \qquad \mathbf{P}_{\mathbf{n}A} = (H_{\mathbf{n}A}/c)\hat{\mathbf{k}}; \qquad \mathbf{M}_{\mathbf{n}A} = A(H_{\mathbf{n}A}/\omega_n)\hat{\mathbf{k}}. \tag{3.59}$$

Canonical representation

For many purposes it is more convenient to express the radiation field in terms of a set of canonical variables for every field mode considered as an elementary oscillator. With this aim we introduce the definitions

$$a_\alpha \equiv a_\alpha(t) = a_{\mathbf{n}\lambda} e^{-i\omega_n t} \equiv a_{\mathbf{n}\lambda}(t). \tag{3.60}$$

Although these may seem to be too many names for the same quantity, the various notations will be useful in different circumstances; in each case it should be clear from the context whether a_α carries with it the time factor. Note that we are again combining the indices $\mathbf{n}$ and λ into a single one α, to simplify the notation. With the above definition the amplitudes refer to elementary oscillators, which are advantageously described by the set of canonical variables

$$q_\alpha = i\sqrt{\frac{\mathcal{E}_\alpha}{2\omega_\alpha^2}}\left(a_\alpha - a^*_\alpha\right), \quad p_\alpha = \sqrt{\frac{\mathcal{E}_\alpha}{2}}\left(a_\alpha + a^*_\alpha\right) \tag{3.61}$$

or, inverting,

$$a_\alpha = \frac{1}{\sqrt{2\mathcal{E}_\alpha}}\left(p_\alpha - i\omega_\alpha q_\alpha\right), \quad a^*_\alpha = \frac{1}{\sqrt{2\mathcal{E}_\alpha}}\left(p_\alpha + i\omega_\alpha q_\alpha\right). \tag{3.62}$$

In terms of these new variables the Hamiltonian of the radiation field becomes

$$H = \sum_\alpha H_\alpha = \sum_\alpha \mathcal{E}_\alpha a^*_\alpha a_\alpha = \sum_\alpha \frac{1}{2}\left(p^2_\alpha + \omega^2_\alpha q^2_\alpha\right), \tag{3.63}$$

corresponding to a set of independent linear oscillators of unit mass. The fields are now expressed as follows:

$$\mathbf{A}(\mathbf{x},t) = \sum_{\mathbf{n},\lambda} \sqrt{\frac{4\pi c^2}{\omega_n^2 V}} \mathbf{e}_\mathbf{n}^\lambda \left(p_{\mathbf{n}\lambda} \cos \mathbf{k}\cdot\mathbf{x} + \omega_n q_{\mathbf{n}\lambda} \sin \mathbf{k}\cdot\mathbf{x}\right), \tag{3.64}$$

$$\mathbf{E}(\mathbf{x},t) = \sum_{\mathbf{n},\lambda} \sqrt{\frac{4\pi}{V}} \mathbf{e}_\mathbf{n}^\lambda \left(-p_{\mathbf{n}\lambda} \sin \mathbf{k}\cdot\mathbf{x} + \omega_n q_{\mathbf{n}\lambda} \cos \mathbf{k}\cdot\mathbf{x}\right), \tag{3.65}$$

$$\mathbf{B}(\mathbf{x},t) = \sum_{\mathbf{n},\lambda} \sqrt{\frac{4\pi c^2}{\omega_n^2 V}} \left(\mathbf{k}\times\mathbf{e}_\mathbf{n}^\lambda\right) \left(-p_{\mathbf{n}\lambda} \sin \mathbf{k}\cdot\mathbf{x} + \omega_n q_{\mathbf{n}\lambda} \cos \mathbf{k}\cdot\mathbf{x}\right). \tag{3.66}$$

Transition to the continuum

When working in free space it is frequently convenient to consider the limiting case $L \longrightarrow \infty$, for which the Fourier series development becomes a continuous integral transform. From equation (3.42) it follows that $\Delta k/2\pi = (1/L)\,(\Delta n)_{\min} = 1/L \longrightarrow_{L\to\infty} dk/2\pi$ along each of the three Cartesian coordinates; this leads in the three-dimensional case to the following correspondences, where the l.h.s. refers to the discrete variables and the r.h.s. to the continuous ones:

$$\frac{1}{V}\sum_{\mathbf{n},\lambda} f\left(\mathbf{k_n}, \mathbf{e}_\mathbf{n}^\lambda\right) \leftrightarrow \frac{1}{8\pi^3}\sum_\lambda \int d^3k\, f\left(\mathbf{k}, \mathbf{e}^\lambda(\mathbf{k})\right), \tag{3.67}$$

$$\sqrt{V} a_{\mathbf{n}\lambda} \leftrightarrow \sqrt{8\pi^3} a_\lambda(\mathbf{k}). \tag{3.68}$$

By combining the two expressions one gets the useful relation

$$\frac{1}{\sqrt{V}}\sum_{\mathbf{n},\lambda} f\left(\mathbf{k_n}, \mathbf{e}_\mathbf{n}^\lambda\right) a_{\mathbf{n}\lambda} \leftrightarrow \frac{1}{\sqrt{8\pi^3}}\sum_\lambda \int d^3k\, f\left(\mathbf{k}, \mathbf{e}^\lambda(\mathbf{k})\right) a_\lambda(\mathbf{k}). \tag{3.69}$$

Thus the continuous and discrete amplitudes have different dimensions, just as the discrete (and dimensionless) Kronecker $\delta_{\mathbf{nn}'}$ has different dimensions from those of the corresponding Dirac $\delta(\mathbf{k}-\mathbf{k}')$, for which $\int d^3k\, \delta(\mathbf{k})=1$.[4]

Sums over polarization

Specific calculations frequently involve a sum over the polarization states; such calculations can be simplified with the aid of the following expressions, where the summation is understood as being performed over the two transverse components.

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[4] A more detailed discussion of this point can be seen in Cohen-Tannoudji et al. (1989).

To calculate

$$\sum_\lambda \left(\mathbf{e}_\mathbf{n}^\lambda\right)_i \left(\mathbf{e}_\mathbf{n}^\lambda\right)_j \equiv \sum_\lambda e_i^\lambda e_j^\lambda = e_i^1 e_j^1 + e_i^2 e_j^2$$

one considers the third unit vector $\mathbf{e}_\mathbf{n}^3 \equiv \hat{\mathbf{k}}$ that completes an orthonormal basis, so that by adding and subtracting $e_i^3 e_j^3 = \left(\mathbf{e}_i \cdot \hat{\mathbf{k}}\right)\left(\mathbf{e}_j \cdot \hat{\mathbf{k}}\right) = \hat{k}_i \hat{k}_j$ one gets

$$\sum_\lambda \left(\mathbf{e}_\mathbf{n}^\lambda\right)_i \left(\mathbf{e}_\mathbf{n}^\lambda\right)_j = \left(e_i^1 e_j^1 + e_i^2 e_j^2 + e_i^3 e_j^3\right) - e_i^3 e_j^3 = \delta_{ij} - \hat{k}_i \hat{k}_j. \tag{3.70}$$

With the help of this expression and equations (3.45) one further obtains

$$\sum_\lambda \left(\mathbf{e}_\mathbf{n}^\lambda\right)_i \left(\hat{\mathbf{k}} \times \mathbf{e}_\mathbf{n}^\lambda\right)_j = \sum_k \varepsilon_{ijk} \hat{k}_k, \tag{3.71}$$

$$\sum_\lambda \left(\hat{\mathbf{k}} \times \mathbf{e}_\mathbf{n}^\lambda\right)_i \left(\hat{\mathbf{k}} \times \mathbf{e}_\mathbf{n}^\lambda\right)_j = \delta_{ij} - \hat{k}_i \hat{k}_j. \tag{3.72}$$

Given a vector $\mathbf{g}(\mathbf{x})$, its longitudinal and transverse components are defined with respect to the vector $\mathbf{k}$ in Fourier space, so that the transverse component of $\tilde{\mathbf{g}}(\mathbf{k})$ is obtained by subtracting from it its longitudinal component; thus,

$$\tilde{\mathbf{g}}^\perp(\mathbf{k}) = \tilde{\mathbf{g}}(\mathbf{k}) - (\hat{\mathbf{k}} \cdot \tilde{\mathbf{g}})\hat{\mathbf{k}}, \quad \text{or} \quad \tilde{g}_i^\perp(\mathbf{k}) = (\delta_{ij} - \hat{k}_i \hat{k}_j)\tilde{g}_j. \tag{3.73}$$

This discloses the expression $\sum_j (\delta_{ij} - \hat{k}_i \hat{k}_j)$ as the projection operator for the transverse components in k-space. Going back to x-space one gets $g_i^\perp(\mathbf{x}) = \int d^3y \delta_{ij}^\perp(\mathbf{x} - \mathbf{y}) g_j(\mathbf{y})$, with the transverse delta function given by

$$\delta_{ij}^\perp(\mathbf{x}) = \frac{1}{8\pi^3} \int d^3k (\delta_{ij} - \hat{k}_i \hat{k}_j) \exp(i\mathbf{k} \cdot \mathbf{x}) \tag{3.74}$$

and

$$\delta_{ij}^\parallel(\mathbf{x}) + \delta_{ij}^\perp(\mathbf{x}) = \delta_{ij}\delta(\mathbf{x}). \tag{3.75}$$

After some integrations and regularizations one is led to the following explicit expressions (see Cohen-Tannoudji et al. 1989, Compl. A.I for the details)

$$\delta_{ij}^\perp(\mathbf{x}) = \frac{2}{3}\delta_{ij}\delta(\mathbf{x}) - \frac{1}{4\pi x^2}\left(\delta_{ij} - \frac{3x_i x_j}{x^2}\right), \tag{3.76}$$

$$\delta_{ij}^\parallel(\mathbf{x}) = \frac{1}{3}\delta_{ij}\delta^3(\mathbf{x}) + \frac{1}{4\pi x^2}\left(\delta_{ij} - \frac{3x_i x_j}{x^2}\right). \tag{3.77}$$

Note that neither $\delta_{ij}^\perp(\mathbf{x})$ nor $\delta_{ij}^\parallel(\mathbf{x})$ vanish for $\mathbf{x} \neq 0$.

From (3.73) it is clear that $\mathbf{k}\cdot\tilde{\mathbf{g}}^{\perp} = 0$ and $\mathbf{k}\times\tilde{\mathbf{g}}^{\parallel} = 0$, which translated into x-space become $\nabla\cdot\mathbf{g}^{\perp}=0$, $\nabla\times\mathbf{g}^{\parallel}=0$. This is the basis for the assertion that any vector may be uniquely decomposed into a longitudinal part (which is curlless, hence given by a gradient) and a transverse part (which is solenoidal or divergenceless, hence given by a curl); thus, $\mathbf{g} = \mathbf{g}^{\parallel} + \mathbf{g}^{\perp} = \nabla\Phi + \nabla\times\mathbf{A}$, say. This result is known as Helmholtz's theorem, and explicitly it states that [see, e.g., Milonni 1994, Appendix F]

$$\mathbf{g}^{\parallel}(\mathbf{x}) = -\frac{1}{4\pi}\nabla\int d^3x\frac{\nabla'\cdot\mathbf{g}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \tag{3.78}$$

$$\mathbf{g}^{\perp}(\mathbf{x}) = \frac{1}{4\pi}\nabla\times\nabla\times\int d^3x\frac{\mathbf{g}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \tag{3.79}$$

or alternatively,

$$g_i{}^{\parallel}(\mathbf{x}) = \int d^3x'\delta^{\parallel}_{ij}(\mathbf{x}-\mathbf{x}')g_j(\mathbf{x}'), \tag{3.80}$$

$$g_i{}^{\perp}(\mathbf{x}) = \int d^3x'\delta^{\perp}_{ij}(\mathbf{x}-\mathbf{x}')g_j(\mathbf{x}'). \tag{3.81}$$

The following formulas, which can be obtained from considerations of symmetry and rotational invariance, are useful when performing angular integrations

$$\int d\Omega_{\mathbf{k}}\hat{k}_i\hat{k}_j = \frac{4\pi}{3}\delta_{ij}, \tag{3.82}$$

$$\int d\Omega_{\mathbf{k}}\hat{k}_i\hat{k}_j\hat{k}_l\hat{k}_m = \frac{4\pi}{15}(\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}). \tag{3.83}$$

In particular, from equations (3.73), (3.82) it follows that for any vector $\mathbf{f}$ that does not depend on the polarization vectors,

$$\sum_{\lambda}\int d\Omega_{\mathbf{k}}\mathbf{e}_{\mathbf{k}\lambda}(\mathbf{f}\cdot\mathbf{e}_{\mathbf{k}\lambda}) = \sum_{\lambda}\int d\Omega_{\mathbf{k}}\mathbf{f}^{\perp} = \frac{8\pi}{3}\mathbf{f}. \tag{3.84}$$

3.2. Electrodynamics of the point charge

The nonrelativistic Hamiltonian that describes a charged particle in interaction with the radiation field is

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + e\Phi + \frac{1}{8\pi}\int d^3x\left(\mathbf{E}^{\perp 2} + \mathbf{B}^2\right). \tag{3.85}$$

In writing this expression the electric vector has been separated into its longitudinal and transverse parts, $\mathbf{E} = \mathbf{E}^{\parallel} + \mathbf{E}^{\perp}$, and the contribution from the longitudinal part has been written as the Coulomb potential $e\Phi$ (of course, any other potential function present should be added to $e\Phi$). This has been done as follows. In the Coulomb gauge $\mathbf{E}^{\|} = -\nabla\Phi$, so that

$$\int d^3x\, \mathbf{E}^{\|2} = \int d^3x\, (\boldsymbol{\nabla}\Phi)^2$$

$$= \int d^3x \boldsymbol{\nabla} \cdot (\Phi\boldsymbol{\nabla}\Phi) - \int d^3x\ \Phi\nabla^2\Phi = 4\pi \int d^3x\, \rho(\mathrm{x})\Phi; \tag{3.86}$$

the last equality follows from the Gauss law $\nabla^2\Phi = -\nabla\cdot\mathbf{E}^{\|} = -4\pi\rho(\mathrm{x})$ (cf. equation (3.90) with $\epsilon = 1$). Further, integrating this last expression one gets $\Phi\,(\mathrm{x}) = \int d^3y\rho(\mathrm{y},t)/\,|\mathrm{x}-\mathrm{y}|$; for charged point particles at positions x_i the charge density is $\rho(\mathrm{x}) = \sum_i e_i\delta^3\,(\mathrm{x}-\mathrm{x}_i)$, whence combining,

$$\frac{1}{8\pi}\int d^3x\, \mathbf{E}^{\|2} = \frac{1}{2}\int d^3x\rho(\mathrm{x})\Phi(\mathrm{x}) = \frac{1}{2}\int d^3xd^3x'\frac{\rho(\mathrm{x})\rho(\mathrm{x}')}{|\mathrm{x}-\mathrm{x}_i|}$$

$$= \frac{1}{2}\sum_{i\neq j}\frac{e_ie_j}{|\mathrm{x}_i-\mathrm{x}_j|} + \text{self-interaction terms.} \tag{3.87}$$

This leads to equation (3.85) when the (infinite) self interaction terms are neglected; thus H refers to a regularized Hamiltonian.

From the Hamiltonian (3.85) one obtains the equations of motion for the particle

$$m\dot{\mathrm{x}} = \mathbf{p} - \frac{e}{c}\mathbf{A}, \tag{3.88}$$

$$\dot{\mathbf{p}} = \frac{e}{c}\left[\dot{\mathrm{x}} \times \mathbf{B} + (\dot{\mathrm{x}}\cdot\boldsymbol{\nabla})\,\mathbf{A}\right] - e\boldsymbol{\nabla}\Phi, \tag{3.89}$$

and also the following Maxwell equations in presence of the charge and its current (cf. equations (3.1) and (3.2); the other two Maxwell equations remain unchanged):

$$\boldsymbol{\nabla}\cdot\mathbf{D} = 4\pi\rho, \tag{3.90}$$

$$\boldsymbol{\nabla}\times\mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}, \tag{3.91}$$

where

$$\rho(\mathrm{x},t) = e\delta^3\left(\mathrm{x}-\mathrm{x}_p(t)\right), \quad \mathbf{J}(\mathrm{x},t) = e\dot{\mathrm{x}}_p\delta^3\left(\mathrm{x}-\mathrm{x}_p(t)\right), \tag{3.92}$$

and $\mathrm{x}_p(t)$ is the actual position of the particle. Of course, by combining the above equations one obtains Newton's Second Law with the well-known expression for the Lorentz force,

$$m\ddot{\mathrm{x}} = e\mathbf{E} + e\frac{\dot{\mathrm{x}}}{c}\times\mathbf{B}. \tag{3.93}$$

The complicated structure of the set of equations (3.88)-(3.92) makes it difficult to solve them, even for simple problems. It is therefore important to look for alternative expressions that can lead to a simplified, even if approximate, description of the motion. A convenient procedure is devised by expressing the Hamiltonian in terms of the complete set of canonical variables $(\mathbf{x}(t), \mathbf{p}(t))$ of the particle and $\{q_\alpha(t), p_\alpha(t)\}$ of the field modes, as was explained in section 3.1 (equations (3.63)-(3.66)), and deriving the corresponding set of Hamilton equations,

$$m\dot{\mathbf{x}} = \mathbf{p} - \frac{e}{c}\mathbf{A}, \tag{3.94}$$

$$\dot{\mathbf{p}} = \mathbf{F} + e' \sum_\alpha (\dot{\mathbf{x}} \cdot \mathbf{e}_\alpha)\mathbf{k} \left(q_\alpha \cos \mathbf{k} \cdot \mathbf{x} - \frac{p_\alpha}{\omega_\alpha} \sin \mathbf{k} \cdot \mathbf{x} \right), \tag{3.95}$$

$$\dot{q}_\alpha = p_\alpha - e'(\dot{\mathbf{x}} \cdot \mathbf{e}_\alpha)\frac{1}{\omega_\alpha} \cos \mathbf{k} \cdot \mathbf{x}, \tag{3.96}$$

$$\dot{p}_\alpha = -\omega_\alpha^2 q_\alpha + e'(\dot{\mathbf{x}} \cdot \mathbf{e}_\alpha) \sin \mathbf{k} \cdot \mathbf{x}, \tag{3.97}$$

where $\mathbf{F} = -\boldsymbol{\nabla}(e\Phi)$ (without the Coulomb self-interaction) and $e' \equiv e\sqrt{4\pi/V}$. Of course, both sets of equations are fully equivalent, only the field is described differently. Note carefully that the energy of each field mode is distributed throughout the volume V; there is nothing suggesting localized field oscillators that could be compared with mechanical oscillators. A separation of variables can be performed in the last two equations by introducing the set of field variables

$$c_\alpha(t) = \frac{1}{\sqrt{2\mathcal{E}_\alpha}} \left(p_\alpha - i\omega_\alpha q_\alpha \right), \tag{3.98}$$

$$c_\alpha^*(t) = \frac{1}{\sqrt{2\mathcal{E}_\alpha}} \left(p_\alpha + i\omega_\alpha q_\alpha \right). \tag{3.99}$$

Upon substitution and simplification one gets indeed

$$\dot{c}_\alpha = -i\omega_\alpha c_\alpha + ie\sqrt{\frac{2\pi}{\mathcal{E}_\alpha V}}(\dot{\mathbf{x}} \cdot \mathbf{e}_\alpha)e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{3.100}$$

and its complex conjugate. These equations can be immediately integrated to give

$$c_\alpha(t) = a_\alpha(t) - ie\sqrt{\frac{2\pi}{\mathcal{E}_\alpha V}} e^{-i\omega_\alpha t} \int_0^t (\dot{\mathbf{x}}' \cdot \mathbf{e}_\alpha) e^{-i\mathbf{k}\cdot\mathbf{x}' + i\omega_\alpha t'} dt', \quad \mathbf{x}' = \mathbf{x}(t') \tag{3.101}$$

and its complex conjugate. Here $a_\alpha(t)$ are the free-field amplitudes (cf. (3.61)), which satisfy the corresponding equations without sources,

$$\dot{a}_\alpha = -i\omega_\alpha a_\alpha, \quad \dot{a}^*_\alpha = i\omega_\alpha a^*_\alpha. \tag{3.102}$$

The amplitudes $c_\alpha \equiv c_{\mathbf{k}\lambda}$ differ from a_α by the contribution from the field radiated by the particle, with its velocity as source. By combining (3.94) and (3.95) and using (3.101) one gets the equation of motion

$$m\ddot{\mathbf{x}} = = \mathbf{F} + \mathbf{F}^{(\text{free})}_{\text{Lorentz}} + \mathbf{F}_{\text{self}}, \tag{3.103}$$

where

$$\mathbf{F}^{(\text{free})}_{\text{Lorentz}} = ie\sum_\alpha \sqrt{\frac{2\pi\mathcal{E}_\alpha}{V}} \left(\mathbf{e}_\alpha + \frac{\dot{\mathbf{x}}}{c} \times (\hat{\mathbf{k}} \times \mathbf{e}_\alpha)\right) \times \left(a_\alpha e^{-i\mathbf{k}\cdot\mathbf{x}} - a^*_\alpha e^{i\mathbf{k}\cdot\mathbf{x}}\right) \tag{3.104}$$

represents the Lorentz force due to the free field and $\mathbf{F}_{\text{self}}$ is the self-force on the particle mediated by the field, i.e., the reaction back on the particle due to its own radiation:

$$\mathbf{F}_{\text{self}} = -\frac{4\pi e^2}{V}\sum_\alpha \left(\mathbf{e}_\alpha + \frac{\dot{\mathbf{x}}}{c} \times (\hat{\mathbf{k}} \times \mathbf{e}_\alpha)\right) \times$$

$$\times \int_0^t dt' (\dot{\mathbf{x}}' \cdot \mathbf{e}_\alpha) \cos\left(\omega_\alpha(t'-t) - \mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})\right). \tag{3.105}$$

3.2.1. THE ABRAHAM-LORENTZ EQUATION OF MOTION

The result obtained for the self-force, equation (3.105), is too complicated to be of practical value; some simplifications are required to transform it into a useful expression. In the rest of this section we briefly discuss the standard nonrelativistic treatment of the self-force, and in section 3.3 we present a revised version of it.

In the standard treatment two approximations are introduced. Firstly, recall that the magnetic component of the force is of order $(v/c)^2$ and hence for nonrelativistic motions it is small compared with terms that are linear in the velocity; neglecting this term one is left with

$$\mathbf{F}_{\text{self}} = -\frac{4\pi e^2}{V}\sum_\alpha \mathbf{e}_\alpha \int_0^t dt' (\dot{\mathbf{x}}' \cdot \mathbf{e}_\alpha) \cos\left(\omega_\alpha(t'-t) - \mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})\right). \tag{3.106}$$

A further simplification is made by noting that the argument of the cosine can be written as $\omega_\alpha(t'-t) - \mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}) = \omega_\alpha(t'-t)\left(1 - \hat{\mathbf{k}}\cdot\frac{\mathbf{x}'-\mathbf{x}}{c(t'-t)}\right) \simeq$

$\omega_\alpha(t' - t)$ for nonrelativistic problems;[5] thus,

$$\mathbf{F}_{\text{self}} = -\frac{4\pi e^2}{V}\sum_{\mathbf{k}\lambda}\mathbf{e}_{\mathbf{k}\lambda}\int_0^t dt'(\dot{\mathbf{x}}'\cdot\mathbf{e}_{\mathbf{k}\lambda})\cos\omega_k(t'-t). \tag{3.107}$$

This expression can now be evaluated without difficulty, by first rewriting it in the continuum limit, with the substitution $(1/V)\sum_{\mathbf{k}} \rightarrow (1/8\pi^3)\int d^3k$,

$$\mathbf{F}_{\text{self}} = -\frac{e^2}{2\pi^2}\int dk k^2 \int_0^t dt' \cos\omega_k(t'-t)\sum_\lambda \int d\Omega_{\mathbf{k}}\mathbf{e}_{\mathbf{k}\lambda}(\dot{\mathbf{x}}'\cdot\mathbf{e}_{\mathbf{k}\lambda}) \tag{3.108}$$

and then performing the angular integration using (3.84)

$$\sum_\lambda \int d\Omega_{\mathbf{k}}\mathbf{e}_{\mathbf{k}\lambda}(\dot{\mathbf{x}}'\cdot\mathbf{e}_{\mathbf{k}\lambda}) = \frac{8\pi}{3}\dot{\mathbf{x}}', \tag{3.109}$$

to get

$$\mathbf{F}_{\text{self}} = -\frac{4e^2}{3\pi c^3}\int_0^\infty d\omega\omega^2\int_0^t dt'\dot{\mathbf{x}}(\mathbf{t}')\cos\omega(t'-t). \tag{3.110}$$

Now, since

$$\begin{aligned}
&\int_0^\infty d\omega\omega^2\int_0^t dt'\dot{\mathbf{x}}(\mathbf{t}')\cos\omega(t'-t)\\
&= \frac{d}{dt}\int_0^\infty d\omega\left[\frac{d}{dt}\int_0^t dt'\dot{\mathbf{x}}(\mathbf{t}')\cos\omega(t'-t) - \dot{\mathbf{x}}(\mathbf{t})\right]\\
&= -\ddot{\mathbf{x}}(\mathbf{t})\int_0^\infty d\omega + \pi\frac{d^2}{dt^2}\int_0^t dt'\dot{\mathbf{x}}(\mathbf{t}')\delta(t'-t),
\end{aligned} \tag{3.111}$$

one can write

$$\mathbf{F}_{\text{self}} = -\frac{4e^2}{3c^3}\left(\frac{1}{2}\dddot{\mathbf{x}}\,(t) - \ddot{\mathbf{x}}(\mathbf{t})\frac{1}{\pi}\int_0^\infty d\omega\right) = m\tau\,\dddot{\mathbf{x}}\,(t) - \delta m\ddot{\mathbf{x}}(\mathbf{t}), \tag{3.112}$$

---

[5] This approximation is frequently used and is worth a further comment. Assume that the system under study (the charge, in the present case) moves in a bounded region, of characteristic dimensions a; then the values of interest of $|\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})|$ are not larger than $ak = 2\pi a/\lambda$, where $\lambda = 2\pi c/\omega$ is the wavelength. Hence if the relevant wavelengths are large compared with the dimensions of the system one gets $ak << 1$. For this reason the above approximation, i.e., the neglect of the term $|\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})|$, is known as the *long-wavelength approximation.* It is also known as *dipolar approximation* because the substitution $\exp(-i\mathbf{k}\cdot\mathbf{x}) = 1 - i\mathbf{k}\cdot\mathbf{x} + \cdots$ by 1 in the description of the interaction between field and atom, amounts to considering only the effects of the atomic electric dipole. Note that the neglect of the terms $\mathbf{k}\cdot\mathbf{x}$ amounts to considering the fields $\mathbf{A}, \mathbf{E}$ and $\mathbf{B}$ as functions of time only; this means, in particular, that $\mathbf{B} = \nabla\times\mathbf{A} = 0$. Hence, neglecting the magnetic contribution to the Lorentz force, as was done above, is consistent with the dipole approximation.

where

$$\tau = \frac{2e^2}{3mc^3}, \quad \delta m = \frac{4e^2}{3\pi c^3}\int_0^\infty d\omega. \tag{3.113}$$

The radiation of the particle has therefore two self-effects, within the present approximation.[6] First there is a reaction force on the particle, proportional to the time derivative of the acceleration; this is the well-known *radiation reaction.* The parameter τ determining the strength of this force is, up to a factor of 2/3, equal to the time needed by a light signal to traverse the particle's classical radius $r_c = e^2/mc^2$. For an electron, $\tau \sim 10^{-23}$s, or in atomic units, $\tau = \frac{2}{3}c^{-3} = \frac{2}{3}137^{-3} \sim 10^{-7}$au. The second effect is a contribution to the term $m\ddot{\mathbf{x}}(t)$ in the equation of motion, giving a total or *dressed mass* of the particle of value $m_T = m + \delta m$; however, the value of this electromagnetic contribution to the mass is infinite for the point particle, since $\int_0^\infty d\omega$ is divergent. This well-known infinity can be cured by taking the integral up to a very high but finite frequency ω_c, to get

$$\delta m = \frac{4e^2}{3\pi c^3}\int_0^{\omega_c} d\omega = \frac{4e^2\omega_c}{3\pi c^3} = \frac{2}{\pi} m\tau\omega_c. \tag{3.114}$$

Even for huge values for ω_c, $\delta m/m$ remains much smaller than one, owing to the very small value of τ. For instance, in a nonrelativistic description an (extreme) natural upper frequency limit is given by the *Compton frequency* $\omega_C = mc^2/\hbar \sim 10^5$ au, at which relativistic effects are very significant (also quantum effects are in full activity at the corresponding small time intervals); for this value one has $\tau\omega_C = (2/3)(e^2/\hbar c) = (2/3)\alpha \sim 5 \times 10^{-3}$, where $\alpha = e^2/\hbar c = 1/137$ is the fine-structure constant. One may therefore conjecture that in a more precise calculation the mass correction would be at most of order α.

Collecting results, equations (3.103) and (3.112) give the *Abraham-Lorentz equation* of motion for a point charge in interaction with the electromagnetic field,

$$m_T\ddot{\mathbf{x}} \equiv (m + \delta m)\,\ddot{\mathbf{x}} = \mathbf{F} + \mathbf{F}^{(\text{free})}_{\text{Lorentz}} + m_T\tau\,\dddot{\mathbf{x}}, \tag{3.115}$$

where $\mathbf{F}^{(\text{free})}_{\text{Lorentz}}$ is the Lorentz force due to the external fields and $\mathbf{F}$ stands for any non electromagnetic force acting on the particle. It should be stressed that the self-field terms have the simple form given above only in free space; in the presence of macroscopic bodies that impose boundary conditions on the fields, more general expressions are required, involving the basis functions $\mathbf{G}_\mathbf{n}(\mathbf{x})$ defined in §3.1.3.

In equation (3.115) the bare mass m has been replaced by the dressed mass m_T in the radiation reaction term, since $m_T\tau$ is independent of m

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[6] Similar results apply to the quantum case; see, e.g., Milonni (1994).

and the same change is assumed to have been made in τ. In what follows we will write m instead of m_T everywhere and take the dressed value as the experimental mass, on the basis of the fundamental nonobservability of the bare mass. This point of view, now widely accepted, was suggested for the first time by Kramers (1944, 1950), and may be considered the outset of the renormalization program of modern quantum physics.[7]

3.2.2. THE RADIATION REACTION: A DISCUSSION

We have just seen that to obtain a meaningful result for the mass correction, a regularization is needed; now we will show that the radiation reaction is not less beset by fundamental problems, although from a practical —or rather, pragmatic— point of view they are taken as minor (as is the case with the mass). To show the kind of problems to which we refer, let us consider the simple case of an electron acted on by a homogeneous time-dependent force; the Abraham-Lorentz equation is

$$m\ddot{\mathbf{x}} = \mathbf{F}(t) + m\tau\,\dddot{\mathbf{x}}, \tag{3.116}$$

or

$$\mathbf{a} - \tau\dot{\mathbf{a}} = \mathbf{F}(\mathbf{t})/m, \quad \text{with} \quad \mathbf{a} = \ddot{\mathbf{x}}, \tag{3.117}$$

and its general solution is

$$\mathbf{a}(t) = e^{t/\tau}\left(\mathbf{a}(0) - \frac{1}{\tau m}\int_0^t e^{-t'/\tau}\mathbf{F}(t')dt'\right). \tag{3.118}$$

The first strange feature to be noticed is that since the Abraham-Lorentz equation is of third order, there is need to specify an initial acceleration in addition to the customary initial conditions. However, it happens that the term that depends on the initial acceleration in equation (3.118) grows indefinitely with time, leading to infinite final velocity and infinite energy. The example shows clearly the kind of elementary violations of the laws of physics which the Abraham-Lorentz equation can bring about. A procedure to avoid this difficulty, that apparently goes back to Ivanenko and Sokolov

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[7] An extended review is given by Plass (1961). The radiation reaction force was obtained first by Abraham (1904) and later by Lorentz (1909) in his theory of the classical electron, using the model of an extended particle and taking the point-mass limit. One observes that the radiation reaction force is independent of the structure assumed, whereas the coefficients of all other correction terms depend on the specific structural details. For the extended particle the mass term is finite (see section 3.4), whereas in the point-particle limit all derivative terms higher than the third vanish, and the mass correction becomes infinite, reproducing the previous results. The relativistic version proposed by Dirac (the so-called *Lorentz-Dirac equation*) solves the problem of the infinite mass at the expense of introducing a mixture of advanced and retarded potentials, thus aggravating the problems with causality [see Rohrlich 1965 for an extensive discussion].

(1953) and is of frequent use despite its artificiality, consists in selecting $\mathbf{a}(0)$ such that the *final* acceleration is zero, $\mathbf{a}(\infty) = 0$, which requires setting $\mathbf{a}(0) = (1/\tau m)\int_0^\infty e^{-t'/\tau}\mathbf{F}(t')dt'$. This selection leads to

$$\mathbf{a}(t) = \frac{1}{\tau m}\int_t^\infty dt' e^{-(t'-t)/\tau}\mathbf{F}(t') = \frac{1}{m}\int_0^\infty ds\mathbf{F}(t+\tau s)e^{-s}. \tag{3.119}$$

Now the acceleration remains finite at all times for well-behaved forces, but $\mathbf{a}(t)$ depends on the value of the force at all *future* times $t' \geq t$ up to $t = \infty$; thus, *preacceleration*, another form of noncausal behaviour, arises instead of infinite accelerations.[8] The conclusion is that the Abraham-Lorentz equation is noncausal and contrary to the principles of mechanics; this is sometimes expressed by stating that classical electrodynamics and the notion of point charge are incompatible. However, even though the last conclusion is correct (recall the infinite mass correction and self-energies in equation (3.87) predicted for the point charge) one should not forget that the Abraham-Lorentz equation is only approximate and that the problem with causality may be due to the approximations. Indeed, the original full Hamiltonian theory is causal and thus any exact description should be causal.

To understand the origin of the difficulties of classical electrodynamics with the notion of a point charge, recall that the interaction of the point particle with the electromagnetic field (including its own one) dresses the particle not only by changing its mass, but also by endowing it with an *effective structure*. Although considerations of this sort are not common in classical physics, it is a well-known result of quantum electrodynamics that a point charge acquires an effective extension of the order of Compton's wavelength [see, e.g., Milonni 1994, pp. 71, 402]. Thus, modern physics can easily accept the result that there is no (classical or relativistic) equation of motion for a radiating point particle in interaction with the radiation field, that is free of conceptual difficulties. One is forced to resort to models of particles with structure, either intrinsic or effective, to recover causality in the classical theory. This is the reason we pay attention to the subject in the next sections.

Closer inspection of the solution (3.119) reveals that preacceleration occurs effectively only for very tiny time intervals, not much bigger than τ. Such times are so small indeed that they lie outside the range of present-day empirical tests, and it can therefore be argued that nobody really knows what happens during these intervals. For example, the adherents of the absorber theory of Wheeler and Feynman (1945), which contains advanced and retarded interactions, hold that nothing happens contrary to facts, since for times larger than τ only retarded interactions take place. It has been contended also that already for time intervals of order $\Delta t \approx \hbar/mc^2 \simeq \tau/\alpha = 137\tau$ the Heisenberg energy-time inequality predicts energy fluctuations of the order of the proper energy, so that time intervals of order τ are deep within the quantum domain and any noncausal effect becomes unobservable. Moreover, for energies comparable to mc^2 relativistic effects such as pair creation, vacuum fluctuations, etc., become important and the nonrelativistic theory ceases to be applicable. Thus the classical rendering fails much earlier than the above description, and the whole discussion becomes byzantine. However, as has been stressed by Levine et al. (1977), what is under discussion is the internal consistency of the theory, not its domain of applicability or testability, which is a different question.

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[8] The phenomenon of preacceleration can be exhibited as intrinsic to the Abraham-Lorentz equation and thus independent of the selection of 'initial' conditions, by recasting the equation into the approximate form $m\,(\mathbf{a}(t) - \tau\dot{\mathbf{a}}(t)) \approx m\mathbf{a}(t-\tau) = \mathbf{F}(t)$, or $m\mathbf{a}(t) = \mathbf{F}(t+\tau)$. Specific examples of solutions that exhibit preacceleration may be seen in Plass (1961) and Levine et al. (1977).

3.2.3. THE NEED FOR A CUTOFF

We are now in good condition to discuss the divergent mass and the need for a frequency cutoff. We have seen that a cutoff is required to get a finite mass correction, and for a particle with a finite extension the mass correction is finite, which means that an effective cutoff is introduced by the structure. Qualitatively this can be understood by observing that the extended particle performs an average of the radiation field within its extension, thus cutting off the effects of the components of wavelengths equal to or smaller than the radius of the structure. To look at this in a more quantitative way, assume that the particle has an electric structure represented by a charge density (normalized to unity) and write this density in terms of its Fourier transform, thus[9]

$$\rho_s(\mathbf{r}) = \int d^3k\, \tilde{\rho}_s(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}, \qquad \int d^3r\, \rho_s(\mathbf{r}) = 1. \tag{3.120}$$

Similarly, write the local electric field as

$$\mathbf{E}(\mathbf{r}) = \int d^3k\, \tilde{\mathbf{E}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{3.121}$$

The effective electric field on the particle is given by this field averaged over the structure,

$$\mathbf{E}_{\text{eff}}(\mathbf{r}) = \int d^3r' \rho_s(\mathbf{r}') \mathbf{E}(\mathbf{r}+\mathbf{r}') = 8\pi^3 \int d^3k\, \tilde{\rho}_s(\mathbf{k}) \tilde{\mathbf{E}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{3.122}$$

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[9] Wherever possible we will use the notation introduced here, so that $f(x)$ and $\tilde{f}(k)$ denote functions related by a Fourier transformation, with $f(x) = \int dk\, \tilde{f}(k) e^{ikx}$ and $\tilde{f}(k) = (1/2\pi) \int dx\ f(x) e^{-ikx}$.

To arrive at the last equality the simplest way is to make use of the convolution theorem for Fourier transforms (here written in three-dimensional space)

$$\int d^3r' F(\mathbf{r}+\mathbf{r}')G^*(\mathbf{r}') = (2\pi)^3 \int d^3k\, \tilde{F}(\mathbf{k})\tilde{G}^*(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{3.123}$$

Note incidentally that for $r = 0$ one gets the useful Parseval-Plancherel identity for Fourier transforms (in three-dimensional space)

$$\int d^3r\, F(\mathbf{r})G^*(\mathbf{r}) = (2\pi)^3 \int d^3k\, \tilde{F}(\mathbf{k})\tilde{G}^*(\mathbf{k}). \tag{3.124}$$

A comparison of equations (3.121) and (3.122) shows that the consideration of a structure represented by $\rho_S(\mathbf{r})$ is equivalent to the introduction of a cutoff factor $K(\mathbf{k}) = 8\pi^3\tilde{\rho}_S(\mathbf{k})$. Assume for simplicity a spherically symmetrical charge distribution, so that this factor depends only on the magnitude of $\mathbf{k}$ and not on its direction, $K(c\mathbf{k}) = K(\omega)$. Then, for example, for the distribution $\rho(r) = (\pi a^2)^{3/2}e^{-r^2/a^2}$ one gets $K(\omega) = e^{-a^2\omega^2/4c^2} = e^{-(\omega/\omega_c)^2}$ with $\omega_c = 2c/a$. The cutoff frequency is thus inversely related to the size of the charge structure; of course the specific coefficient depends on the distribution, but up to a factor of order unity the disclosed relationship is general. We will have opportunity to use it several times, and particularly in chapter 12.

The introduction of the cutoff means that the field components of very high frequencies $\omega \gtrsim c/a$, where a represents the extent of the structure, uncouple from the particle and become ineffective. These frequencies correspond to time intervals shorter than a/c, so that if $a > \tau c \approx r_c$, that is, if the particle extends over a radius larger than its classical radius, then all noncausal effects associated with the Abraham-Lorentz equation disappear. Thus the introduction of the structure, or equivalently, of the cutoff, restores causality if $a > \tau c$. According to this discussion, even a cutoff frequency as high as $c/r_c \sim mc^2/\alpha\hbar = 137\omega_C$ (ω_C is again the Compton frequency) would solve the problem of causality, besides giving a very reasonable value for the mass correction, as was seen earlier. However, the nonrelativistic description already fails at frequencies ω such that the energy $\hbar\omega$ becomes of order mc^2 (or equivalently, the de Broglie wavelength becomes of the order of the Compton wavelength); at the velocities associated with such wavelengths the magnetic effects previously neglected also become appreciable. Considerations of this kind suggest rather a cutoff frequency in the region of $\omega_C = mc^2/\hbar$; in fact, within SED this selection will prove to be the appropriate one. Thus, from the present point of view, the cutoff is introduced not because it is possible or practical, but because it *must* be introduced to recover the consistency of the description.

Of course, the cutoff can have its origin in a combination of several factors, the structure of the radiating particle being just one of them; others may be generated by dynamical effects [for an example see Goedecke 1983a, Appendix B]. Notice that in the case of the electron, the distribution ρ_s does not refer to an extended charge but to an *effective structure* originating in the rapid fluctuations of the position of the *point* charge. Indeed, from high-energy scattering experiments probing very small distances we know that the electron is certainly not bigger than $\sim 10^{-15}$ cm, which is two orders of magnitude smaller than its classical radius. To closely pack the hypothetical constituents of the electron into such a small structure would require fantastic kinetic energies, exceeding $10^4\, mc^2$, as follows from the Heisenberg inequalities, so the picture of a *jiggling* point electron seems more natural. This is the point of view usually adopted in QED [see, e.g., Milonni 1994, p. 403].

3.3. Causal version of the Abraham-Lorentz equation

Before embarking ourselves in the study of the electrodynamics of particles with structure, let us return for a moment to our earlier expressions for the point particle and investigate how a more careful approach to the dynamical problem can help us recover the lost causality. To avoid the problems of infinities and acausality, we go back to equation (3.105), where still no long-wave approximation has been made. Assuming the motion to be independent of the polarization —which is a reasonable assumption at least as long as the spinning motion of the particle is neglected—, we sum over the polarizations using (3.70) and take the continuous limit, to get

$$\mathbf{F}_{\text{self}} = -\frac{4\pi e^2}{8\pi^3 c^3}\int d\omega\,\omega^2 \int d\Omega_{\mathbf{k}} \int_0^t dt'\mathbf{u}_{\mathbf{k}} \cos\Big(\omega(t'-t) - \mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})\Big);$$

the angular integration can now be performed and it gives

$$\mathbf{F}_{\text{self}} = -\frac{4e^2}{3\pi c^3}\int_0^\infty d\omega\,\omega^2 \int_0^t dt' K(\omega s)\dot{\mathbf{x}}(t')\cos\omega(t-t'), \tag{3.125}$$

where

$$\mathbf{u}_{\mathbf{k}} = \left(1 - \frac{\hat{\mathbf{k}}\cdot\dot{\mathbf{x}}}{c}\right)\dot{\mathbf{x}}' + \left(\frac{\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}'}{c} - \hat{\mathbf{k}}\cdot\dot{\mathbf{x}}'\right)\hat{\mathbf{k}} \tag{3.126}$$

and

$$K(\omega s) = -3\left(\frac{\cos\omega s}{\omega^2 s^2} - \frac{\sin\omega s}{\omega^3 s^3}\right), \quad s = \frac{|\mathbf{x}-\mathbf{x}'|}{c}. \tag{3.127}$$

In integrating over the solid angle it has been assumed that the vector $\mathbf{x} - \mathbf{x}'$ maintains its direction during the interval $0 < t' < t$; for nonrelativistic motions this should be a good approximation. The dependence of (3.125) on the trajectory of the particle is still too complicated to allow for an explicit evaluation, even in the simplest cases. To simplify it we observe that the factor $K(\omega s)$, which originates in the spatial structure of the electric and magnetic fields previously ignored, plays the role of a structure factor or cutoff function, just as discussed above.[10] For low frequencies such that $\omega s \ll 1$, $K(\omega s)$ has a long plateau of value $K(0) = 1$, whereas for high frequencies it oscillates very fast, with an amplitude that decreases as ω^{-2}. The roughest approximation, $K = 1$, takes us back to the Abraham-Lorentz expression (3.110). A somewhat finer description is achieved by observing that the variable s in (3.127) never attains negative values, so that any average of it is a positive number; therefore we introduce an *effective time* s_0 (corresponding to an *effective radius* $a = cs_0$) and approximate the function $K(\omega s)$ by its value at s_0:$K(\omega s) \Rightarrow K(\omega s_0) \equiv K_0(\omega)$ [Cetto and de la Peña 1983b, de la Peña and Cetto 1993b]. Since s_0 can be taken as the minimum time required for the effects of the self-field to build up, for any $t > s_0$ the self-force can be approximated by

$$\mathbf{F}_{\text{self}}= -\frac{4e^2}{3\pi c^3}\int_0^\infty d\omega\omega^2 K_0(\omega)\int_0^t dt'\dot{\mathbf{x}}(t')\cos\omega(t-t'), \tag{3.128}$$

with

$$K_0(\omega) = \frac{3\sin\omega s_0}{\omega^3 s_0^3} - \frac{3\cos\omega s_0}{\omega^2 s_0^2}. \tag{3.129}$$

Alternatively one can write

$$\mathbf{F}_{\text{self}}= -\int_0^t G_0(t-t')\dot{\mathbf{x}}(t')dt', \quad G_0(t)=\frac{2m\tau}{\pi}\int_0^\infty d\omega\omega^2 K_0(\omega)\cos\omega(t). \tag{3.130}$$

Equations (3.130) are explicitly causal because the self-force depends only on the past trajectory. From the previous considerations it follows that the parameter a playing the role of an effective radius is greater than zero; the point particle is therefore excluded by the theory. This confirms that the introduction of the cutoff function is not merely optional. Also, as the self-force has the form of a retarded velocity–dependent friction, the theory may contain nonlocal (though still causal) 'memory' effects.[11]

Performing the integrations indicated in equations (3.130) one obtains

$$\mathbf{F}_{\text{self}}= -\frac{3m\tau}{s_0^3}\left[\mathbf{x}(t) - \mathbf{x}(t-s_0) - s_0\dot{\mathbf{x}}(t-s_0)\right] \tag{3.131}$$

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[10] This same structure factor was already studied in Bohm and Weinstein (1948).

[11] Observe, however, that an expression of the form $y(t)=\int_0^t dt'G(t,t')x(t')$ does not necessarily imply memory effects; for example, these are absent when $G(t,t')$ is a transition probability for a Markovian process.

for $t > s_0$. This explicitly causal expression for the self-force is quite unusual. Contact with more familiar results can be made by performing a Taylor series development of $\mathrm{x}(t - s_0)$ and $\dot{\mathrm{x}}(t - s_0)$ around $\mathrm{x}(t)$, whence

$$\mathbf{F}_{\text{self}} = -\frac{3m\tau}{s_0}\ddot{\mathrm{x}} + m\tau\,\dddot{\mathrm{x}} + \ldots \tag{3.132}$$

where the dots represent time derivatives of order higher than three: $\mathrm{x}^{(3+n)}$, multiplied respectively by $(s_0)^n$. This is equivalent to the Abraham-Lorentz result with a cutoff (cf. equations (3.112, 3.113)); a comparison with (3.114) shows that the effect of the acquired structure on the mass correction corresponds to a cutoff at $\omega_c' = (3\pi/2s_0)$. This conclusion is confirmed by noting that (3.131) describes a damped oscillator, with solutions of the form $\exp(-\sigma + i\omega)t$ where σ, $\omega \simeq \pi/s_0$, if $\kappa \equiv s_0/\tau \geq 3$ (but κ not too large). Consequently, one can consider the particle as having acquired an effective structure of size $cs_0 \geq 3\tau c$ and performing violent oscillations with a frequency of the order of

$$\omega' = \frac{2\pi}{3\kappa}\frac{c}{r_c}, \tag{3.133}$$

which is not too far from the Compton frequency. Observe that these general features are essentially independent of the detailed motions driven by the external forces; this observation will be recalled later on in chapter 12. Other details of the calculation and comments on the meaning of these results are left for the next section, where a similar situation is met with the extended charge.

3.4. The extended charge

Let us now consider a rigid extended charge in interaction with the electromagnetic field, and determine the radiation reaction force upon it. Various relativistic models for the study of extended charges are known [see, e.g., Nodvik 1964, Rohrlich 1965, Kaup 1966, França et al. 1978]; but the present calculation will be nonrelativistic, so that the notion of a rigid charge distribution $\rho_s(\mathbf{r})$ can still be used and only the electric contribution to the self-force needs to be considered. The distribution is assumed to be spherically symmetric and normalized to 1, to simplify the calculations. The field radiated by the extended particle is then given by the following equations for the electromagnetic potentials:

$$\nabla^2\Phi = -4\pi\rho, \qquad \Box^2\mathbf{A}^{\perp} = \frac{4\pi}{c}\mathbf{J}^{\perp}, \tag{3.134}$$

with solution, with $\mathbf{R} = \mathrm{x} - \mathrm{x}'$,

$$\Phi(\mathrm{x}, t) = \int d^3x \frac{\rho_s(\mathrm{x}', t)}{R}, \tag{3.135}$$

$$\mathbf{A}^{\perp}(\mathrm{x},t)=\mathbf{A}(\mathrm{x},t)=\frac{1}{c}\int d^3x\frac{\mathbf{J}^{\perp}(\mathrm{x}',t-\frac{R}{c})}{R}. \tag{3.136}$$

The self-field at the position of the particle is given in terms of these potentials by

$$\mathbf{E}_s^{\|}(\mathrm{x},t)=-\nabla\Phi(\mathrm{x},t),\quad \mathbf{E}_s^{\perp}(\mathrm{x},t)=-\frac{1}{c}\frac{\partial\mathbf{A}(\mathrm{x},t)}{\partial t}. \tag{3.137}$$

The electrostatic potential $\Phi(\mathrm{x},t)$ of (3.135) is nonretarded, so that it cannot produce a net force over a spherically symmetric charge distribution; hence the radiation reaction force acting on the extended charge is

$$\begin{aligned}\mathbf{F}_{\mathrm{rr}}(\mathrm{x},t)&=e\mathbf{E}_{\mathrm{rr}}(\mathrm{x},t)=e\int d^3x\rho_s(\mathrm{x})\mathbf{E}_s^{\perp}(\mathrm{x},t)\\&=-\frac{e}{c^2}\int d^3x\rho_s(\mathrm{x})\frac{\partial}{\partial t}\int d^3x'\frac{\mathbf{J}^{\perp}(\mathrm{x}',t-\frac{R}{c})}{R}.\end{aligned} \tag{3.138}$$

The current is given according to equations (3.92) by

$$\mathbf{J}(\mathrm{x},t)=e\rho_s(\mathrm{x})\dot{\mathrm{x}}(t), \tag{3.139}$$

so that its transverse part is

$$J_i{}^{\perp}(\mathrm{x},t)=\int d^3x'\delta_{ij}^{\perp}(\mathrm{x}-\mathrm{x}')J_j(\mathrm{x}',t). \tag{3.140}$$

We introduce here the expression (3.76) for the transverse delta-function, to obtain

$$J_i{}^{\perp}(\mathrm{x},t)=\frac{2}{3}J_i(\mathbf{r})-\frac{1}{4\pi}\int d^3x'\frac{1}{R^3}\left(\delta_{ij}-\frac{3R_iR_j}{R^2}\right)J_j(\mathrm{x}',t)=\frac{2}{3}J_i(\mathbf{r}), \tag{3.141}$$

since the integral vanishes due to the assumed spherical symmetry. Therefore,

$$\mathbf{F}_{\mathrm{rr}}(\mathrm{x},t)=-\frac{2e^2}{3c^2}\int d^3x\rho_s(\mathrm{x})\frac{\partial}{\partial t}\int d^3x'\frac{1}{R}\rho_s(\mathrm{x}')\dot{\mathrm{x}}(t-\frac{R}{c}). \tag{3.142}$$

Neglecting the magnetic force as we have done here means that contributions to the force quadratic in the velocity are not taken into account; hence for consistency one should discard in (3.142) all terms that are nonlinear in the velocity, which is equivalent to considering that the operator $\partial/\partial t$ acts only on the velocity. In this linear approximation one gets

$$\mathbf{F}_{\mathrm{rr}}(\mathrm{x},t)=-\frac{2e^2}{3c^2}\int d^3x\int d^3x'\frac{1}{R}\rho_s(\mathrm{x})\rho_s(\mathrm{x}')\ddot{\mathrm{x}}(t-\frac{R}{c}). \tag{3.143}$$

In writing the corresponding equation of motion, it is convenient to separate the mass correction term

$$\delta m = \frac{2e^2}{3c^2} \int d^3r \int d^3r' \frac{1}{R} \rho_s(r) \rho_s(r'), \tag{3.144}$$

so that we add and subtract it, to get the integro-differential equation

$$\begin{aligned} m_0 \ddot{\mathrm{x}} &= \mathbf{F}(\mathrm{x}, t) - \delta m \ddot{\mathrm{x}}(t) \\ &- \frac{2e^2}{3c^2} \int d^3x \int d^3x' \frac{1}{R} \rho_s(\mathrm{x}) \rho_s(\mathrm{x}') \left[\ddot{\mathrm{x}}(t - \frac{R}{c}) - \ddot{\mathrm{x}}(t) \right]. \end{aligned} \tag{3.145}$$

Finally, in terms of the dressed mass $m = m_0 + \delta m$ the equation of motion becomes (with m entering in the definition of τ)

$$m \ddot{\mathrm{x}} = \mathbf{F}(\mathrm{x}, t) - mc\tau \int d^3x \int d^3x' \frac{1}{R} \rho_s(\mathrm{x}) \rho_s(\mathrm{x}') \left[\ddot{\mathrm{x}}(t - \frac{R}{c}) - \ddot{\mathrm{x}}(t) \right]. \tag{3.146}$$

It can be readily verified that in the limit of a point particle this equation reduces to the Abraham-Lorentz equation. The motion described by (3.146) involves memory effects: the solution at time t depends on the past accelerations and previous trajectory. A detailed analysis confirms that this equation predicts causal behaviour under very general conditions, provided only that the extent of the charge exceeds a value of the order of τc [Erber 1961, Moniz and Sharp 1977, França et al. 1978]. This last condition results from the requirement that the mass correction should not exceed the proper mass, or, equivalently, that the bare mass should be nonnegative, as will be seen in a moment. This important result verifies that there is no point limit for the classical theory under the demand of causality; the charge distribution must have a minimum nontrivial size (corresponding essentially to the classical radius) for the theory to be causal.

For a simplification of the analysis it is convenient to express the structure in terms of form factors,

$$\begin{aligned} &\int d^3x \int d^3x' \frac{1}{R} \rho_s(\mathrm{x}) \rho_s(\mathrm{x}') \\ &= \int d^3x \int d^3x' \int d^3k \int d^3k' \frac{1}{R} \tilde{\rho}_s(\mathrm{k}) \tilde{\rho}_s^*(\mathrm{k}') e^{i(\mathrm{k} \cdot \mathrm{x} - \mathrm{k}' \cdot \mathrm{x}')} \\ &= \int d^3x \int d^3R \int d^3k \int d^3k' \frac{1}{R} \tilde{\rho}_s(k) \tilde{\rho}_s^*(k') e^{i(\mathrm{k} - \mathrm{k}') \cdot \mathrm{x}} e^{ik'R\cos\theta_R} \\ &= 4\pi \int d^3x \int dR \int d^3k \int d^3k' \tilde{\rho}_s(k) \tilde{\rho}_s^*(k') e^{i(\mathrm{k} - \mathrm{k}') \cdot \mathrm{x}} \frac{1}{k'} \sin k'R \end{aligned}$$

$$
\begin{aligned}
&= 32\pi^4 \int dR \int d^3k \int d^3k' \tilde{\rho}_s(k)\tilde{\rho}_s^*(k')\delta(\mathbf{k}-\mathbf{k}')\frac{1}{k'}\sin k'R \\
&= 128\pi^5 \int dR \int dkk\, |\tilde{\rho}_s(k)|^2 \sin kR = 16\pi^2 \int dRg(R). \qquad (3.147)
\end{aligned}
$$

After introducing the Fourier transforms we changed an integration variable from $\mathbf{x}'$ to $\mathbf{R}$ and performed the angular integration in $\mathbf{R}$ space; then we integrated over $\mathbf{x}$, over $\mathbf{k}'$ and over the solid angle in k-space and introduced the structure factor

$$g(r) = 8\pi^3 \int dkk\, |\tilde{\rho}_s(k)|^2 \sin kr. \qquad (3.148)$$

With the final change of variable $R = c(t-t')$ the equation of motion becomes [Kaup 1966]

$$m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x},t) - 16\pi^2 m\tau c^2 \int_{-\infty}^{t} dsg\,(c(t-s))\,[\ddot{\mathbf{x}}(s) - \ddot{\mathbf{x}}(t)]\,. \qquad (3.149)$$

The mass correction can be recast in terms of a characteristic radius $\mathcal{R}$ of the charge distribution:

$$\delta m = 16\pi^2 m\tau c \int_0^\infty drg\,(r) \equiv \frac{m\tau c}{\mathcal{R}}. \qquad (3.150)$$

Thus the condition that $\delta m < m$ (or that $m_0 > 0$) is equivalent to

$$\mathcal{R} > \tau c, \quad \mathcal{R}^{-1} = 16\pi^2 \int_0^\infty drg\,(r)\,. \qquad (3.151)$$

Another form of arriving at this condition is by writing $m = m_0 + m\tau c\mathcal{R}^{-1}$, with τ expressed in terms of the observed mass; it follows that

$$m = \frac{m_0}{1 - (\tau c/\mathcal{R})}, \qquad (3.152)$$

which obviously must be positive.

For the case in which the external force $\mathbf{F}(t)$ depends only on time, equation (3.149) can be rewritten in a more interesting form as follows. First the time integral is extended to infinity by introducing the function $g_0(t) = g(t)\Theta(t)$, where $\Theta(t)$ is the Heaviside step function, $\Theta(t) = 1$ for $t > 0$ and $\Theta(t) = 0$ for $t < 0$; by taking the Fourier transform and inverting one gets then

$$\tilde{\mathbf{a}}(\omega) = \frac{\tilde{\mathbf{F}}(\omega)}{m_0\,[1 + 2\pi\eta\tilde{g}_0(\omega)]}; \quad \eta = 16\pi^2\tau c^2 \frac{m}{m_0}, \qquad (3.153)$$

whence $\tilde{G}(\omega) \equiv [m_0(1+2\pi\eta\tilde{g}_0(\omega))]^{-1}$ plays the role of an inverse-mass operator. In terms of the original time variable,

$$\mathbf{a}(t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dsG(t-s)\mathbf{F}(s). \tag{3.154}$$

The requirement of causality entails $G(t) = 0$ for $t < 0$; with $G(t)$ given by the inverse transform of $\tilde{G}(\omega)$, this condition leads to the previous results, including equation (3.151).

Further insight can be obtained by applying the theory to a specific example. Some simple models have been studied in detail, particularly the spherical shell [Bohm and Weinstein 1948, Milonni 1994] and the Yukawa distribution $(\beta^2/4\pi r)\exp(-\beta r)$ [de la Peña et al. 1982]. Here we take the first one due to its simplicity. For a spherical shell of radius a one has

$$\rho_s(r) = \frac{1}{4\pi a^2}\delta(r-a), \qquad \tilde{\rho}_s(k) = \frac{1}{8\pi^3}\frac{\sin ka}{ka}. \tag{3.155}$$

Using this in (3.148) gives after straightforward calculations

$$g(R) = \frac{1}{64\pi^2 a^2}\left[2\text{sign}\left(\frac{R}{a}\right) - \text{sign}\left(2+\frac{R}{a}\right) + \text{sign}\left(2-\frac{R}{a}\right)\right]. \tag{3.156}$$

In the absence of external forces, one thus obtains from (3.149) the equation of motion

$$\ddot{\mathbf{x}} = \frac{\tau c^2}{2a^2(1-\tau c/a)}\left[\dot{\mathbf{x}}(t-\frac{2a}{c}) - \dot{\mathbf{x}}(t)\right]. \tag{3.157}$$

The correction to the mass comes out as $\delta m/m = \tau c/a$, as could be expected; assuming this quantity to be less than one, we look for solutions of (3.157) in the form $\dot{\mathbf{x}}(\mathbf{t}) = \dot{\mathbf{x}}(\mathbf{0})\exp(i\omega - \sigma)t$ with $\sigma \geq 0$. After separating real and imaginary parts the following set of equations are obtained, with $T_0 = (2a^2/\tau c^2)(1-\tau c/a)$, namely,

$$T_0\sigma = 1 - e^{2a\sigma/c}\cos\frac{2a}{c}\omega, \quad T_0\omega = -e^{2a\sigma/c}\sin\frac{2a}{c}\omega. \tag{3.158}$$

From the second equation we see that the sine factor must be negative, so that the minimum value allowed for ω is given by $2a\omega_{\text{min}}/c = \pi$, or

$$\omega_{\text{min}} \geq \frac{\pi}{2}\frac{c}{a}. \tag{3.159}$$

In other words, the particle oscillates with a characteristic frequency of the order of c/a. For a of the order of the Compton wavelength this frequency corresponds to Compton's value $\omega_C = mc/\hbar$, whereas for the classical radius $r_c = \alpha\lambda_C$ the characteristic frequency is α^{-1} times larger.

The amplitude of oscillation can be written as the product of an appropriate average velocity $\bar{v}$ times half the period of oscillation, so that $\bar{v}(2a/c) = 2a(\bar{v}/c) < 2a$, a result that shows that the motion is confined to within the region of space occupied by its structure. In particular, assuming the characteristic frequency to be of the order of ω_C, an oscillation amplitude of the order of the Compton wavelength is obtained. The appearance of oscillatory motions for the free particle is a universal phenomenon, i.e., it occurs for almost any structure, the details of which only fix the specific frequency [de la Peña et al. 1982]. Since similar oscillations are predicted by the causal theory of the 'point' particle, and in both cases they occur even if the structure is merely effective, we conclude that quite generally the radiating electron oscillates with frequencies of the order of the Compton frequency.[12]

One could argue that these oscillations are unimportant, in view of their high frequency and their very short life time, which is of the order of σ^{-1}. However, in this book we assume (as in quantum theory) that the 'free' electron is permanently subject to the action of the fluctuating zero point field, which stimulates by resonance the constant reconstruction of this otherwise 'transient' behaviour; thus the particle ends up performing a permanent vibration with frequency of order ω_C and amplitude λ_C. These are just the features of the *zitterbewegung*,[13] so that it becomes difficult to refrain from considering the motion under discussion as a nonrelativistic version of the zitterbewegung of the electron. Our argument is strengthened by recalling that in QED the effective structure of the electron is ascribed to zitterbewegung [see, e.g., Milonni 1994, chapter 9]. With this identification, the zitterbewegung becomes a prediction of classical electromagnetism, once the vacuum field is incorporated to ensure the necessary ubiquity and persistence of the phenomenon. However, due to the value of the characteristic parameters involved, it can be described correctly and in detail only within a relativistic treatment.

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[12] A further example of this phenomenon can be seen in the numerical solution discussed by Levine et al. (1977); see particularly their figure 2.

[13] The zitterbewegung of the electron was discovered by Schrödinger (1930) while studying the Dirac equation. It consists of a violent vibration with frequency around $2\omega_C$ and amplitude λ_C, usually interpreted as the result of interference between the positive- and negative-energy solutions of the Dirac equation. Since packets with only positive-energy component-states cannot have spreads smaller than λ_C, this value can be seen as an effective radius of the electron. The zitterbewegung can be related to the localization of the wave packet over distances of order λ_C, also by observing that such tight localization implies energy spreads due to the momentum fluctuations large enough to allow for pair creation [see, e.g., Milonni 1994, chapter 9]. The zitterbewegung gives rise to observable effects, a well-known one being the Darwin term that enters as part of the relativistic corrections to the atomic levels. There are also proposals to link the spin and the zitterbewegung [Huang 1952, Feshbach and Villars 1958, Hestenes 1985, etc.]; we met a link of this sort when discussing the theory of Schiller and Tesser (1971) in section 2.4.

The importance for quantum theory of the appearance of fast vibrations that can possibly be identified with the zitterbewegung has been emphasized by Cavalleri (1985) and Rueda (1993a). We will have opportunity to find important applications of the above results within SED in chapters 8 and 12.

Part II

Theme

CHAPTER 4

THE ZEROPOINT RADIATION FIELD

In this chapter we introduce the fundamental principle of stochastic electrodynamics, namely, that space is not empty but occupied by a collection of random zeropoint fields, which together constitute the physical vacuum. Further, we propose that of all these vacuums its electromagnetic component, the *zeropoint radiation field*, is especially relevant for an understanding of atomic and quantum physics. SED explores thus the quantum domain in an attempt to find out to what extent it can be understood to be a consequence of the interaction of matter with the zeropoint radiation field. The basic properties of this field are discussed in detail.

4.1. Discovery and nature of the zeropoint field

In classical electrodynamics a strong assumption is customary, namely, that in the absence of sources no electromagnetic field exists. Since the fields are generated by charged matter, such an assumption is *apparently* natural; however, after a moment's reflection one perceives that it is unnecessarily restrictive. As argued for instance by Boyer (1975a), instead of the null classical solution to the source-free Maxwell equations, a more natural and general boundary condition would be a nonzero random field at infinity.

As is well known, however, it was only with the advent of quantum theory that the idea of a zeropoint field began to take shape. While elaborating his second theory of the blackbody law, Planck (1911, 1912) arrived at the expression (in modern notation)

$$\mathcal{E}(\omega, T) = \frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} \tag{4.1}$$

for the average energy of an elementary radiator or oscillator (or molecule, for practical purposes) of natural frequency ω in equilibrium with the radiation field at temperature T. The zeropoint energy appeared thus for the first time, represented by the term $\mathcal{E}(\omega, 0) = \frac{1}{2}\hbar\omega$.

Contrary to classical physics, where all motions are assumed to freeze at absolute zero, this result shows that in the atomic world fluctuations continue to take place even at $T = 0$. Several authors, notably Einstein and Stern, immediately assigned an importance to this discovery and set out to build upon the new theory, as is discussed in section 5.2. Thereafter, Debye (1914) observed that the fluctuations of the zeropoint energy should produce a blurring of the X-ray diffraction patterns similar to that produced by the thermal motion of the atoms in a crystal, but persisting down to the absolute zero of temperature; this is now called the Debye-Waller effect [for a more complete discussion see Milonni 1994, chapter 1].[1]

Now, Planck's theory predicted the zeropoint energy for the mechanical oscillators, but for the electromagnetic field modes the spectral energy distribution was still given by (4.1) *without* the zeropoint contribution. It was Nernst (1916) who argued that the difference between field and matter oscillators is inadmissible if these systems are to attain statistical equilibrium when in thermal contact, and that equation (4.1) should therefore hold for both cases. The concept of the *zeropoint field* was thus born. Nernst considered this field as an all-pervasive, minimum fluctuating electromagnetic field of unknown origin, and he even conjectured on the need to introduce a cutoff in order to maintain its total energy content finite. This field was so real for him, that he ventured the idea of seeing it as a possible source of the elementary interaction potentials between particles [Nernst 1916, Appendix] and as a possible source of useful energy, a proposal to which several people have come back in more recent times [e.g., Forward 1984; a brief discussion of these ideas is presented in §6.2.5].

Nernst also saw that the zeropoint field could help explain atomic stability by providing a mechanism to compensate for the energy lost through radiation by the orbiting electrons, and he speculated that this field could well be the source of the quantum properties of matter. Physics went along a different course and Nernst's ideas were soon forgotten; however, we will see them recover their intrinsic value once the zeropoint field is taken seriously into consideration.

With time, the idea of a zeropoint fluctuating radiation field has found support, both on the theoretical and on the experimental side. There are several phenomena widely considered to be caused by the interaction of matter with the zeropoint field. Some of the most frequently adduced ones are the van der Waals and Casimir forces between macroscopic bodies [e.g.,

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[1] Direct evidence of the term $\frac{1}{2}\hbar\omega$ in the energy levels of the molecular vibrational spectra was obtained by Mulliken from his studies of the spectrum of boron monoxide, as early as 1924, before the advent of the modern quantum description. This is usually considered the first empirical evidence of the reality of the zeropoint energy; since then, a continuous string of experimental results have been reported to verify the presence of the zeropoint fluctuations in different instances. Some of them are: the homopolar binding between the H atoms in the hydrogen molecule, due to the decrease in zeropoint energy when the electrons move in the broader potentiall well of the two protons; the failure of liquid He_4 to solidify at normal pressures as the temperature approaches zero [Finkeenburg 1963, p. 468], and the diffuse scattering of X rays [Peierls 1955, chapter III and p.135; Ewald 1962, pp. 26, 230] and of neutrons [Beacon and Pease 1955] by crystals at very low temperatures. Additional examples are mentioned in Boyer (1970c) and Sciama (1991).

Power 1965, chapter 3, Boyer 1970c, Milonni 1976, Spruch and Kelsey 1978, Weinberg 1989; these are discussed in chapter 6], and the natural atomic linewidth, which has been proved to be due in equal parts to the zeropoint field and radiation reaction [e.g., Milonni et al. 1973, Senitzky 1973; see also chapters 7 and 11]. Further, the suggestion by Welton (1948) to view the Lamb shift as due to fluctuations impressed on the electron by the zeropoint field has met wide acceptance. Specifically, following a suggestion by Feynman (1961), Power (1966b) obtained the atomic Lamb shift as the change in the zeropoint energy arising from the dielectric effect of the H atoms on the vacuum (see section 6.3). More recently, attention has been focussed on the squeezed states of light as another effect due to the zeropoint field (Santos 1990a). We recall also that in his classical review article, Weisskopf (1949) treats the QED vacuum field as an entirely real entity, as is taken for granted by many physicists in their daily work [see, e.g., Yuen 1988].[2]

A frequently raised objection against the existence of a real zeropoint field is that it implies huge effects —both electromagnetic and gravitational ones— which could not possibly pass unnoticed. It is well known that for this reason Pauli strongly opposed the notion of a real zeropoint field, and this line has been explicitly followed by some contemporary authors [see, e.g., Enz 1974, Onley 1973 and rebuttal in Boyer 1974b]; in fact, the prevalent picture in present-day QED is that of a merely virtual field. A partial answer proposed by SED with respect to the electromagnetic effects is that they do not at all pass unnoticed, but quite on the contrary, they are being systematically observed in the form of quantum properties of matter. Of course this does not mean a direct observation of the field, but an indirect one through its effects (such as, perhaps, the structure and stability of atoms, the nature of spectra, and phenomena such as van der Waals forces and the like).

Even so, it is true that the zeropoint field would seem to produce other effects (such as the blackening of photographic emulsions in the absence of light, dark current in photodetectors, and so on) which are simply not observed. These problems are not clearly solved in SED, which means that our understanding of the excitation mechanisms in presence of the zeropoint field is still far from satisfactory. A more detailed discussion of these matters is given in chapter 13. The gravitational consequences associated with the immense —indeed, formally infinite— energy content of the vacuum field, will be commented upon in some detail in §4.2.4 below.

[2] The idea of a real zeropoint field is certainly much more general than a few examples may suggest, and it reappears time and again in almost all contexts of contemporary physics; see, e.g., Sakharov (1968), Misner et al. (1973). For a nontechnical introduction to the quantum vacuum see Aitchison (1985); for the SED vacuum see Boyer (1985).

4.1.1. THE ORIGINS OF STOCHASTIC ELECTRODYNAMICS

Nernst's proposal that the zeropoint radiation field is real and produces observable effects to be identified with quantum phenomena is so intuitively appealing that it has been rediscovered and reformulated independently several times on different grounds. Particularly interesting is a paper by Park and Epstein (1949) were the proposal was made explicit once again and used perhaps for the first time to perform a quantum calculation. The essence of their argument is as follows. Consider the vacuum field modes of QED; according to quantum theory, their (ground state) energy $\frac{1}{2}\hbar\omega$ is due to the unavoidable quantum fluctuations of the vacuum field amplitudes, which satisfy the set of Heisenberg inequalities

$$\Delta E_x \Delta B_y \geq 2\pi\hbar c/(\delta l)^4, \tag{4.2}$$

and so on, where ΔE_x, ... are the fluctuations of the field components within a region of space of linear dimensions δl. Indeed, from (4.2) it follows that the average energy within the volume $(\delta l)^3$ due to these fluctuations is $\delta\mathcal{E} \approx 2 \cdot 3(\Delta E_x)^2(\delta l)^3/8\pi \geq 3\hbar c/2\delta l \approx (3/2)\hbar\omega$, where the last equality is obtained assuming that the most important contributions come from frequencies of order $\omega \sim c/\delta l$. Since this is just the energy of the vacuum field, the fluctuations should be considered as real; however, since the total energy of the field $\frac{3}{2}\int_0^\infty d\omega\, \hbar\omega$ is infinite, they are at the same time unobservable.

As an application, Park and Epstein considered the spontaneous atomic emission as stimulated by the vacuum field and used this idea to derive the Planck distribution following a modified form of the Einstein A-B argument of 1917. Here we find for the first time the interesting explanation of the atomic 'spontaneous' emissions as stimulated emissions, but due to the vacuum field and not to an external field. This point of view fits well with the proposal made almost simultaneously by Welton (1948) to consider the Lamb shift of the atomic levels as a result of the additional fluctuations impressed by the zeropoint field. Both proposals afford a rich heuristic picture of the phenomenon under consideration that allows for a (semi)quantitative treatment. In later chapters we study such problems in detail from the point of view of SED and get thus a better appraisal of these early proposals. For example, we will see that the vacuum fluctuations and the radiation reaction contribute with equal shares to the spontaneous emissions, so that the above identification, though nowadays relatively extended, is not entirely correct.[3]

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[3] A discussion of these same topics from the point of view of QED can be seen in Milonni 1994, chapter 1. A revised form of the Park-Epstein argument with due consideration to the radiation reaction contribution, is given in Jiménez et al. (1980).

What may perhaps be considered the first calculation within SED was made by Kalitsin (1953), who used the points of view of Welton and Weisskopf to determine the dispersions of $x(t)$ and $p(t)$ for a radiating harmonic oscillator subject to the action of the (nonquantized) zeropoint field, and to derive approximate expressions for the Lamb shift and the radiative correction to the mass. Kalitsin's model stimulated further research on the harmonic oscillator and its radiative corrections, notably by Adirovich and Podgoretskii (1954), Braffort et al. (1954) and Sokolov and Tumanov (1956).

It is interesting to observe that the views of these authors on the origin and nature of the zeropoint field vary widely. Where Kalitsin sees a classical random field, Adirovich and Podgoretskii consider both a random and the second quantized zeropoint fields as different possibilities and show that only for the second one is the energy of the ground state dispersionless, even though the two are equivalent for other calculations, such as the ground-state energy or the Lamb shift. As discussed in section 2.4, for Sokolov and Tumanov the zeropoint field is quantized and, finally, for Braffort and coworkers it is a classical residual field within the Wheeler-Feynman absorber theory.

The much less known work by Sinelnikov (1956) should be mentioned apart, because of its conceptual richness and the total lack of calculations; the author goes as far as to view the background field as the direct cause of the undulatory properties of matter. This idea is part of the handwaving of SED and is worth closer attention, because it helps to develop a heuristic picture of some of the most obscure properties of quantum systems. For illustrative purposes, assume a typical double slit setup, with the detector far away from the slits; let $P(x|A)$ be the probability that the electron reaches point x on the plate when slit A is open and slit B is closed; let $P(x|B)$ be the complementary probability when B is open and A closed; finally, let the probability that the electron reaches point x with both slits open be denoted by $P(x|A;B)$. Quantum mechanics tells us that $P(x|A;B)$ is different from $P(x|A)+P(x|B)$, but it does not say why. In Feynman's words [Feynman et al. 1965, chapter 1], here lies "the only mystery of atomic behaviour": how is it that the particle when passing through one of the slits 'knows' whether the other one is open or not?

Now take a look at this question from the perspective of SED with its all-pervading random radiation field. Any nearby body modifies the background field, and one may conceive that in the neighbourhood of a periodic structure the radiation is enhanced in the direction of the Bragg angles. Under the assumption that the electron responds mainly to the waves of the zeropoint field of wavelength close to de Broglie's wavelength, their main effect on the particle will be to produce those angular deviations that tend to give shape to the observed interference pattern. Hence the particle needs to 'know' nothing: it is the random background that carries the required 'knowledge' and operates accordingly on the particle.[4]

Braffort et al. (1954) came to the idea of the zeropoint field within the framework of the absorber theory, by considering that there must exist a remnant field, due to the irregular motions of the atoms in the absorber. They argue that, despite the statistical equivalence of their field to the vacuum of QED, the two fields are logically independent, theirs being a classical field produced by the absorber. A more detailed study of this problem by Pegg (1980) shows that even though zeropoint fluctuations are not present in the absorber theory, there exists nevertheless an effective field from the absorber, the fluctuations of which mimic those of the QED vacuum and give the illusion of a quantized vacuum boson field. Although SED is conceptually far from the absorber theory, it is interesting to note how the difficulties associated with an otherwise total infinite energy can be mitigated by considering the vacuum field not as a source-free field, simply added to the universe, so to speak, but rather as a source field generated by matter (the absorber).

In fact, quite apart from the absorber theory the idea that the zeropoint radiation field is not independent of matter has been advanced several times, along the following argument. Classical electrodynamics predicts that charged oscillators radiate, and there is no evidence that this does not extend to the domain of microphysics; thus, a source should exist that is able to restore the otherwise lost atomic stability. It is not difficult to come to the idea that the origin of the field required to restore equilibrium, i.e. of the zeropoint radiation field, lies in all the remaining oscillators of the universe,which constitute a sort of electromagnetic reservoir; the environmental zeropoint radiation at a given point of space is then made up of radiation emitted by distant matter. The complex structure of this field, produced by an enormous number of independent sources, would explain its stochastic nature.[5] When at the beginning of the century the argument of the collapse of the radiating atom was advanced, people were assuming that the atom is an isolated system; but isolated atoms do not exist in SED, and there is no reason a priori to neglect the effects of the vacuum field. It would be better to say that matter is stable because it radiates [de la Peña 1983].

Accepting distant matter as the source of the zeropoint field, and using an argument reminiscent of Olbers' paradox, one can easily see that this background field may produce nonnegligible effects. The total field intensity at a given point of space produced by all matter in a static universe can be written as $I_{\text{tot}} = \int d^3nI(r)$, where $I(r)$ is the average intensity due to a star at a distance r and d^3n is the number of stars in the volume element d^3r. For a homogeneous and isotropic universe, $I(r) \sim r^{-2}$ and $d^3n \sim r^2dr$, so that I_{tot} is roughly proportional to the radius of the visible universe. In other words, the dominant contribution comes indeed from far matter and it may attain an arbitrary value. Of course things become less dramatic in an expanding universe, but the conclusion about the importance of the far source field remains in force.

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[4] The question of the wave behavior of matter is discussed in more detail in chapter 12.

[5] The observation about the mechanism of atomic equilibrium constitutes the fulcrum of two notable papers by Marshall (1963, 1965a) in which the foundations of SED were laid once more. Points of view akin to those of the main text have been expressed in Santos (1968, 1981, 1985a), Theimer (1971), de la Peña and Cetto (1975), Claverie and Diner (1980), Cavalleri (1981), Goedecke (1983a), Julg and Julg (1983), Puthoff (1989b) and França et al. (1992), among others.

The zeropoint field has alternatively been conceived as logically independent from matter, i.e., as an additional entity of our universe.[6] An upholder of this point of view has been Boyer [see, e.g., Boyer 1975a], who argues that in developing his classical theory of electrons, Lorentz (1909) chose a particular boundary condition to solve the Maxwell equations; but this boundary condition can be replaced by a new one that allows for a homogeneous random radiation with a Lorentz-invariant spectrum, leading to a new (classical) theory that hopefully explains far more phenomena than the original one.

It should be stressed that the Lorentz choice is indeed far from obvious. For example, a conventional experiment in classical electrodynamics is normally conducted in a thermal bath at room temperature, which includes a thermal radiation bath that is not taken into account; a choice of the boundary conditions corresponding to this bath would seem more appropriate. Note, however, that according to equation (4.27) the only way to avoid a friction force on classical oscillators that are in equilibrium with a background field and satisfy the equipartition law (so that $(\partial\mathcal{E}/\partial\omega) = 0$), is by cancelling this field, i.e., reducing it to the void. This is just what classical physics does; it has no other choice. From the vantage point of present-day physics one sees here an interesting way to solve the difficulty of classical theory: the introduction of any nontrivial vacuum field and the abandonment of the equipartition law go hand by hand, so the need of the second may be taken as a suggestion towards the first.

4.1.2. THE ZEROPOINT FIELD, CLASSICAL OR QUANTUM?

The term 'classical' has several connotations, and should be used with a grain of salt. In the SED literature it is used frequently with the deep meaning of a phenomenon amenable to a space-time (continuous) description. In this broad sense, SED is unquestionably a classical theory. To avoid confusions, however, it is convenient to recall another important point of view according to which neither the original (c-number) vacuum field nor the ensuing theory can rigorously be considered classical, since such a field constitutes a fundamentally new ingredient, unknown to classical physics and assumed to produce a qualitatively new behaviour of matter. The mechanical system (the atom, say) becomes an open system and acquires properties fixed by the 'external' component, that is, the vacuum field, ceasing to behave classically. In fact, one expects the theory to explain in one way or another the quantum behavior of matter as a result of the interaction with the vacuum field. And at least part of these expectations are met by SED, as we will have occasion to see from chapter 5 on.

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[6] This possibility is common in the quantum literature, and is frequently stretched to its extreme, to conclude that the zeropoint field is but a manifestation of the ultimate quantum nature of the electromagnetic field [see, e.g., Davies and Burkitt 1980].

Further, one can add that any theory in which Planck's constant plays a central and irreducible role is, by definition, nonclassical. This is not a question of semantics, but of the intrinsic nature of the theory; if $\hbar$ enters into the description of the dynamics in an essential way, the system is not classical any more. And this is just the principle of SED: the zeropoint field, with an average energy $\frac{1}{2}\hbar\omega$ per normal mode, is expected to play a central role in determining the dynamics of the small subsystem. Some further remarks are added in §5.1.1.

The works mentioned in the previous section are all based on the idea of a real vacuum field, but both a classical and a quantum version of it have been considered, and the question arises as to which (if any) is the correct selection. This point is crucial, and there will be occasion to discuss it at length in later chapters; here we only add a couple of remarks that help to give shape to the general view we are trying to develop. In SED one wants to find out the extent to which the quantum properties of matter can be considered to arise from its interaction with the vacuum field. A program of this kind should not in principle start by assuming a quantum vacuum whose properties are to be explained by the theory itself (and, even worse, postulated by analogy with the description of quantum material oscillators).

From this point of view, the starting point, the original field of SED, should be a random field described in nonquantized terms. Certainly, the starting assumptions should be carefully distinguished from the (desired) final description, which is expected to lead *in some approximation* to the familiar quantum properties of both matter and field. That the ultimate correspondence between the two theories cannot be exact —even if the mathematical apparatus of quantum mechanics can be eventually recovered in SED under some conditions— follows from general considerations on their different properties as regards realism and locality, say, without the need to enter into details. Recall that the same conclusion was reached previously, in chapter 2. Apart from the differences in physical insight afforded by the two theories, the predicted differences should lead eventually to the design of empirical tests. Even if for the time being the possibilities of carrying out such a program to completion are uncertain, the example afforded by quantum theory, which goes from the simple rules of first quantization of matter to the full self-consistent quantization of everything that interacts with it, somehow mimics what we have in mind. This is but a reformulation of the conclusion reached at the end of chapter 2, that the nature (classical or quantal) of field and matter in interaction must ultimately agree.

An interesting observation regarding the vacuum is the following. As is well known, in QED there is a zeropoint energy $\frac{1}{2}\hbar\omega$ per mode; however, the zeropoint field does not appear explicitly in the formalism, the nearest thing to it being the vacuum *state* $\mid 0 >$. The notion of the zeropoint field appears indirectly, associated with the commutator of the field operators $\hat{a}, \hat{a}^{\dagger}$. This is to be compared with the description of the stochastic field, where the amplitude of the total field of a given mode can be written in the form $c = a_0 + a_e$, the first term describing the vacuum and the second one the external field (see equation (3.60)). Considering these two contributions as statistically independent, one has

$$\left\langle |c|^2 \right\rangle = \left\langle |a_0|^2 \right\rangle + \left\langle |a_e|^2 \right\rangle . \tag{4.3}$$

In the QED description the amplitudes (operators) $\hat{a}$'s cannot be decomposed into a zeropoint part and an external part, but such decomposition still holds for the expectation value of the product. Indeed, the quantum counterpart of the last equation can be taken as

$$\left\langle |\hat{a}|^2 \right\rangle \rightarrow \tfrac{1}{2} \langle n| \hat{a}\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{a} |n\rangle = \tfrac{1}{2} \langle n| [\hat{a}, \hat{a}^{\dagger}] |n\rangle + \langle n| \hat{a}^{\dagger}\hat{a} |n\rangle = \tfrac{1}{2} + n, \tag{4.4}$$

where the first term corresponds to $\langle |a_0|^2 \rangle$ in the SED calculation, and n to $\langle |a_e|^2 \rangle$. Thus, even in the absence of excitations ($n = 0$) a nonzero contribution appears in the quantum calculation in the form of $\frac{1}{2} \langle 0| [\hat{a}, \hat{a}^{\dagger}] |0\rangle = \frac{1}{2}$. The magic here lies in the fact that 'no excitations' does not mean $\hat{a} = 0$; it only means that the state of interest is the vacuum state. Thus, nothing in QED is the direct analog of the zeropoint field of SED, and even less of its energy fluctuations, but nevertheless the commutator-plus-states formalism generates its effects, in an indirect but effective way. SED brings thus to the fore a physical element which is only latent, so to say, in modern quantum theory. It is the potential richness of this step what SED attempts to exploit. The fact that the energy of the zeropoint field of SED is a fluctuating quantity whereas in QED it is fixed, is but one of the many differences between the two theories, as commented above. The issue has remained however largely neglected and one can find only occasional comments on the respective disagreements between the two descriptions.

From the perspective of SED, the fact that QED can do without an explicit zeropoint field can be understood as follows. The complete quantum theory contains both matter and field in a quantized form; this means that the quantum field interacts with an already quantized electron. In SED the quantum properties of matter are expected to arise as an effect —a leading effect, in fact— of the interaction with the field; hence the quantized particle is a particle that already 'knows' about the zeropoint field and acts accordingly. This would explain why QED only needs to deal with the higher-order effects of the vacuum. The question of whether the quantization of matter is a mechanism of 'detection' of the zeropoint field, which can be taken as purely semantic in QED [Sciama 1991], constitutes by contrast a core question for SED.

4.1.3. RECOVERY OF ATOMIC STABILITY

As discussed above, atomic stability is conceived in SED as a result of the balance between the average rates of absorption and emission of energy by the atomic electrons, as proposed originally by Nernst. This point of view is so appealing that it has found its way also into more conventional texts on QED [see, e.g., Sciama 1979]. Qualitatively, the idea is that as the atomic electron falls towards the nucleus due to radiation, its orbital frequency increases, so that the dominant interaction with the field takes place with modes of higher frequencies; since these modes are more energetic, the electron extracts a greater amount of energy from the field [Bergia 1991]. Since the opposite occurs when the electron gains energy, an equilibrium situation should arise in which the rates of radiated and absorbed energy are equal. For the argument to hold it is essential that the spectral energy density of the field is an increasing function of the frequency, i.e., the zeropoint field cannot be identified with a white noise, and the possibility of atomic equilibrium for Brownian systems is thus eliminated for good.

Considering the importance of the argument, it is convenient to put it into a more quantitative form. We restrict the following heuristic analysis to the circular orbits of the hydrogen atom, treated as a classical Kepler problem.[7] For an orbit of energy $\mathcal{E}$ the orbital frequency is of the form $\omega = A_1|\mathcal{E}|^{3/2}$, the radius and energy are related by $|\mathcal{E}| = A_2/r$ and the power radiated is given by Larmor's formula as $\dot{W}_r = A_3 a^2 = A_3\omega^4 r^2 = B_1|\mathcal{E}|^4$, with A_i, B_i suitable constants and a the acceleration. To write a simple expression for the power absorbed from the field we observe that it must be proportional to its spectral density; a dimensional analysis leads to $\dot{W}_a = \text{const}(e^2/m)\rho(\omega)$, where the constant is a number of order unity. We set $\rho(\omega)$ proportional to ω^s, with s a positive integer. Then $\dot{W}_a = A_4\omega^s = B_2|\mathcal{E}|^{3s/2}$, and finally we get

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[7] This analysis appears for the first time in Claverie and Diner (1976a); related considerations are given in Boyer (1975a) and Surdin (1975b).

$$\eta \equiv \frac{\dot{W}_a}{\dot{W}_r} = \left|\frac{\mathcal{E}}{\mathcal{E}_e}\right|^{(3s-8)/2} = \left(\frac{r_e}{r}\right)^{(3s-8)/2}, \tag{4.5}$$

where $\mathcal{E}_e$, r_e are the equilibrium energy and radius defined by $\dot{W}_r = \dot{W}_a$. We want to know under what conditions can the system be stable? When the electron is losing energy and falling towards the nucleus, $|\mathcal{E}| \to \infty$, and $\eta > 1$ is required for stability to exist; this implies that $3s > 8$, or $s \geq 3$. We verify that with this selection, when the electron gains energy ($|\mathcal{E}| \to 0$), $\eta < 1$, as is required to recover the equilibrium situation. The result was encouraging when it was first obtained, since $s = 3$ for the SED field, which appears thus as the simplest field able to stabilize the atom. When it became possible to make more exact calculations, it turned out that this nice picture is spoiled by the elliptical orbits. It was found that the electron absorbs energy in excess; not only does it not collapse toward the nucleus, but it escapes and ionizes the system. The detailed discussion of this problem is left for an appropriate place in chapter 9.

The above discussion has been made in terms of charged particles, and indeed in the literature this is practically the only case that has been studied within SED. The question of how to deal with neutral particles requires thus special consideration, and several suggestions have been made to address it. Except for the neutrinos, neutral particles can be treated as systems composed of charged constituents, each of which is subject to the action of the background field [Santos 1968, Theimer and Peterson 1975]; quite similarly, Surdin (1979a) considered that the fluctuating field polarizes the particle, whereas de la Peña and Cetto (1977a) suggested that the electric charge could itself be a fluctuating quantity. In §12.2.2 an explicit calculation is made showing that the coupling of the neutral system to the field through higher moments (the dipole electric moment, in the example) leads to results equivalent to those predicted for the charged particle. This result is conceptually in agreement with the more formal consideration by Boyer (1975b) —as well as with the nonperturbative scheme developed in chapter 10— in the sense that quantum mechanics should correspond to the limit $e \to 0$ and thus be independent of the charge.

4.1.4. FURTHER COMMENTS ON THE ZEROPOINT FIELD

The existence of a balance between $\dot{W}_r$ and $\dot{W}_a$ implies some mechanism of regeneration of the vacuum field by matter on the microscopic scale;

indeed, the mechanism is provided by radiation due to acceleration, so that the field plays the role of a reservoir.[8] A similar idea, but on a completely different scale and more speculative, has been recently considered by Puthoff (1989b), who examines the possibility of predicting the correct spectrum for the vacuum field from the hypothesis of regeneration on a cosmic scale within an inflationary scenario. In addition to the ω^3-dependence he gets a numerical relationship between cosmological and atomic constants, which roughly coincides with one of the well-known Dirac large numbers relations.[9] This compelling idea, namely that the intensity of the background field (the value of Planck's constant) can be determined from joint cosmological and atomic considerations, has been around for many years, without taking yet its final shape.[10]

As was already remarked, one should think in principle of the possibility that the equilibrium situation results from the interaction of matter with each and every one of the existing vacuum fields. Santos (1974c, 1981) refers to this more general scheme as 'stochastic theory' [see also Moore 1983b]. One can, for example, speculate that the different vacua have all the same energy $\frac{1}{2}\hbar\omega$ per mode, leading thus to similar conditions of equilibrium. However, the difficulties for the development of such a broad theory are evident, and the effort has consistently been focussed on SED under the tacit assumption that the electromagnetic component of the vacuum is the main determinant of the atomic structure.

4.2. Properties of the zeropoint field

4.2.1. DENSITY OF STATES AND SPECTRAL DENSITY

One of the most important and distinctive properties of the zeropoint field is its frequency spectrum. This spectrum can be derived on the basis that the minimum energy of a quantum oscillator is $\frac{1}{2}\hbar\omega$; indeed, this was the method used by Kalitsin (1953) and Adirovitch and Podgoretskii (1954). However, considering the central theoretical role played by this spectrum, it is rather unsatisfactory to merely import the datum from quantum theory. We shall therefore consider in detail the problem of determining the spectral density of the vacuum field from the most general possible arguments.

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[8] A related, but more formal argument, based on the fluctuation-dissipation theorem to associate a fluctuating field with the radiation reaction, has been expressed on several occasions. See, e.g., Braffort et al. (1954), Surdin (1977a), Brody et al. (1979), Claverie and Diner (1980), Milonni (1981).

[9] For criticism of this work and reply by the author see Santos (1991d), Wesson (1991b) and Puthoff (1991a).

[10] Two early attempts, both of a tentative nature, are due to Hobart (1976) and Surdin (1976). Using a much more cautious language, Theimer (1971) speaks analogously of a selfconsistent field, without specifying whether he refers to a local or cosmic scale.

The simplest possibility that comes to mind, and that was indeed exploited by Braffort and Tzara (1954), is to resort to an application of Wien's displacement law at zero temperature. We recall that by considering the radiation enclosed in a cylinder with a movable piston, and taking into account the possible energy exchange between different modes as a result of the Doppler shift, Wien established that the equilibrium spectral density must be of the general form

$$\rho(\omega,T)=\omega^3\Phi(T/\omega), \tag{4.6}$$

with Φ a *universal* but undetermined function.[11] The usual derivations of Wien's law do not guarantee its applicability at $T = 0$. However, a continuity argument involving the requirement that the specific heat of the radiation of a given frequency remains finite at all temperatures, shows that it should be applicable down to the limit $T = 0$ [Jiménez et al. 1980]. Further, by applying the Second Law of thermodynamics to several specific systems, Cole (1990c, 1992a) has been able to derive a generalized Wien displacement law for the equilibrium radiation spectrum, which must hold even if the spectrum does not vanish at $T = 0$, at which Φ attains an unknown but constant value $\Phi(0) \equiv \lambda$ (a more explicit version of Cole's argument is offered at the end of §5.3.4). This gives

$$\rho_0(\omega)\equiv\rho(\omega,T=0)=\lambda\omega^3, \tag{4.7}$$

with λ to be specified.

To better grasp the meaning of this result, let us consider the relationship between the spectral density $\rho(\omega,T)$ and the average energy $\mathcal{E}(\omega,T)$ of the field oscillators of frequency ω. We note that $\rho(\omega,T)d\omega$ is the electromagnetic energy per unit volume within a bandwidth $d\omega$ that contains ω, and if $\mathcal{N}(\omega)d\omega$ represents the number of modes of frequency ω per unit volume contained in the same frequency interval, this energy can alternatively be written as $\mathcal{N}(\omega)\mathcal{E}(\omega,T)d\omega$; it follows that

$$\rho(\omega,T)=\mathcal{N}(\omega)\mathcal{E}(\omega,T). \tag{4.8}$$

The *density of states* within the cavity, $\mathcal{N}(\omega)$, does not depend on the temperature and can be determined from simple geometrical considerations. A way to calculate it is the following [Jeans 1905]. From equation (3.42) for the field modes that can be accommodated inside a box of size L, $k_i = (2\pi/L)n_i$, it follows that the minimal increments in k correspond to $\Delta n_i = 1$, which gives $dk_i = 2\pi/L$. The number of modes per unit volume within the interval $d\omega = cdk$ with both polarizations and all directions included is therefore

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[11] If Φ were a different function for different bodies at the same temperature, it would be possible to transfer energy between two systems in equilibrium with the help of appropriate filters.

$$\mathcal{N}(\omega)d\omega = 2\frac{1}{L^3}\int_\Omega d^3k \cdot \frac{L^3}{(2\pi)^3} = 2\int_\Omega \frac{d\Omega}{8\pi^3}dk\,k^2 = \frac{\omega^2}{\pi^2c^3}d\omega, \tag{4.9}$$

or[12]

$$\mathcal{N}(\omega) = \frac{\omega^2}{\pi^2c^3}. \tag{4.10}$$

Introducing this important result into (4.8) one gets *Planck's formula* for the spectral energy density of the field in terms of the average energy of its modes

$$\rho(\omega,T) = \frac{\omega^2}{\pi^2c^3}\mathcal{E}(\omega,T). \tag{4.11}$$

This formula played an important role in the early years of the development of quantum theory. In his first studies of the blackbody problem, Planck derived it as a relationship between the mean equilibrium energy $\mathcal{E}(\omega,T)$ of mechanical oscillators of frequency ω and the equilibrium spectral density $\rho(\omega,T)$; his result had therefore a dynamical sense. In the present derivation (4.11) gives merely an account of the allowed states; it can be used equally well within classical or quantum physics, and, of course, within SED, since it applies to any set of oscillators that describe a vector field fulfilling the assumed periodic boundary conditions. Nernst's consideration of the equilibrium condition between matter and field oscillators implies that (4.11) applies to either of them; hence the equivalence between the formula derived by Planck and the one derived here. By combining equations (4.7) and (4.11) for $T=0$, it follows that

$$\mathcal{E}_0 \equiv \mathcal{E}(\omega,0) = \frac{\pi^2c^3}{\omega^2}\rho_0(\omega) = \pi^2c^3\lambda\omega, \tag{4.12}$$

meaning that the energy of the zeropoint field oscillators is indeed linear in the frequency. With $\mathcal{E}_0 = \frac{1}{2}\hbar\omega$ one gets $\lambda = \hbar/(2\pi^2c^3)$, whence

$$\rho_0(\omega) = \frac{\hbar}{2\pi^2c^3}\omega^3. \tag{4.13}$$

This is the doorway for Planck's constant in SED, as the measure of the intensity of the fluctuations of the zeropoint field. The appearance of $\hbar$

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[12] In an alternative derivation one counts the number Γ of standing waves of two polarizations allowed in an octant of k-space: $\Gamma(k) = 2 \times \frac{1}{8}(\frac{4}{3}\pi k^3)(L/\pi)^3 = V\omega^3/3\pi^2c^3$, and uses $\mathcal{N}(\omega) = \frac{1}{V}(d\Gamma/d\omega)$. How to derive this factor directly from the quantum rules was learned only much later, with the advent of Bose-Einstein statistics. (The details can be seen in Jammer (1966), §5.3.)

anywhere should be taken as evidence of the presence or action of this field.

The above result for the energy can be given a more suggestive form by recalling that for any harmonic oscillator of frequency ω, the Hamiltonian is expressed in terms of the action variable as $H = \omega J/2\pi$ [see, e.g., Goldstein 1980, chapter 10] so that the average energy of the oscillator at temperature T is $\mathcal{E}(\omega, T) = \omega \bar{J}(\omega, T)/2\pi$, where $\bar{J}$ represents the thermal average of J. From (4.6) and (4.11) one obtains

$$\bar{J}(\omega, T) = 2\pi^3 c^3 \Phi(T/\omega). \tag{4.14}$$

The universal Wien function turns out to be essentially the thermal average of the oscillator's action variable; it follows that at $T = 0$, $\bar{J}_0 \equiv \bar{J}(\omega, 0)$ must be a universal constant (hence independent of ω), so that the energy of the oscillator is indeed linear in ω,

$$\mathcal{E}_0 = \frac{\bar{J}_0}{2\pi}\omega \tag{4.15}$$

and $\bar{J}_0$ has a universal value, which is fixed (e.g., by comparison with experiment) at $\hbar$.

4.2.2. LORENTZ INVARIANCE OF THE ZEROPOINT FIELD SPECTRUM

Let us reinforce the previous arguments on the shape of the spectral density, by recalling a well-known variant that may be more appealing to some readers. Since the complete derivation is somewhat lengthy and the details can be seen in the appropriate textbooks, we give here only a sketch of it, indicating the assumptions made along the calculations. Consider an atom, to be modelled as a very small dipole oscillator of natural frequency ω_0, that vibrates on the z axis and moves with a constant velocity v along the x axis through a homogeneous and isotropic radiation field of spectral density $\rho(\omega)$. As an effect of the translational motion, the field detected by the dipole ceases to be isotropic and therefore exerts a friction force on it; the calculation of this force is our present task. Let the dipole moment be $\mathbf{q} = e\xi\hat{\mathbf{k}}$, where ξ denotes the instantaneous displacement of the dipole along the direction $\hat{\mathbf{k}}$ of the z axis. We distinguish the variables as seen from the moving frame of the atom by a prime; then the expression for $\mathbf{q}'(t)$ is obtained by solving the Abraham-Lorentz equation of motion in the dipole approximation (cf. equation (7.1))

$$m\ddot{\mathbf{q}}' = -m\omega_0^2\mathbf{q}' + e^2\mathbf{E}'(t) + m\tau\,\dddot{\mathbf{q}}', \tag{4.16}$$

where $\mathbf{E}'$ is the electric component of the radiation field and $\tau = 2e^2/3mc^3$. The force on the dipole $\mathbf{q}'$ is given by

$$\mathbf{F}'_q = (\mathbf{q}' \cdot \nabla')\,\mathbf{E}' + \frac{1}{c}\frac{d\mathbf{q}'}{dt'} \times \mathbf{B}' = q'\frac{\partial \mathbf{E}'}{\partial z'} + \frac{1}{c}\dot{q}'\left(-\hat{\imath}B'_y + \hat{\jmath}B'_x\right). \qquad (4.17)$$

The primed and unprimed (lab) fields are related by the usual relativistic transformation laws [see, e.g., Jackson 1975, chapter 11; Barut 1980, chapter 3]

$$E'_x = E_x, \qquad E'_y = \gamma\left(E_y - \frac{v}{c}B_z\right), \quad \ldots \quad B'_z = \gamma\left(B_z - \frac{v}{c}E_y\right), \quad (4.18)$$

with the coordinates transformed by a Lorentz boost,

$$x' = \gamma\,(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \qquad (4.19)$$

Thus, in particular,

$$\dot{q} = \frac{dq}{dt'} = \gamma\frac{\partial q}{\partial t} + \gamma v\frac{\partial q}{\partial x}. \qquad (4.20)$$

The radiation field is written in terms of its Fourier components, as in equations (3.46)-(3.48), and the solution $\mathbf{q}(t)$ of (4.16) is introduced in $\mathbf{F}_q$. In calculating the average force on the dipole it is assumed as usual that the amplitudes $a_{\mathbf{k}\lambda}$, $a^*_{\mathbf{k}\lambda}$ referring to different modes are statistically independent, so that their average product is $\langle a^*_{\mathbf{k}\lambda} a_{\mathbf{k}'\lambda'}\rangle = \left\langle |a_{\mathbf{k}\lambda}|^2 \right\rangle \delta_{\lambda\lambda'}\delta_{\mathbf{k}\mathbf{k}'}$, with the quantities $\left\langle |a_{\mathbf{k}\lambda}|^2 \right\rangle$ proportional to the spectral density of the field at the corresponding frequency, $\rho(\omega_{\mathbf{k}\lambda}) = \rho(\omega_k)$ (see section 4.3 below). A calculation to first order in the velocity proves to be sufficient. Further, due to the very low value of τ, the response of the particle to the field is sharply peaked around ω_0, so that the usual (narrow-linewidth) approximation can be taken in performing the required integrations; the details can be seen in chapter 7, where similar calculations are performed. Solving equation (4.16) and using the result to calculate $\langle \mathbf{F}' \rangle$, one gets after a relatively long but straightforward calculation

$$\langle \mathbf{F}' \rangle = \frac{3}{4}\tau^2 c^8 \int d^3k' \frac{\rho\,(\mathbf{k}, \omega_k)}{\omega_k^3} \frac{\omega_k'^2}{\left(\omega_0^2 - \omega_k'^2\right)^2 + \tau^2\omega_k'^6}\mathbf{k}'\left(k'^2 - k_z'^2\right). \qquad (4.21)$$

Note carefully that the factor ρ/ω^3 in the integrand depends on the unprimed variables, and that the possibility of an anisotropic field is taken into consideration through its dependence on $\mathbf{k}$. For an isotropic field as seen from the laboratory, $\rho = \rho\left(\omega_k\right)$, and from $\omega = \omega' + vk'_x = \omega'\left(1 + \frac{v}{c}\frac{k'_x}{k'}\right)$ it follows that

$$\begin{aligned}\frac{\rho\left(\omega\right)}{\omega^3} &\simeq \frac{\rho\left(\omega'\right)}{\omega'^3} + \left(\omega - \omega'\right)\left(-3\frac{\rho\left(\omega'\right)}{\omega'^4} + \frac{1}{\omega'^3}\frac{\partial\rho\left(\omega'\right)}{\partial\omega'}\right)\\ &= \frac{\rho\left(\omega'\right)}{\omega'^3} - \frac{3}{\omega'^3}\left(\rho(\omega') - \frac{1}{3}\omega'\frac{\partial\rho\left(\omega'\right)}{\partial\omega'}\right)\frac{k'_x}{k'}\frac{v}{c}.\end{aligned} \tag{4.22}$$

This same result can alternatively be obtained by noting that the spectral energy density transforms as

$$\rho'(\omega', \partial')d\omega' d\Omega' = (1 - 2\frac{v}{c}\cos\varphi)\rho(\omega)d\omega d\Omega, \tag{4.23}$$

where φ is the angle between the direction of propagation **k** and the axis of motion x, and recalling that the Doppler shift of ω and the angular aberration are given by

$$\omega' = \omega\left(1 - \frac{v}{c}\cos\varphi\right), \quad \cos\varphi' = \cos\varphi - \frac{v}{c}\sin^2\varphi. \tag{4.24}$$

With the help of these equations and $d\Omega'/d\Omega = d\cos\theta'/d\cos\theta$, and keeping only terms up to first order in (v/c), one gets

$$\begin{aligned}\rho'(\omega', \varphi') &= (1 - 3\frac{v}{c}\cos\varphi')\rho(\omega) = (1 - 3\frac{v}{c}\cos\varphi')\rho(\omega' + \frac{v}{c}\omega'\cos\varphi')\\ &= (1 - 3\frac{v}{c}\cos\varphi')\left[\rho(\omega') + \frac{\partial\rho}{\partial\omega'}\frac{v}{c}\omega'\cos\varphi')\right]\\ &= \rho(\omega') - 3\frac{v}{c}\left[\rho(\omega') - \frac{1}{3}\frac{\partial\rho}{\partial\omega}\omega'\right]\cos\varphi'.\end{aligned} \tag{4.25}$$

For the calculation of the force we insert the above result in (4.21) and perform the angular integrations with $\int d\Omega\cos^2\varphi\sin^2\theta\left(1 - \cos^2\theta\right) = 16\pi/15$, to get

$$F = -A\left[\rho(\omega') - \frac{1}{3}\omega'\frac{\partial\rho}{\partial\omega'}\right]v. \tag{4.26}$$

Notice that the whole force is directed along the direction of motion and comes entirely from the field anisotropy induced by the motion. Here A represents a positive constant which in the specific case of the dipole has the value $A = 4\pi^2e^2/5mc^2 = (6/5)\pi^2\tau c$, but which may change with the details of the problem. As follows from the derivation, this result is quite general and can be applied under a wide range of situations, for any $\rho(\omega)$, the main assumptions being the isotropy of the random field and the statistical independence of its Fourier components. Alternative expressions for the force are

$$F = -\frac{A}{3\pi^2 c^3}\omega^2 \left[\mathcal{E}(\omega,T) - \omega\frac{\partial \mathcal{E}(\omega,T)}{\partial \omega}\right] v = A\omega^4 \frac{\partial \Phi}{\partial \omega} v. \tag{4.27}$$

Now we come back to our problem. Assume an atom embedded in the radiation field and moving uniformly with respect to the laboratory. If there is any force on it of the type (4.26), then one can tell from it the velocity of this motion; hence relativity demands that $F = 0$ for the vacuum field, and from equation (4.26) the spectrum of this field must be $\rho_0 = \lambda\omega^3$. Alternatively, from (4.27) one gets $\mathcal{E}(\omega) \sim \omega$ and Φ independent of ω. Stated in simpler terms, relativistic invariance of ρ means $\rho'(\omega', \theta') = \rho(\omega')$ and hence, from equation (4.25), $\rho \sim \omega^3$. This is equivalent to saying that the zeropoint field appears as isotropic in all inertial frames. This fundamental peculiarity of the spectrum (4.24) distinguishes it from any other spectrum and makes it unique.[13] For a white noise, for instance, equation (4.26) gives $F = -A\rho v = -\text{const}\, v$, which means that in this case the friction ceases only when the noise vanishes.

Recapitulating, we observe from (4.25) that the demand of isotropy $\rho'(\omega', \theta') = \rho'(\omega')$, along with the obvious requirement that the energy and momentum of the free field have the correct relativistic form, suffice to derive the spectral density of the vacuum [Boyer 1969b, 1980b, Goedecke 1983a; see also de la Peña 1983]. This can be seen also from considering the average energy and momentum of a mode of the zeropoint field as components of a 4-vector related by $\mathcal{E}^2 = c^2\mathbf{p}^2$ (as follows from (3.49) and (3.50)) and writing them in the most general possible form in terms of $k_\mu = (k_i, \omega/c)$, namely, $p_i = k_i f(k_\mu)$, $\mathcal{E} = \omega f(k_\mu)$, with $f(k_\mu)$ a Lorentz scalar. The demand of isotropy means that $f(k_\mu)$ can be a function of k only through ω and $k_\mu k_\mu$; moreover, since f should be expressible in terms of invariants, it can depend only on the single invariant $k_\mu k_\mu = 0$, which means that it is a number. Hence $\mathcal{E} = \text{const}\, \omega$, which introduced in (4.11) gives (4.13), after a proper choice for the constant. The fact that the conditions imposed on the field to deduce its spectral energy density have nothing to do with its classical or quantum nature, explains why the zeropoint fields of QED and SED have the same spectrum.

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[13] Use of the argument of Lorentz invariance to determine the form of $\rho(\omega)$ for the zeropoint radiation field was made independently by Marshall (1963) and Boyer (1969b); see also Santos (1968). More recently, Santos (1993) has demonstrated that the cubic spectrum is the only invariant one in a static gravitational field, and Rueda (1990c) argues that it is invariant with respect to the expansion of the universe. In cosmology it is usual to define the vacuum state as the general relativistic invariant state, which accords well with the present treatment.

4.2.3. SOME SEQUELS OF THE WIEN AND STEFAN-BOLTZMANN LAWS

Here we allow ourselves a short digression on some general thermodynamic relations that must be satisfied by the thermal part of the radiation field in equilibrium at $T > 0$. By integrating equation (4.6) applied to the purely thermal part of ρ over all frequencies and using the Stefan-Boltzmann law for the energy density of a thermal field, $u(T) = aT^4$, one has

$$u(T) = \int_0^\infty d\omega \rho_T(\omega, T) = \int_0^\infty d\omega \omega^3 \Phi_T(T/\omega) = aT^4, \tag{4.28}$$

where a stands for a universal constant (initially known only empirically). To examine some consequences of the above relation, it is useful to introduce the variable $x = k_B T/\omega$, with k_B the Boltzmann constant, whereby $\int_0^\infty d\omega\omega^3\Phi_T(k_BT/\omega) = -k_B^4T^4\int_0^\infty dx\Phi_T(x)/x^5$; thus,

$$a = k_B^4 \int_0^\infty dx \frac{\Phi_T(x)}{x^5}. \tag{4.29}$$

Since a is finite, this integral must exist, which means, in particular, that $\lim_{x\to 0}\Phi_T(x)/x^5 < \infty$. Now from the classical law of energy equipartition $\mathcal{E}(\omega, k_BT) = k_BT$, one has $\Phi_T(x) \sim x$ and the integral becomes strongly divergent (this is the so-called ultraviolet catastrophe). The result is of course unphysical, and the calculation shows that the problem in classical physics is caused by the assumed equipartition. Let us now make a slightly more general consideration. Combining the Planck formula (4.11) with Wien's law one may write, with b an appropriate constant,

$$\mathcal{E}_T(\omega, k_BT) = \frac{\rho_T(\omega)}{b\omega^2} = \frac{\omega}{b}\Phi_T(k_BT/\omega) = k_BTf(k_BT/\omega), \tag{4.30}$$

where $f(x) = (1/x)\Phi_T(x)$ represents a new dimensionless universal function. Since k_BT/ω has physical dimensions but f is universal and dimensionless, there are only two possibilities:[14] either $f =$ const, which would take us back to the equipartition law, or else there exists a universal constant (let us call it $\hbar$) such that $k_BT/\hbar\omega$ is dimensionless. The argument allows for a quite arbitrary f, as long as it remains finite for any value of ω and T [see Sommerfeld 1956, chapter 2].

From this analysis it follows that equipartition may be at most an approximation, and that the more general result must contain $\hbar$. The average equilibrium energy of an oscillator will therefore be a function of $\hbar\omega$. The classical theory of heat is thus seen to already contain elements pointing toward the existence of an additional universal constant that can serve to restore internal consistency. SED allows for this constant to enter into the description through the vacuum field, which has a universal nature. In the next chapter we will have opportunity to verify that the introduction of this missing universal constant through the zeropoint field, in combination with very general arguments, suffices to predict the correct thermal behaviour. Some care is still required because the zeropoint contribution obviously violates the Stefan-Boltzmann law, since $\int \rho_0(\omega)d\omega = \lambda \int \omega^3 d\omega = \infty$. Here we are confronted with a different divergency problem, unknown to classical physics.

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[14] With k_BT, ω and c alone it is not possible to construct a dimensionless quantity.

4.2.4. ENERGY CONTENT OF THE ZEROPOINT FIELD

Two arguments are frequently raised in QED, both applicable to SED, to use a cutoff for the spectrum of the zeropoint field and thus assign a finite value to the quantity $\int \rho_0(\omega)d\omega$. One is that there is no evidence that the rules of electrodynamics can be extended to arbitrarily high frequencies. Indeed, as was observed in chapter 3, there is evidence to the contrary, since for wavelengths shorter than the Compton wavelength the description (even the relativistic one) is clearly inadequate. The other is that the spectrum (4.7) has been derived from the demand of Lorentz invariance, which is of course only an approximation to the deeper demand of general covariance. It is reasonable to assume that a spectrum fulfilling the rules of general relativity must be integrable, and that the one used here is only a low-frequency local approximation to it. Of course, from a pragmatic point of view, the problem is irrelevant for the present nonrelativistic description, since, as was seen in chapter 3, a cutoff must be introduced anyway to recover internal consistency.

The problem of the divergence associated with the spectral energy density $\lambda\omega^3$ of the zeropoint field is one of the main reasons that have led the majority of physicists to consider the electromagnetic vacuum of QED as a virtual field —though able to produce observable effects. As was already said, particularly distressing are the foreseeable dramatic gravitational consequences the huge energy content of this field is bound to produce. To get a feeling of the magnitude of the problem, let us estimate the energy density contained within a given frequency band, given by

$$u_{21} = \int_{\omega_1}^{\omega_2} d\omega \rho(\omega) = \frac{\hbar}{8\pi^2 c^3}(\omega_2^4 - \omega_1^4) = \frac{\hbar c}{8\pi^2}(k_2^4 - k_1^4). \tag{4.31}$$

For a narrow band within the optical spectrum this gives an already enormous energy density of order $10^2 - 10^3$ erg/cm^3. For a 'reasonable' cutoff as might be $k_C = mc/\hbar$, say, one gets an equivalent mass density $\hbar k_C^4/8\pi^2 c$ of the order 2×10^{15} g/cm^3 $\sim 10^{42}$ electrons/cm^3! This problem, common to all field theories with a nontrivial vacuum, is set aside in the case of QED by considering the vacuum as virtual, which is equivalent to redefining its energy as effectively zero. Of course in a fully relativistic treatment the zero of the energy cannot be defined arbitrarily, so it seems difficult to escape from the feeling that both the energy and the problem are real. Indeed, the question is considered by many as open and continues to attract now and then the attention of researchers, particularly within general relativity. A recent very illustrative review on the subject and the related issue of the cosmological constant can be found in Weinberg (1989).

The huge equivalent mass of the zeropoint field should be expected to produce big gravitational effects on a cosmic scale, which are simply not observed. Several mechanisms of compensation with other effects have been suggested, such as the enormous stresses implied [Feynman and Hibbs 1965], the negative gravitational mass densities predicted by inflationary theories [see, e.g., Gron 1986], the vacuum polarization as speculated by Zel'dovich (1967), etc. The variety of solutions simply shows that the solution is not known, as can be seen in Weinberg (1989).

The vacuum energy is tightly linked to the cosmological constant in the Einstein equations, since anything that contributes to the energy density of the vacuum is equivalent to a cosmological constant. This can be easily seen starting from the Einstein equations [see, e.g., Misner et al. 1973, §17.3]

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{4.32}$$

(where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the stress-energy tensor, R is the scalar curvature, G is Newton's gravitational constant and $c = 1$) and adding the cosmological term $\Lambda g_{\mu\nu}$ to the left-hand side; moving this term to the right and rewriting it as $-8\pi G T^{\mathrm{vac}}_{\mu\nu}$, gives $T^{\mathrm{vac}}_{\mu\nu} = (\Lambda/8\pi G) g_{\mu\nu}$. It follows that $\rho^{\mathrm{vac}} = T^{\mathrm{vac}}_{00} = \Lambda/8\pi G$, which verifies the assertion. But the introduction of the cosmological constant has its drawbacks, since it leads at least to two problems. The first one is of principle, since according to the altered equations space-time cannot be asymptotically flat. This is contrary to the essence of general relativity, although in a moment of weakness Einstein himself accepted the possibility of a cosmological constant, committing what he called his biggest blunder.

The second problem is that the value of Λ called for by any one of the known alternatives happens to be 100–120 *orders* of magnitude above any value consistent with observation. It seems therefore that for the solution to these problems a revision somewhere in contemporary physics would be required, as has been remarked more than once [see, e.g., Weinberg 1989, Penrose 1991, p. 25]. In the applications dealing with quantum fields, instead of the tensor $T_{\mu\nu}$ the vacuum expectation $<\hat{T}_{\mu\nu}>_0$ is usually considered; this expression does not contain contributions from the vacuum, which can be explained in two different forms. The direct one is that the vacuum contribution has been compensated with a cosmological term, as was just explained, with all the noted problems, plus that of the implicit fine tuning. The other, much more speculative but not less plausible, is that the homogeneous vacuum fields do not generate curvature of spacetime as usual matter and fields do. When one takes $T_{\mu\nu}$ in the Einstein equations as containing everything (including the vacuum), one is postulating that the zeropoint field generates curvature; but this is just a postulate, for the validity of which we have no evidence. It would be more realistic to say that the facts seem to speak against it, even if we do not know why. If such an observation makes sense, eliminating the zeropoint field from the Einstein equations does not amount to introducing a cosmological constant, but to removing one that has been improperly introduced and finely tuned.[15]

4.3. Statistical description of the zeropoint field

This and the following sections are dedicated to a more complete characterization of the zeropoint field; in particular, some of its most important statistical properties will be discussed.[16]

In free space the zeropoint field is usually considered as maximally disordered, under the natural assumption that it has been generated by an immense number of independent sources. Alternatively, one can say that this field is highly incoherent, which helps to explain why it is essentially unobservable on the macroscopic level [Santos 1981; see also Boyer 1975a]. Roughly speaking, on the macroscopic scale only the very low frequencies are effective; but for such frequencies it is negligibly small by relative factors with respect to the microworld, of the order of $(\lambda_{\text{micro}}/\lambda_{\text{macro}})^3 \lesssim 10^{-24}$. On the other hand, microscopic objects may be affected by the short wavelengths (and high frequencies), at which the vacuum field dominates over the thermal contribution. Thus, one may expect in general to observe the zeropoint field only indirectly through its action in the microscopic domain.

One may consider that during a measurement that takes a minimum time T, a time averaging of the field is being performed; the direct nonobservability of the field means then that this time average gives zero, i.e.,

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[15] Gonzalez-Diaz (1989) has recently considered the gravitational analog of stochastic electrodynamics, which he calls *stochastic gravitodynamics*. He assumes the existence of a real zeropoint gravitational field interacting with all observable matter in our universe, and proposes to use it as a source to construct quantum gravity.

[16] For general discussions on the theory of random processes and basic applications to physics the reader is referred to any appropriate textbook on the subject. Classical examples are Papoulis (1965), van Kampen (1981), Stratonovich (1963).

$$\frac{1}{T}\int_0^T dt\mathbf{E}(\mathbf{x},t)=0,\quad \frac{1}{T}\int_0^T dt\mathbf{B}(\mathbf{x},t)=0 \tag{4.33}$$

at any point $\mathbf{x}$ in space, where T is a large quantity compared with relevant atomic periods. One can alternatively assume that in each instance $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ represent one realization out of the infinity of possibilities contained in the statistical ensemble; in this case the purely random character of the field means that its average over all possible realizations is zero. Thus, using $\langle f\rangle$ to denote the average of f over the ensemble of realizations of the random variables, we write

$$\langle\mathbf{E}(\mathbf{x},t)\rangle=0,\quad \langle\mathbf{B}(\mathbf{x},t)\rangle=0. \tag{4.34}$$

In writing the vector potential field in the form of equation (3.46), i.e.,

$$\mathbf{A}(\mathbf{x},t)=c\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\mathcal{E}_k}{\omega_k^2V}}\mathbf{e}_{\mathbf{k}}^{\lambda}\left(a_{\mathbf{k}\lambda}e^{-i\omega_kt+i\mathbf{k}.\mathbf{x}}+a_{\mathbf{k}\lambda}^*e^{i\omega_kt-i\mathbf{k}.\mathbf{x}}\right) \tag{4.35}$$

where $\mathbf{k}$ is the wave vector and λ the polarization index, the amplitudes $a_{\mathbf{k}\lambda}, a_{\mathbf{k}\lambda}^*$ are taken to be random variables and all the stochasticity of the field is contained in the set $\{a_{\mathbf{k}\lambda}, a_{\mathbf{k}\lambda}^*\}$. For simplicity, in this section the time factors $e^{-i\omega_kt}$ are not included in the amplitudes $a_{\mathbf{k}\lambda}$, which are therefore constant complex numbers. According to (3.49), the average energy of a mode is given by $\langle H_{\mathbf{k}\lambda}\rangle=\mathcal{E}_k\left\langle|a_{\mathbf{k}\lambda}|^2\right\rangle=\mathcal{E}_k$, which means that the scale of the $a_{\mathbf{k}\lambda}$ has been chosen such that

$$\left\langle|a_{\mathbf{k}\lambda}|^2\right\rangle=1. \tag{4.36}$$

With $\mathcal{E}_k=\frac{1}{2}\hbar\omega_k$ for the zeropoint field, we therefore get

$$\mathbf{A}(\mathbf{x},t)=c\sum_{\mathbf{k},\lambda}\sqrt{\frac{\pi\hbar}{\omega_kV}}\mathbf{e}_{\mathbf{k}}^{\lambda}\left(a_{\mathbf{k}\lambda}e^{-i\omega_kt+i\mathbf{k}.\mathbf{x}}+a_{\mathbf{k}\lambda}^*e^{i\omega_kt-i\mathbf{k}.\mathbf{x}}\right) \tag{4.37}$$

and similarly,

$$\mathbf{E}(\mathbf{x},t)=i\sum_{\mathbf{k},\lambda}\sqrt{\frac{\pi\hbar\omega_k}{V}}\mathbf{e}_{\mathbf{k}}^{\lambda}a_{\mathbf{k}\lambda}e^{-i\omega_kt+i\mathbf{k}.\mathbf{x}}+\text{c.c.}, \tag{4.38}$$

$$\mathbf{B}(\mathbf{x},t)=i\sum_{\mathbf{k},\lambda}\sqrt{\frac{\pi\hbar\omega_k}{V}}(\hat{\mathbf{k}}\times\mathbf{e}_{\mathbf{k}}^{\lambda})a_{\mathbf{k}\lambda}e^{-i\omega_kt+i\mathbf{k}.\mathbf{x}}+\text{c.c.} \tag{4.39}$$

Any other choice $\left\langle |a_{\mathbf{k}\lambda}|^2 \right\rangle = \eta$, with η an arbitrary positive number, would entail the substitution $\mathcal{E}_k \longrightarrow \mathcal{E}_k/\eta$ in order to preserve the value of the product $\mathcal{E}_k \left\langle |a_{\mathbf{k}\lambda}|^2 \right\rangle$. This arbitrariness is of no practical consequence, but it has given place to different conventions. With the help of equations (3.67)-(3.69) it follows that in the continuous description the vector potential of the vacuum field takes the form

$$\mathbf{A}(\mathbf{x},t) = c\sum_{\lambda}\int d^3k\sqrt{\frac{\hbar}{8\pi^2\omega_k}}\mathbf{e}^{\lambda}(\mathbf{k})\left(a_\lambda(\mathbf{k})e^{-i\omega_k t + i\mathbf{k}.\mathbf{x}} + a_\lambda^*(\mathbf{k})e^{i\omega_k t - i\mathbf{k}.\mathbf{x}}\right). \tag{4.40}$$

Since the fields (4.37)-(4.39) describe the vacuum, their average must be zero. The amplitudes $a_{\mathbf{k}\lambda}$ corresponding to different modes of the free field are assumed to be statistically independent; thus, each amplitude averages separately to zero,

$$\langle a_{\mathbf{k}\lambda}\rangle = 0, \quad \langle a_{\mathbf{k}\lambda}^*\rangle = 0, \tag{4.41}$$

and the product of any two amplitudes corresponding to different modes also averages to zero. Thus, consistency with (4.36) leads to

$$\langle a_{\mathbf{k}\lambda}^* a_{\mathbf{k}'\lambda'}\rangle = \delta_{\lambda\lambda'}\delta_{\mathbf{k}\mathbf{k}'}, \tag{4.42}$$

whereas

$$\langle a_{\mathbf{k}\lambda} a_{\mathbf{k}'\lambda'}\rangle = 0, \quad \langle a_{\mathbf{k}\lambda}^* a_{\mathbf{k}'\lambda'}^*\rangle = 0. \tag{4.43}$$

In the limit $V \longrightarrow \infty$ the random amplitudes transform into $a_{\mathbf{k}\lambda} \longrightarrow a_\lambda(\mathbf{k})$, with

$$\langle a_\lambda^*(\mathbf{k}) a_{\lambda'}(\mathbf{k}')\rangle = \delta_{\lambda\lambda'}\delta(\mathbf{k}-\mathbf{k}'). \tag{4.44}$$

Notice by the way that this result goes hand in hand with the transformation property of the Kronecker delta into the corresponding Dirac delta, to wit,

$$\text{for } V \longrightarrow \infty: \qquad \delta_{\mathbf{k}\mathbf{k}'} \longrightarrow \delta(\mathbf{k}-\mathbf{k}')/\left\langle |a_\lambda(\mathbf{k})|^2 \right\rangle. \tag{4.45}$$

For many purposes it is convenient to introduce the set of canonical variables $\{q_{\mathbf{k}\lambda}, p_{\mathbf{k}\lambda}\}$ defined as in (3.61),

$$\omega_k q_{\mathbf{k}\lambda} = i\sqrt{\frac{\mathcal{E}_k}{2}}\left(a_{\mathbf{k}\lambda} - a_{\mathbf{k}\lambda}^*\right), \quad p_{\mathbf{k}\lambda} = \sqrt{\frac{\mathcal{E}_k}{2}}\left(a_{\mathbf{k}\lambda} + a_{\mathbf{k}\lambda}^*\right). \tag{4.46}$$

From (4.41)-(4.43) it follows that the random variables $q_{\mathbf{k}\lambda}$, $p_{\mathbf{k}\lambda}$ have zero average

$$\langle q_{\mathbf{k}\lambda}\rangle = 0, \qquad \langle p_{\mathbf{k}\lambda}\rangle = 0, \tag{4.47}$$

and are uncorrelated

$$\langle q_{\mathbf{k}\lambda} p_{\mathbf{k}\lambda}\rangle = 0; \tag{4.48}$$

the second-order moments, which coincide with the variances, are given by

$$\sigma_{p_k}^2 = \sigma_{p_{\mathbf{k}\lambda}}^2 \equiv \left\langle p_{\mathbf{k}\lambda}^2 \right\rangle = \mathcal{E}_k = \tfrac{1}{2}\hbar\omega_k, \quad \sigma_{q_k}^2 = \sigma_{q_{\mathbf{k}\lambda}}^2 \equiv \left\langle q_{\mathbf{k}\lambda}^2 \right\rangle = \mathcal{E}_k/\omega_k^2 = \frac{\hbar}{2\omega_k}. \tag{4.49}$$

One can alternatively express the complex amplitudes $a_{\mathbf{k}\lambda}$ in terms of a new pair of real, independent random variables $r_{\mathbf{k}\lambda}$, $\varphi_{\mathbf{k}\lambda}$ in the form

$$a_{\mathbf{k}\lambda} = \frac{1}{\sqrt{2\mathcal{E}_k}} \left(p_{\mathbf{k}\lambda} - i\omega_k q_{\mathbf{k}\lambda} \right) = r_{\mathbf{k}\lambda} e^{i\varphi_{\mathbf{k}\lambda}}, \tag{4.50}$$

where $r_{\mathbf{k}\lambda} = |a_{\mathbf{k}\lambda}| \geq 0$. One gets

$$\omega_k q_{\mathbf{k}\lambda} = -\sqrt{2\mathcal{E}_k}\, r_{\mathbf{k}\lambda} \sin\varphi_{\mathbf{k}\lambda}, \quad p_{\mathbf{k}\lambda} = \sqrt{2\mathcal{E}_k}\, r_{\mathbf{k}\lambda} \cos\varphi_{\mathbf{k}\lambda} \tag{4.51}$$

and the energy of the mode assumes the simple form

$$H_{\mathbf{k}\lambda} = \mathcal{E}_k \left| a_{\mathbf{k}\lambda} \right|^2 = \mathcal{E}_k r_{\mathbf{k}\lambda}^2. \tag{4.52}$$

From (4.41)-(4.43) we see that

$$\langle \sin\varphi_{\mathbf{k}\lambda} \rangle = 0, \qquad \langle \cos\varphi_{\mathbf{k}\lambda} \rangle = 0, \tag{4.53}$$

which means that the phases $\varphi_{\mathbf{k}\lambda}$ are uniformly distributed in the interval $(0, 2\pi)$, and that

$$\left\langle r_{\mathbf{k}\lambda}^2 \right\rangle = 1. \tag{4.54}$$

4.3.1. TWO-POINT CORRELATIONS

With the results of the preceding section we are in a position to calculate two-point correlations involving different field components.[17] For the two-point covariances of the electric and magnetic field components one gets using (4.38)-(4.44) and summing over λ [see, e.g., Bourret 1960]

$$\langle E_i(\mathrm{x}, t) E_j(\mathrm{x}', t') \rangle = \langle B_i(\mathrm{x}, t) B_j(\mathrm{x}', t') \rangle =$$

$$= \frac{2\pi\hbar}{V} \sum_{\mathbf{k}} \omega_k \left(\delta_{ij} - \hat{k}_i \hat{k}_j \right) \cos(\omega_k s - \mathbf{k}_{\mathbf{n}} \cdot \mathbf{r}), \tag{4.55}$$

$$\langle E_i(\mathrm{x}, t) B_j(\mathrm{x}', t') \rangle = \frac{2\pi\hbar}{V} \sum_{\mathbf{k}} \omega_k \left(\delta_{ij} - \hat{k}_i \hat{k}_j \right) \epsilon_{ijl} \hat{k}_l \cos(\omega_k s - \mathbf{k}_{\mathbf{n}} \cdot \mathbf{r}), \tag{4.56}$$

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[17] Admittedly, we are calculating two-point covariances; however, in the literature they are commonly referred to as correlations. Other names given to them are first-order correlations and linear correlations.

with $s \equiv t-t'$ and $\mathbf{r} \equiv \mathbf{x}-\mathbf{x}'$.[18] The fact that these expressions depend only on the differences $\mathbf{r}$ and s is due to the homogeneity and stationarity of the fields. For their explicit evaluation it is convenient to take the continuum limit $V \to \infty$, in which the triple sums transform into triple integrals; one then obtains, e. g.,

$$\Gamma_{EE'} \equiv \langle \mathbf{E}(\mathbf{x},t) \cdot \mathbf{E}(\mathbf{x}',t') \rangle = \frac{\hbar c}{2\pi^2} \int d^3k \, k \, \cos(\omega_k s - \mathbf{k_n} \cdot \mathbf{r}). \tag{4.57}$$

Choosing $\mathbf{r}$ along the z axis, one gets after integrating over the solid angle

$$\Gamma_{EE'} = -\frac{\hbar c}{\pi r} \frac{d^2}{dr^2} \int_0^\infty dk \left[\sin k(r-cs) + \sin k(r+cs) \right]. \tag{4.58}$$

Direct evaluation gives

$$\Gamma_{EE'} = -\frac{4\hbar c}{\pi} \frac{r^2 + 3c^2 s^2}{\left(r^2 - c^2 s^2\right)^3}, \quad r \neq cs. \tag{4.59}$$

Note that the sign of the covariance changes when going from space-like $(r > cs)$ over to time-like $(r < cs)$ intervals. If in (4.58) the integration is performed only up to a cutoff value k_c, the alternative result

$$\Gamma_{EE'}(k_c) = \Gamma_{EE'} + \frac{2\hbar c}{\pi r} \left[\left(\frac{1}{r_-^3} - \frac{k_c^2}{2r_-} \right) \cos k_c(r_-) \right.$$

$$\left. + \left(\frac{1}{r_+^3} - \frac{k_c^2}{2r_+} \right) \cos k_c r_+ + \frac{k_c}{r_-^2} \sin k_c r_- + \frac{k_c}{r_+^2} \sin k_c r_+ \right], \tag{4.60}$$

is obtained, with $\Gamma_{EE'}$ given by (4.59) and $r_\pm \equiv r \pm cs$. The additional terms oscillate with the high wave number k_c, so that in general they do not represent a meaningful contribution to the covariance. On the light cone $(r = cs)$ the expression simplifies to

$$\Gamma_{EE'}(k_c) = -\frac{\hbar c}{4\pi r^4} \left[1 - \left(1 - 2k_c^2 r^2\right) \cos 2k_c r - 2k_c r \sin 2k_c r \right], \tag{4.61}$$

---

[18] To get the four-dimensional form of the above expressions one may use the transformation (we set $\hbar = c = 1$)

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta\left(k^2 - m^2\right) \theta(k^0).$$

This allows for a direct comparison with the expressions used in relativistic QED for the expectations of ordered products and Green functions of the electromagnetic field. For details refer to Davies and Burkitt (1980).

which for $r = cs = 0$ gives

$$\Gamma_{EE}(k_c) = \sigma_E^2(k_c) = \frac{\hbar c k_c^4}{2\pi} \tag{4.62}$$

for the variance of **E**.

To get an idea of the distances and times at which the free field still maintains its correlation, it is convenient to calculate the correlation coefficient $\gamma_{EE'}$, obtained by dividing the covariance by the variance, $\gamma_{EE'} = \Gamma_{EE'}/\sigma_E^2$. Thus, for instance, by taking $r = 0$, $cs \neq 0$ one gets for the two-time correlation of the electric field

$$\gamma_{EE'} = 24(sck_c)^{-4}; \tag{4.63}$$

alternatively, for $cs = 0$, $r \neq 0$, one gets for the two-point (equal times) correlation

$$\gamma_{EE'} = -8(rk_c)^{-4}. \tag{4.64}$$

These results show that the correlation time of the field is determined by its higher frequencies and is of the order of $(ck_c)^{-1} = \omega^{-1}$, and similarly the correlation length is of the order of k_c^{-1}. Note that only with the introduction of a cutoff are these correlations well defined; they depend actually in an essential way on the cutoff, so the results have physical sense only to the extent to which the cutoff is justified. For $k_c = mc/\hbar$ the correlation length of the zeropoint field is of the order of the Compton wavelength and its correlation time is of the order of $\hbar/mc^2 \sim \tau/\alpha$, i. e., it is larger than τ, but still very small if measured in terms of atomic units.

A useful expression is obtained by taking $\mathrm{x} = \mathrm{x}'$ directly in equation (4.57); the result can be written in the general form

$$\langle E_i(\mathrm{x}, t) E_j(\mathrm{x}, t')\rangle = \delta_{ij} \int_0^\infty d\omega S(\omega) \cos\omega(t - t'), \tag{4.65}$$

where the *power spectrum* $S(\omega)$ is related to the spectral density by

$$S(\omega) = \frac{4\pi}{3}\rho(\omega), \tag{4.66}$$

as is readily seen by considering the case $t = t'$. The Fourier transform of equation (4.65) is the SED version of the Wiener-Khintchine theorem, which gives the power spectrum in terms of the Fourier transform of the two-time covariance [see, e.g., McQuarrie 1973; Papoulis 1965, chapter 10]. In terms of the Fourier transform of the field,

$$\tilde{E}_i(\mathrm{x}, \omega) = (2\pi)^{-1} \int_{-\infty}^\infty dt E_i(\mathrm{x}, t) e^{i\omega t}, \tag{4.67}$$

and with the help of the one-dimensional version of formula (3.124), equation (4.65) gives the useful result

$$\left\langle \tilde{E}_i(\mathrm{x},\omega)\tilde{E}_j^*(\mathrm{x},\omega')\right\rangle = \delta_{ij}S(\omega)\delta(\omega-\omega'). \tag{4.68}$$

Once more, in the case of the zeropoint field it is necessary to introduce a cutoff to give a definite meaning to these expressions.[19]

4.3.2. DISTRIBUTIONS OF RANDOM FIELD VARIABLES

For the calculations made to this point, only the first- and second-order moments of the distribution of the random variables have been required, (4.41)-(4.44), and for many purposes this information suffices, as will be seen along the text. However, to construct all higher-order correlations it is in principle necessary to have a full knowledge of the distributions. A most frequent additional assumption in the SED literature is that the amplitudes of the (statistically independent) field modes are normally distributed random variables. This means in particular that each $q_{\mathbf{k}\lambda}$ and $p_{\mathbf{k}\lambda}$ has a Gaussian distribution, so that the corresponding probability densities are (recall from equation (4.47) that they have zero mean)[20]

$$P_q(q_{\mathbf{k}\lambda}) = \frac{1}{\sqrt{2\pi}\sigma_q}e^{-q_{\mathbf{k}\lambda}^2/2\sigma_q^2}, \quad P_p(p_{\mathbf{k}\lambda}) = \frac{1}{\sqrt{2\pi}\sigma_p}e^{-p_{\mathbf{k}\lambda}^2/2\sigma_p^2}. \tag{4.69}$$

The respective variances are given by equations (4.49); we used the condensed notation $\sigma_q \equiv \sigma_{q_k}, \sigma_p \equiv \sigma_{p_k}$. Since $q_{\mathbf{k}\lambda}$ and $p_{\mathbf{k}\lambda}$ are uncorrelated, as follows from (4.48), these probability densities can be simply multiplied to obtain the phase-space distribution (omitting the indices $\mathbf{k}, \lambda$, for simplicity)

$$P_{qp}(q,p) = \frac{1}{2\pi\sigma_q\sigma_p}e^{-q^2/2\sigma_q^2}e^{-p^2/2\sigma_p^2} = \frac{2}{\sqrt{2\pi}\hbar}e^{-(\omega^2q^2+p^2)/\hbar\omega}. \tag{4.70}$$

The Hamiltonian of each mode is $H = \frac{1}{2}\left(p^2+\omega^2q^2\right)$; therefore the (reduced) density for the energy of mode $(\mathbf{k},\lambda)$ is

$$P_{E_k}(H) = \frac{2\pi}{\omega_k}P_q(q)P_p(p) = \frac{1}{\mathcal{E}_k}e^{-H/\mathcal{E}_k}, \tag{4.71}$$

---

[19] Results as the nonexistence of the correlations or the power spectrum, etc. in the absence of cutoff, imply that the zeropoint radiation field without cutoff does not enjoy ergodic properties [see, e.g., Onicescu and Guiasu 1971, section 2.3; Papoulis 1965, section 9.8].

[20] The harmonic representation of the radiation field with normally distributed independent amplitudes goes back to Einstein and Hopf (1910); a short discussion on the subject from a historical perspective is given in Rice (1954). For a transparent introductory discussion on normal variables and Gaussian distributions the reader is invited to refer to chapters 4-8 of Papoulis (1965).

with $\mathcal{E}_k = \frac{1}{2}\hbar\omega_k$. Further, from (4.69) one gets

$$\omega_k^{2n}\left\langle q_{\mathbf{k}\lambda}^{2n}\right\rangle = \left\langle p_{\mathbf{k}\lambda}^{2n}\right\rangle = (2n-1)!!\left\langle p_{\mathbf{k}\lambda}^{2}\right\rangle^n = (2n-1)!!\sigma_p^{2n} = (2n-1)!!\mathcal{E}_k^n \tag{4.72}$$

for any positive integer n, as corresponds to a Gaussian distribution. We see that all even moments are determined by the central moments of second order, σ_q^2 and σ_p^2 (the odd moments are all zero). The amplitudes $a_{\mathbf{k}\lambda}$ are in their turn

$$\langle a_{\mathbf{k}\lambda}^{*m} a_{\mathbf{k}\lambda}^{n}\rangle = \left\langle r_{\mathbf{k}\lambda}^{n+m}\right\rangle\left\langle e^{i(n-m)\varphi}\right\rangle = \delta_{nm}\left\langle r_{\mathbf{k}\lambda}^{2n}\right\rangle. \tag{4.73}$$

We used equations (4.53) to check that $< e^{i(n-m)\varphi} >= \delta_{nm}$. On combining (4.46) and the two latter expressions one gets ($C_{n,k}$ is a binomial coefficient)

$$\left\langle (a_{\mathbf{k}\lambda} + a_{\mathbf{k}\lambda}^*)^{2n}\right\rangle = \left(\frac{2}{\mathcal{E}_k}\right)^n \left\langle p_{\mathbf{k}\lambda}^{2n}\right\rangle = 2^n\,(2n-1)!! =$$

$$= \sum_k C_{2n,k}\left\langle a_{\mathbf{k}\lambda}^{2n-k} a_{\mathbf{k}\lambda}^{*k}\right\rangle = C_{2n,n}\left\langle a_{\mathbf{k}\lambda}^{n} a_{\mathbf{k}\lambda}^{*n}\right\rangle = 2^n\frac{(2n-1)!!}{n!}\left\langle r_{\mathbf{k}\lambda}^{2n}\right\rangle, \tag{4.74}$$

from which it follows that

$$\langle a_{\mathbf{k}\lambda}^{*n} a_{\mathbf{k}\lambda}^{n}\rangle = \left\langle r_{\mathbf{k}\lambda}^{2n}\right\rangle = n!. \tag{4.75}$$

Thus the variables r are not normally distributed under the above assumptions (cf. equation (4.72)); moreover, all moments of $r_{\mathbf{k}\lambda}$ are different from zero, and in particular,

$$\langle r_{\mathbf{k}\lambda}\rangle = \sqrt{\frac{2}{\pi}}, \quad \left\langle r_{\mathbf{k}\lambda}^2\right\rangle = 1, \qquad \text{whence} \quad \sigma_r^2 = 1 - \frac{2}{\pi}. \tag{4.76}$$

The last result follows from considering that $r^2 = H/\mathcal{E}$ and that according to (4.71), H has a Laplace distribution with $H \geq 0$. From (4.71) it also follows that

$$\langle H^n\rangle = n!\mathcal{E}_k^n. \tag{4.77}$$

In particular, $\langle H^2\rangle = 2\mathcal{E}_k^2$, so the energy of each independent mode has a dispersion equal to the square of its average value

$$\sigma_E^2 = \mathcal{E}_k^2. \tag{4.78}$$

An alternative construction for the zeropoint field can be frequently found in the SED literature, in which the modulus $r_{\mathbf{k}\lambda}$ is taken as *fixed* and equal to 1 for any $(\mathbf{k}, \lambda)$. One has thus instead of equations (4.50) and (4.75)

$$a_{\mathbf{k}\lambda} = \sqrt{\frac{1}{2\mathcal{E}_k}}\,(p_{\mathbf{k}\lambda} - i\omega_k q_{\mathbf{k}\lambda}) = e^{i\varphi_{\mathbf{k}\lambda}} \tag{4.79}$$

and

$$\langle a^{*n}_{\mathbf{k}\lambda} a^{n}_{\mathbf{k}\lambda} \rangle = \left\langle r^{2n}_{\mathbf{k}\lambda} \right\rangle = 1. \tag{4.80}$$

This *fixed-modulus* (or *random-phases*) *representation*, which goes back to Planck (1911) [see also Planck 1959], has been used extensively by Boyer for the description of the free zeropoint field. In this case, $q_{\mathbf{k}\lambda}$ and $p_{\mathbf{k}\lambda}$ cease to be independent and become related by the fixed value of the energy, which implies that their absolute value is bounded,

$$\left\langle (p^2_{\mathbf{k}\lambda} + \omega^2_k q^2_{\mathbf{k}\lambda})^n \right\rangle = \left\langle p^2_{\mathbf{k}\lambda} + \omega^2_k q^2_{\mathbf{k}\lambda} \right\rangle^n = (2\mathcal{E}_k)^n \tag{4.81}$$

and they cannot be normally distributed. Physically this corresponds to the assumption that the total energy of each individual mode of the field within the normalization volume is a nonfluctuating quantity, so that

$$\langle H^n \rangle = \mathcal{E}^n_k = (\tfrac{1}{2}\hbar\omega_k)^n, \tag{4.82}$$

as is the case in QED. Below it is shown that the two representations (fixed and stochastic modulus) become equivalent in the limit of infinite volume. However, it can still be argued that the demand of invariance of the statistical properties of the field modes after repeated reflections by mirrors or similar optical transformations, favours a Gaussian distribution for the single modes.[21] This point will be further discussed below.

4.3.3. STATISTICS OF THE ZEROPOINT FIELD

Let us consider once more a single normal mode $\mathbf{k}\lambda$, as in the preceding section. The total energy of this mode within the normalization volume is given by (4.52), and the dispersion of this energy is therefore

$$\sigma^2_{E_k} = \mathcal{E}^2_k \left[\left\langle |a_{\mathbf{k}\lambda}|^4 \right\rangle - 1 \right]. \tag{4.83}$$

This gives $\sigma^2_{E_k} = \mathcal{E}^2_k$ for the Gaussian distribution and $\sigma^2_{E_k} = 0$ for the random-phases distribution, as before. The difference between the predictions of the two representations for the statistical properties of each mode becomes also manifest in the energy density

$$U_{\mathbf{k}\lambda} = \frac{1}{8\pi}(\mathbf{E}^2_{\mathbf{k}\lambda} + \mathbf{B}^2_{\mathbf{k}\lambda}). \tag{4.84}$$

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[21] This argument has been advanced to the authors on several occasions by E. Santos. On the other hand, in modern quantum theory it is possible to find, for the study of certain applications, a stochastic field with random phases and fixed moduli as a model for the quantum radiation field. An example can be seen in Haken (1981), section 4.2.

One obtains $< U_{\mathbf{k}\lambda} >= \mathcal{E}_k/V$, as expected, and

$$\sigma^2_{U_k} = \langle U_{\mathbf{k}\lambda}\rangle^2 \left[\tfrac{3}{2}\left\langle |a_{\mathbf{k}\lambda}|^4\right\rangle - 1\right], \tag{4.85}$$

which means that the energy density of a mode is always a dispersive quantity, regardless of the distribution of the random amplitudes, since $\left\langle |a_{\mathbf{k}\lambda}|^4\right\rangle \geq 1$. In particular, $\sigma^2_{U_k} = 2\langle U_{\mathbf{k}\lambda}\rangle^2$ for the Gaussian distribution, and $\sigma^2_{U_k} = \frac{1}{2}\langle U_{\mathbf{k}\lambda}\rangle^2$ for the random-phases distribution.

The contrast between the results for the two distributions is, however, lost when one considers the totality of modes comprising the zeropoint field. For the total energy density $U = \sum U_{\mathbf{k}\lambda}$ one obtains from (4.31) with $\omega_1 = 0$ and $\omega_2 = \omega_c$

$$\langle U\rangle = \frac{1}{\pi^2 c^3}\int_0^{\omega_c} d\omega\, \omega^2 \mathcal{E}(\omega) = \frac{\hbar\omega_c^4}{8\pi^2 c^3}. \tag{4.86}$$

For the calculation of the dispersion of the energy it is convenient to write explicitly (with $\alpha = \mathbf{k}\lambda$)

$$\left\langle |a_\alpha|^2 |a_\beta|^2\right\rangle = \left\langle |a_\alpha|^2\right\rangle \left\langle |a_\beta|^2\right\rangle (1 + \eta\delta_{\alpha\beta}), \tag{4.87}$$

to take into account the two possible random-variable distributions under discussion, namely, the Gaussian one with $\eta = 1$, and the random-phases one with $\eta = 0$. In the continuum limit the Kronecker delta in (4.87) transforms into the corresponding Dirac delta function, as follows from (4.45), according to $\delta_{\alpha\beta} = \delta_{\mathbf{k}_\alpha \mathbf{k}_\beta}\delta_{\lambda_\alpha\lambda_\beta} \to \delta(\mathbf{k}_\alpha - \mathbf{k}_\beta)\delta_{\lambda_\alpha\lambda_\beta}/\left\langle |a_\lambda(\mathbf{k}_\alpha)|^2\right\rangle$. A somewhat lengthy but straightforward procedure gives for the dispersion of the energy density

$$\sigma^2_U = \tfrac{1}{3}\langle U\rangle^2, \tag{4.88}$$

independently of the value of η, due to the fact that the term depending on η in (4.87) gives a negligible contribution to the double integral over the frequencies ω_α, ω_β. Hence, the statistical properties of a field with an infinite number of modes are independent of η. In other words, in the limit $V \to \infty$ the two representations become equivalent.

Let us now consider the *total* energy of the field, $H_T = \int d^3x\, U$. Its average value can be obtained from (4.86),

$$\langle H_T\rangle = \int d^3x \,\langle U\rangle = V\langle U\rangle,$$

and is of course an infinitely large quantity. More interesting, however, is the result obtained for the dispersion of H_T. This can be most easily calculated by taking the mean square of $H_T = \sum_{\mathbf{k}\lambda} E_k |a_{\mathbf{k}\lambda}|^2$ in the continuum limit and using (4.87). Again due to the fact that the η-dependent term gives a negligible contribution to the double integral, for any value of η one obtains

$$\left\langle H_T^2 \right\rangle = \langle H_T \rangle^2 , \tag{4.89}$$

i.e., $\sigma_{H_T} = 0$. Hence the total energy of the zeropoint field is a fixed quantity in both representations.

The above results can be seen to fully correspond to the concept of the zeropoint field as a stationary field that is maximally disordered, as was mentioned at the beginning of section 4.3. In general, a random stationary field with a large number (infinite, in principle) of statistically independent Fourier amplitudes is known to be Gaussian.[22] Applied to the present case this means that every electric and magnetic component (E_i or B_i) of the field is a normal random variable, with a distribution of the form

$$P_i(E_i) = \frac{1}{\sqrt{2\pi}\sigma_{E_i}} e^{-E_i^2/2\sigma_{E_i}^2}, \quad P_i(B_i) = \frac{1}{\sqrt{2\pi}\sigma_{E_i}} e^{-B_i^2/2\sigma_{E_i}^2} \tag{4.90}$$

whence from (4.72) with $n = 2$, $\langle E_i^4 \rangle = \langle B_i^4 \rangle = 3\sigma_{E_i}^4$, since all components have the same dispersion σ_{E_i}. The reduced distribution for the energy-density term $U_i = \frac{1}{8\pi}({E_i}^2 + {B_i}^2)$ is therefore the Laplace distribution (with $\langle U_i \rangle = \frac{1}{4\pi}\sigma_{E_i}^2$)

$$P_i(U_i) = \frac{1}{\langle U_i \rangle} e^{-U_i/\langle U_i \rangle}. \tag{4.91}$$

It is easy to verify that this distribution gives an extremum for the entropy $S_i = -\int P(U_i) \ln P(U_i) dU_i$, subject to the conditions of fixed normalization and mean energy, $\int dU_i \, P(U_i) = 1$, $\langle U_i \rangle = \int dU_i \, U_i P(U_i)$, as corresponds to a maximally disordered field. Since the different components are uncorrelated one obtains for the full field $\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle = 6\sigma_i^2$, $\langle (\mathbf{E}^2 + \mathbf{B}^2)^2 \rangle = 48\sigma_i^4 = \frac{4}{3} \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle^2$, whence it follows that the dispersion of the total energy density $U = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2)$ is indeed $\sigma_U^2 = \frac{1}{3} \langle U \rangle^2$, in agreement with (4.88).

4.3.4. COMPARISON WITH THE SECOND-QUANTIZED RADIATION FIELD

The connection between the statistical properties of the zeropoint field of SED and the expectation values of the QED electromagnetic field in vacuo has been studied from different angles.[23] In particular, Boyer (1975b) has shown by direct calculation that the N-point correlation functions of the fields in both theories are identical if all products of the quantum operators are symmetrized. Thus, for instance, for $N = 2$ the covariances calculated in §4.3.1 can be reproduced within QED by taking the vacuum expectation value of the Hermitian combination of the corresponding operators. Indeed, with[24]

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[22] This follows directly from the central-limit-theorem [see, e.g., Bartlett 1966, p. 199; Papoulis 1965, section 8.6].

[23] Some references are: Segal (1963), Bourret (1964, 1966), Marshall (1965a), Santos (1974c, 1975d), Boyer (1975b), Goedecke (1983a), Peřina (1985) chapter 14 and Landau (1988).

$$\mathbf{E}(\mathbf{x},t) = i\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar\omega_k}{V}}\,\mathbf{e}_{\mathbf{k}}^{\lambda}\left(\hat{a}_{\mathbf{k}\lambda}e^{-i\omega_k t+i\mathbf{k}.\mathbf{x}} - \hat{a}^{\dagger}_{\mathbf{k}\lambda}e^{i\omega_k t-i\mathbf{k}.\mathbf{x}}\right), \qquad (4.92)$$

where $\hat{a}^{\dagger}_{\mathbf{k}\lambda}, \hat{a}_{\mathbf{k}\lambda}$ are the photon creation and annihilation operators, respectively, with commutator $[\hat{a}_{\mathbf{k}\lambda}, \hat{a}^{\dagger}_{\mathbf{k}\lambda}] = 1$, and $|0\rangle$ the vacuum state, so that $\hat{a}\,|0\rangle = 0$, $\langle 0|\,\hat{a}^{\dagger} = 0$, the expectation value over the vacuum state of the symmetrized product

$$\Gamma^{s}_{E_iE'_j} = \langle 0|\,\tfrac{1}{2}[\hat{E}_i(\mathbf{x},t)\hat{E}_j(\mathbf{x}',t') + \hat{E}_j(\mathbf{x}',t')\hat{E}_i(\mathbf{x},t)]\,|0\rangle \qquad (4.93)$$

gives the same result as equation (4.55). This problem has been analysed by Landau (1988) following a somewhat different approach. To illustrate his results with a simple example let us consider again the two-point correlations. Let $\Gamma^{q}_{kl} \equiv \langle A_k A_l\rangle = \Gamma^{s}_{kl} + i\Gamma^{a}_{kl}$ be the matrix elements of the 2-point correlation functions of a pair of observables A_k, A_l pertaining to the second quantized field; the matrix Γ^{a} is real and antisymmetric, whereas Γ^{s} is real, symmetric and positive. The corresponding correlations of the classical field turn out to be given by Γ^{s}, so that $\Gamma^{s}_{kl} = \frac{1}{2}(\Gamma^{q}_{kl} + \Gamma^{q}_{lk}) = \frac{1}{2}\langle A_kA_l + A_lA_k\rangle$, in agreement with the previous result. This correspondence will be studied with some more detail in section 13.5.

To illustrate yet another approach to the problem, let us briefly review the method developed by Bourret (1964). Consider a single plane wave of the random field, and write it rather schematically as

$$E = \frac{1}{\sqrt{2}}(\tilde{E}a + \tilde{E}^{*}a^{*}). \qquad (4.94)$$

Assume the mode to be Gaussian, with variance $\sigma^2 = \left|\tilde{E}\right|^2$; then the characteristic (moment-generating) function is

$$\left\langle e^{\beta E}\right\rangle = \sum_{n}\frac{\beta^n}{2^{n/2}n!}\left\langle\left(\tilde{E}a + \tilde{E}^{*}a^{*}\right)^n\right\rangle = \sum_{n,k}\frac{\beta^n}{2^{n/2}n!}C_{n,k}\tilde{E}^{n-k}\tilde{E}^{*k}\left\langle a^{n-k}a^{*k}\right\rangle$$

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[24] There appears an additional factor $\sqrt{2}$ in this expression for the quantum field as compared with the SED expression (4.38). The reason is that in the SED case the scale of the $a_{\mathbf{k}\lambda}$ is chosen such that $\left\langle|a_{\mathbf{k}\lambda}|^2\right\rangle = 1$, whereas in QED, $\left\langle\hat{a}_{\mathbf{k}\lambda}\hat{a}^{\dagger}_{\mathbf{k}\lambda} + \hat{a}^{\dagger}_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda}\right\rangle = 1$.

$$= \sum_n \frac{\beta^{2n}}{2^n(2n)!} C_{2n,n} \left|\tilde{E}\right|^{2n} n! = \sum_n \frac{\beta^{2n}}{2^n n!} \left|\tilde{E}\right|^{2n} = e^{\beta^2\sigma^2/2}, \tag{4.95}$$

where equation (4.75) was used to write the third equality. This is indeed the generating function of a Gaussian distribution [see, e.g., Papoulis 1965, section 5.5], so that by successively differentiating with respect to β and putting $\beta = 0$ in each result, one gets for the moments of the field

$$\langle E \rangle = 0, \quad \left\langle E^2 \right\rangle = \sigma^2, \quad \left\langle E^4 \right\rangle = 3\left\langle E^2 \right\rangle^2, \cdots \left\langle E^{2n} \right\rangle = (2n-1)!!\left\langle E^2 \right\rangle^n. \tag{4.96}$$

Now consider the corresponding quantized field

$$\hat{E} = \tilde{E}\hat{a} + \tilde{E}^*\hat{a}^\dagger. \tag{4.97}$$

The vacuum expectation value of the square of this field coincides with the variance of the random field, $\langle 0| \hat{E}^2 |0\rangle = |\tilde{E}|^2 = \sigma^2$. The characteristic function is now defined as $\langle 0| e^{\beta\hat{E}} |0\rangle = \langle 0| e^{\beta\tilde{E}\hat{a}+\beta\tilde{E}^*\hat{a}^\dagger} |0\rangle$. With the help of the Baker-Campbell-Hausdorff formula one obtains then

$$\langle 0| e^{\beta\hat{E}} |0\rangle = \langle 0| e^{\beta\tilde{E}^*\hat{a}^\dagger} e^{\beta\tilde{E}\hat{a}} |0\rangle e^{\beta^2\sigma^2/2} = e^{\beta^2\sigma^2/2}, \tag{4.98}$$

in agreement with the result of the c-number field, (4.95). The second-quantized field appears thus as 'normally distributed'. It is important to note that this result is tightly linked to the normal ordering of the operators $\hat{a}^\dagger$, $\hat{a}$ in equation (4.98); a change in the order of the operators leads to a different conclusion.

The above examples illustrate that it is possible to obtain a close resemblance between the SED formalism for the zeropoint field and that of the quantized vacuum field, with operators taking the place of random variables, provided an ordering rule for the operators is specified. Note, however, that the parallelism involves only average values and correlations; the SED formalism developed so far does not contain any stochastic analog to the quantum commutator. From this point of view, the quantized field seems to be richer than its stochastic counterpart, since to each different ordering there corresponds a different process. There will be opportunity in what follows to come back to this and related matters, particularly in chapter 13. In chapter 10 we will introduce a mathematical quantity that has properties similar to those of a commutator and is particularly useful in connection with stochastic processes.

CHAPTER 5

THE EQUILIBRIUM RADIATION FIELD

In the preceding chapter we made ourselves familiar with the zeropoint field and studied some of its most important properties, as a preparation for our inquiry into the effects that such a field is bound to have on matter. In this chapter we direct our attention to the fluctuations of the complete radiation field in equilibrium at a given temperature, and under very general assumptions we show that the presence of the zeropoint field affects most dramatically the thermal equilibrium distribution, which becomes described by Planck's law instead of the classical Rayleigh-Jeans law. The explanation for such astonishing behaviour is found in the extra fluctuations generated by the interference between the thermal and the nonthermal components of the field modes. Other typical quantum properties of this radiation field are studied, although with much less detail and, of course, treating the field everywhere as continuous.

5.1. The Planck distribution

The first problem of physics that found an answer in quantum terms was the determination of the spectrum of blackbody radiation. As was discovered by Planck (1899, 1900) at the turn of the century and confirmed by Einstein (1905) briefly afterwards, the blackbody equilibrium spectral distribution is obtained by allowing for discrete processes of interchange of energy to enter into the description. This discovery gave way to a picture which is now a fundamental part of quantum theory: Planck's law and discrete properties of the radiation field are nowadays synonymous.

The connection between the Planck distribution and the zeropoint field is, however, much less understood. The subject has attracted the attention of SED authors for almost three decades, and an important effort has been made to answer the question: is the hypothesis of the zeropoint field sufficient to derive Planck's law, or must discrete elements be introduced to attain this goal? Quantum theorists favour the latter alternative; SED practitioners would prefer first to explore thoroughly the former one, in accordance with Nernst's insight.

The idea is far from new; it was stated and explored for the first time by Einstein and Stern (1913), who arrived at an affirmative answer, although based on an ad hoc and not quite correct assumption.[1] The Einstein-Stern proposal remained neglected for decades until it was revived by Boyer (1969b), who showed that Planck's law follows from the assumption of the existence of a random zeropoint field with energy $\frac{1}{2}\hbar\omega$ per normal mode, if some assumptions concerning the role of the container walls in restoring and maintaining equilibrium are allowed. Boyer's theory, to be discussed in section 5.2, renewed the interest on the subject and became an important reference for all the work that followed.[2] We here present an alternative procedure, close in spirit to that of Boyer (1969d) and Theimer (1971) and given in detail by de la Peña and Cetto (1993d, 1995b), to show that it is indeed possible to derive Planck's law from the hypothesis of the existence of the zeropoint field with some very reasonable additional assumptions of statistical and thermodynamic nature, without introducing explicitly any discrete property of the field.

5.1.1. NONCLASSICAL NATURE OF STOCHASTIC ELECTRODYNAMICS

Consider the radiation field inside a cavity in equilibrium at temperature T as composed of independent modes of frequency ω. We assume that the state of this field at any temperature is described by a canonical ensemble, so that we write the probability of the state with energy E as

$$W(E) = \frac{1}{Z} g(E) e^{-\beta E}, \tag{5.1}$$

where, as usual, $\beta = 1/k_B T$ with k_B the Boltzmann constant. Equation (5.1) is a generalization of the classical canonical ensemble to include a statistical weight $g(E)$ which can be interpreted as an intrinsic probability for the state with energy E. In classical physics it is assumed that all energy states of the field oscillators have equal intrinsic probabilities, so

$$g(E) = 1 \tag{5.2}$$

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[1] Detailed modern accounts of the Einstein-Stern theory and of the previous attempts by Einstein and Hopf (1910a, b) can be seen in Boyer (1969b) and Milonni (1994). Fully annotated translations into English of these two papers by Einstein and collaborators are given in Bergia et al. (1979, 1980).

[2] See, e.g., Boyer (1969d, 1970a), Theimer (1971), Jiménez et al. (1980, 1982), Marshall (1981), Payen (1984). Other SED papers dealing with the Planck distribution are Park and Epstein (1949) (see §4.1.1), Marshall (1963, 1965a, 1965b), Surdin et al. (1966), Boyer (1980c, 1983, 1984b, d), Theimer and Peterson (1974, 1976), Santos (1975c), Kracklauer (1976), Theimer (1976), Cole (1986, 1990c), França and Maia (1993), de la Peña and Cetto (1993d, 1995b); see also Sachidanandam (1984). A detailed list of references to works up to 1982 may be found in de la Peña (1983).

and one is led directly to the equipartition law, since equations (5.1) and (5.2) together give for the average energy of the field oscillators

$$\overline{E} = \int_0^\infty dE\, E W(E) = Z = k_B T, \quad Z = \int_0^\infty dE\, e^{-\beta E} = \frac{1}{\beta} \quad (g(E) = 1). \tag{5.3}$$

Additionally, one gets

$$\overline{E^r} = r!\overline{E}^r, \quad (g(E) = 1). \tag{5.4}$$

The intrinsic probability factor $g(E)$ has been included in (5.1) in view of the discussion in §4.2.3, where strong arguments were given to revise the equipartition law and, further, the introduction of the zeropoint field into the classical scheme was shown to lead to a violation of this law.[3] To allow for the required departures from the classical laws without having to abandon the canonical distribution, we leave the factor $g(E)$ unspecified for the time being, and to emphasize the fact that this allows for a possible nonclassical behaviour, we refer to (5.1) as a *metaclassical distribution* [Cetto and de la Peña 1989]. This metaclassical law was used for the first time by Einstein (1907) in his famous paper on the specific heats of solids.

From the normalization of (5.1) it follows that the partition function is given in general by

$$Z = \int_0^\infty dE\, g(E)\, e^{-\beta E}, \tag{5.5}$$

its derivative with respect to β being

$$Z' \equiv \frac{dZ}{d\beta} = -Z\overline{E}. \tag{5.6}$$

For any function $f(E)$ of the energy one gets

$$\overline{f(E)}' = \overline{E}\,\overline{f} - \overline{Ef}, \tag{5.7}$$

where the bar means average over the (metaclassical) ensemble; for $f = E^r$, $r = 0, 1, 2, ...,$

$$\overline{E^r}' = \overline{E}\,\overline{E^r} - \overline{E^{r+1}}. \tag{5.8}$$

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[3] A way to arrive at $g(E)$ is by considering first the full distribution for all field oscillators of the given frequency, in the corresponding phase space $\{q_n\}$. The probability for the system to be in a state of energy $E(q_n)$ when the variables q_n are within the elementary volume $dq_1 dq_2 \cdots$ can be written in the form $dw(\beta; q_n) = \Phi(E; \beta) dq_1 dq_2 \cdots$. The integral over all q_n corresponding to an energy between E and $E + dE$ at a given moment, gives for the reduced probability density $dW = \int_{dE} dw(\beta; q_n) = \Phi(E; \beta) \int_{dE} dq_1 dq_2 \cdots = \Phi(E; \beta) g(E) dE$, where $g(E)dE = \int_{dE} dq_1 dq_2 \cdots$.

This important result gives with $r = 1$ the well-known Einstein formula for the thermal energy fluctuations [see, e.g., Kittel 1958]

$$\bar{E}' = \overline{E}^2 - \overline{E^2}. \tag{5.9}$$

Another useful formula derived from (5.5) is

$$\overline{E^r} = (-)^r \frac{1}{Z}\frac{d^r Z}{d\beta^r}, \tag{5.10}$$

of which (5.6) is a particular case. Equations (5.5) to (5.10) hold for any $g(E)$, including of course the classical value $g = 1$.

Einstein's formula (5.9), rewritten as $k_B T^2(\partial\overline{E}/\partial T) = \overline{E^2} - \overline{E}^2 = \sigma_E^2$, gives the well-known relation between the energy fluctuations in the ensemble and the specific heat $c_V = \partial\overline{E}/\partial T$ of the field at temperature T and constant volume

$$k_B T^2 c_V = \sigma_E^2. \tag{5.11}$$

The fact that the specific heat remains finite as $T \longrightarrow 0$, which has been established empirically at least for material systems, means that the energy variance σ_E^2 cannot be different from zero at $T = 0$ (in fact it must go to zero as rapidly as $c_V T^2$), whence

$$\overline{E^2}(0) = \overline{E}^2(0). \tag{5.12}$$

In the presence of a zeropoint field with $\overline{E}(0) \equiv \mathcal{E}_0 > 0$, equation (5.4) with $r = 2$ is incompatible with (5.12); hence $g(E)$ must depend nontrivially on the energy. This verifies that the mere inclusion of a real zeropoint energy into the scheme of theoretical physics entails a modification of the classical laws of statistical physics. Since the basic hypothesis of SED is the existence of the zeropoint field with its correlated energy, it would be inconsistent to regard this theory as classical, at least from the present point of view.[4] In the following it will be shown that the function $g(E)$ becomes essentially fixed by demanding that $\mathcal{E}_0 \neq 0$ and considering that the derivatives of c_V remain finite at $T = 0$.

[4] As discussed in §4.1.2, two basic notions of 'classical' can be distinguished in the SED literature. The one formulated in terms of a space-time description does not apply to the present statistical description of the field; we therefore resort to the second notion as the appropriate one in the present context. Accordingly, since the theory contains in an essential way the zeropoint field and differs qualitatively from classical physics, it should not be considered classical. This criterion can be substantiated by recalling that the zeropoint field operates as a special kind of reservoir, a source of extra nonthermal (quantum) fluctuations, which are totally unknown in classical physics.

5.1.2. MOMENTS OF THE ENERGY DISTRIBUTION

We first set out to prove that equation (5.12) is a particular case of a more general relation, namely,

$$\overline{E^r} = \overline{E}^r = \mathcal{E}_0^r \text{ at } T = 0, \tag{5.13}$$

in contrast to equation (5.4). With this aim we take the derivative of (5.11) with respect to β; since c_V' is finite at $T = 0$ this gives $\overline{E^2}'(0) = 2\mathcal{E}_0 \bar{E}'(0)$. But $\bar{E}'(0) = 0$ from (5.9) and (5.12); hence it follows that $\overline{E^2}'(0) = 0$, and from (5.8) with $r = 2$,

$$\overline{E^3}(0) = \mathcal{E}_0^3. \tag{5.14}$$

By successively differentiating of (5.11) and combining with (5.8) evaluated at $T = 0$ for $r = 3, 4, 5, \ldots$, one arrives at equation (5.13) for any r. According to this equation, the energy associated with the set of all modes of a given frequency ω contained in the cavity does not fluctuate at $T = 0$; it has the fixed value $\mathcal{E}_0$. This result is stronger than the one previously derived, equation (4.89), which refers to the *total* zeropoint electromagnetic energy in the cavity. As will be discussed in §5.1.6, *local* fluctuations and fluctuations of the energy of each separate mode can of course still exist, even at $T = 0$.

In terms of the energy distribution, the previous result becomes

$$W(E) \underset{\beta \to \infty}{\longrightarrow} \delta(E - \mathcal{E}_0), \tag{5.15}$$

whence upon inspection of (5.1) one sees that $g(E)$ must have the form

$$g\,(E) = \delta\,(E - \mathcal{E}_0) + G\,(E - \mathcal{E}_0)\,\Theta\,(E - \mathcal{E}_0) \tag{5.16}$$

where $\Theta(x)$ is the step function ($\Theta(x) = 1$ for $x > 0$, $\Theta(x) = 0$ otherwise). Introducing (5.16) into (5.5) one gets

$$Z = e^{-\beta\mathcal{E}_0}\left[1 + \int_0^\infty dE'\, G(E')e^{-\beta E'}\right] \equiv e^{-\beta\mathcal{E}_0} Z_T$$

and the energy distribution (5.1) takes the form

$$W(E) = \frac{1}{Z_T}\left[\delta(E_T) + G(E_T)\Theta(E_T)\right]e^{-\beta E_T}, \tag{5.17}$$

with

$$E_T = E - \mathcal{E}_0. \tag{5.18}$$

Now we sketch a method for determining all the moments of the distribution at $T > 0$, which is equivalent to finding $G(E)$ and constructing the distribution itself. For this purpose we treat equation (5.8) as a recurrence relation for the energy moments and write

$$\overline{E^{r+1}} = \overline{E}\,\overline{E^r} - \overline{E^r}'. \tag{5.19}$$

Since the zeropoint term does not contribute to the fluctuations, it is convenient to subtract it from the total energy, using (5.18); then (5.19) transforms into[5]

$$\overline{E_T^{r+1}} = \overline{E_T}\,\overline{E_T^r} - \overline{E_T^r}'. \tag{5.20}$$

It is clear that E_T represents the purely thermal contribution to the energy; hence at zero temperature all its moments must vanish, as confirmed by (5.18),

$$\overline{E_T^r}(0) = 0. \tag{5.21}$$

On the other hand, the distribution of E_T should go to the classical one for high temperatures; this means that (5.4) must hold in this limit, so

$$\overline{E_T^r} = r!\overline{E_T}^r, \quad T \to \infty. \tag{5.22}$$

For arbitrary temperatures we therefore propose to write

$$\overline{E_T^r} = r!\overline{E_T}^r + \mathcal{E}_0^r \phi_r \tag{5.23}$$

where ϕ_r, $r = 0, 1, 2, \cdots$, represent functions of β to be determined; the coefficient $\mathcal{E}_0^r$ has been introduced to make the ϕ_r dimensionless. Before obtaining the functions ϕ_r, let us disclose some of their properties. In the first place, integrating (5.9) rewritten in terms of $\overline{E_T}$ instead of $\overline{E}$, one gets

$$\beta = -\int \frac{d\overline{E_T}}{\overline{E_T^2} - \overline{E_T}^2}, \tag{5.24}$$

which says that $\overline{E_T^2}$ can be expressed (in principle) as a function of $\overline{E_T}(\beta)$; hence according to (5.20), every $\overline{E_T^r}$ can in its turn be expressed as a function of $\overline{E_T}$, and so also ϕ_r. Secondly, for $r = 0, 1$ equation (5.23) gives $\phi_0 = 0$, $\phi_1 = 0$, and from (5.20) and (5.23) one gets the recurrence relation

$$\phi_{r+1} = -\frac{1}{\mathcal{E}_0}\frac{d\phi_r}{d\beta} + \frac{\overline{E_T}}{\mathcal{E}_0}\phi_r + rr!\left(\frac{\overline{E_T}}{\mathcal{E}_0}\right)^{r-1}\phi_2,$$

which in terms of the dimensionless variable $\xi \equiv \overline{E_T}/\mathcal{E}_0$ reads

$$\phi_{r+1} = (\xi^2 + \phi_2)\phi_r' + \xi\phi_r + rr!\xi^{r-1}\phi_2, \tag{5.25}$$

---

[5] This equation can be verified by a step-by-step procedure starting from (5.19) and using (5.18), or else applying directly equation (5.17).

where $\phi_r' = d\phi_r/d\xi$. This relation allows us to determine ϕ_r for $r > 2$ when ϕ_2 is known. Note that with $\phi_2 = 0$ one would get $\phi_r = 0$ for $r > 2$, which would take us back to the classical solution, $g(E) = 1$; hence necessarily $\phi_2 \neq 0$.

To obtain $\phi_2(\overline{E_T})$, we first express it as a power series in ξ

$$\phi_2(\xi) = \sum C_k \xi^k \tag{5.26}$$

and further observe from (5.23) with $r = 2$ that

$$\overline{E_T^2} = 2\overline{E}_T^2 + \mathcal{E}_0^2 \phi_2. \tag{5.27}$$

At high temperatures the value of $\overline{E_T}$ grows indefinitely, and the term $2\overline{E}_T^2$ in (5.27) becomes the dominant contribution to $\overline{E_T^2}$, as follows from (5.22); hence there can be no term in (5.26) with $k \geq 2$. On the other hand, since at zero temperature $\overline{E_T} = 0$, there can be no term with $k < 0$; nor can there be any constant contribution to ϕ_2, because it would give a nonzero value for $\overline{E_T^2}$ at $T = 0$, in violation of (5.21). Therefore the most general possible form of ϕ_2 is

$$\phi_2(\xi) = \eta\xi, \qquad \eta \neq 0, \tag{5.28}$$

where η ($\equiv C_1$) is an undetermined dimensionless constant, the only free parameter appearing in the solution. The recurrence relation (5.25) for $r > 2$ becomes now

$$\phi_{r+1} = (\xi^2 + \eta\xi)\phi_r' + \xi\phi_r + \eta r r! \xi^r, \tag{5.29}$$

which gives for $r = 3$, $\phi_3 = 6\eta\xi^2 + \eta^2\xi$; for $r = 4$, $\phi_4 = 36\eta\xi^3 + 14\eta^2\xi^2 + \eta^3\xi$, and so on. At low temperatures ($\xi \ll 1$), equation (5.29) gives

$$\phi_{r+1} \underset{\xi \to 0}{\longrightarrow} \eta\xi\phi_r' = \eta^2\xi\phi_{r-1}' = \ldots = \eta^r\xi; \tag{5.30}$$

hence $\overline{E_T^r} \longrightarrow 0$, in agreement with (5.21), confirming that the energy of the system $E = \mathcal{E}_0 + E_T$ acquires a fixed value as $T \longrightarrow 0$. In conclusion, for each frequency, as $T \longrightarrow 0$ the canonical ensemble of oscillators converges toward a microcanonical ensemble.

5.1.3. THE PLANCK DISTRIBUTION

We focus now on our main task, to construct $\overline{E}(\beta)$. By combining equations (5.27) and (5.28)

$$\overline{E_T^2} = 2\overline{E}_T^2 + \eta\mathcal{E}_0\overline{E_T} \tag{5.31}$$

and using (5.24), one obtains upon integration

$$\beta = \frac{1}{\eta\mathcal{E}_0}\ln\frac{\eta\mathcal{E}_0 + \overline{E_T}}{\overline{E_T}};$$

the integration constant has been chosen so that $\overline{E_T} \to \infty$ when $\beta \to 0$ to meet the classical limit. Inverting, one gets $\overline{E_T} = \eta\mathcal{E}_0\left(e^{\eta\mathcal{E}_0\beta} - 1\right)^{-1}$, which combined with (5.18) gives

$$\overline{E} = \mathcal{E}_0 + \frac{\eta\mathcal{E}_0}{e^{\eta\mathcal{E}_0\beta} - 1} = \left(1 - \frac{\eta}{2}\right)\mathcal{E}_0 + \frac{\eta}{2}\mathcal{E}_0\coth\frac{\eta}{2}\mathcal{E}_0\beta. \tag{5.32}$$

As expected, $\eta = 0$ leads to the classical Rayleigh law; but we have seen that in the presence of the zeropoint field one must set $\eta \neq 0$. With $\mathcal{E}_0 = \frac{1}{2}\hbar\omega$ for the zeropoint energy this equation gives Planck's law for the mean equilibrium energy of the field as a function of the frequency and the temperature, although written still in terms of the parameter η. There will be several occasions to single out the value $\eta = 2$ on the basis of other considerations, but for the moment we merely accept this value as fixed by experiment, and write therefore

$$\overline{E} = \mathcal{E}_0 + \frac{2\mathcal{E}_0}{e^{2\mathcal{E}_0\beta} - 1} = \mathcal{E}_0\coth\mathcal{E}_0\beta, \quad \mathcal{E}_0 = \frac{1}{2}\hbar\omega. \tag{5.33}$$

This result can be rewritten in terms of the spectral density with the help of equation (4.10), giving

$$\rho(\omega,\beta) = \frac{\hbar\omega^3}{2\pi^2c^3}\left(1 + \frac{2}{e^{\hbar\omega\beta} - 1}\right) = \frac{\hbar\omega^3}{2\pi^2c^3}\coth\frac{1}{2}\hbar\omega\beta.. \tag{5.34}$$

Observe that a Taylor series development of the exponential in (5.32) gives

$$\overline{E} = \frac{1}{\beta} + (1 - \frac{\eta}{2})\mathcal{E}_0 + \mathcal{O}(\beta) \qquad \beta \to 0,$$

so that $\eta = 2$ leads indeed to the correct classical limit for the Planck distribution.[6]

Equation (5.31) can now be cast in a form that played a historical role in the hands of Einstein, as we will have soon occasion to recall, namely,

$$\sigma^2_{E_T} = \overline{E_T^2} - \overline{E_T}^2 = \overline{E_T}^2 + 2\mathcal{E}_0\overline{E_T}. \tag{5.35}$$

---

[6] We recall that Einstein and Stern (1913) used a similar consideration regarding the correct classical limit, as an argument in favour of retaining the zeropoint term in the Planck distribution.

Further, by rewriting (5.23) with $r = 2$ in terms of $\overline{E}$

$$\overline{E^2} = 2\overline{E}^2 - \mathcal{E}_0^2, \tag{5.36}$$

one can see how $\overline{E^2}$ passes from its classical, high-temperature value $2\overline{E}^2$, to its dispersion-free value $\overline{E}^2$ near $T = 0$. Equations (5.11) and (5.36) give now

$$c_V = \frac{1}{k_B T^2}(\overline{E}^2 - \mathcal{E}_0^2) = k_B \left(\frac{2\mathcal{E}_0\beta}{e^{2\mathcal{E}_0\beta} - 1}\right)^2 e^{2\mathcal{E}_0\beta}, \tag{5.37}$$

$$c_V \underset{\beta\to\infty}{\longrightarrow} 4k_B\beta^2\mathcal{E}_0^2 e^{-2\mathcal{E}_0\beta}.$$

The exponential decrease of c_V as the temperature approaches zero, indicates that it becomes increasingly difficult to change the energy content of the system by a cooling process. In the derivation of Planck's formula we introduced the requirement that c_V and its derivatives remain finite as $T \to 0$, but we have obtained the much stronger result that the specific heat decreases exponentially, which is faster than needed to guarantee that the radiation field satisfies the third law of thermodynamics [see, e.g., Reif 1965; Boyer 1970a]. We briefly come back to this point in §5.3.4.

5.1.4. DISCRETE ENERGY SPECTRUM OF THE EQUILIBRIUM FIELD

The function $g(E)$ that enters into the metaclassical distribution can now be determined as follows. By using Planck's law (5.33) in (5.6) and integrating, one obtains for the partition function

$$Z = \frac{e^{-\beta\mathcal{E}_0}}{1 - e^{-2\mathcal{E}_0\beta}}. \tag{5.38}$$

A development of the denominator in powers of $e^{-2\mathcal{E}_0\beta}$ gives

$$Z = \sum_{n=0}^{\infty} e^{-(1+2n)\mathcal{E}_0\beta} = \sum_{n=0}^{\infty} e^{-E_n\beta} \tag{5.39}$$

with

$$E_n = (1 + 2n)\,\mathcal{E}_0. \tag{5.40}$$

A comparison of (5.39) with the integral form (5.5) shows now that

$$g\,(E) = \sum_{n=0}^{\infty} \delta\,(E - E_n)\,, \tag{5.41}$$

and equation (5.1) becomes finally

$$W\,(E) = \frac{1}{Z}\sum_{n=0}^{\infty} e^{-E_n\beta}\delta\,(E - E_n)\,. \tag{5.42}$$

Thus, despite the fact that the theory has been formulated solely in terms of continuous quantities, only the discrete energy values E_n given by (5.40) contribute to the average value of any $f(E)$, and they do so with a canonical weight p_n, so that[7]

$$\overline{f(E)} = \sum_{n=0}^{\infty} p_n f_n, \quad p_n = \frac{1}{Z} e^{-E_n \beta}, \quad f_n = f(E_n). \tag{5.43}$$

Observe that the frequency dependence of E_n is directly determined by the zeropoint energy. Indeed, from equation (5.33) it is clear that the dependence of the average energy of an oscillator on its frequency is determined by $\mathcal{E}_0(\omega)$, in sharp contrast to the classical equipartition law, according to which the average energy of an oscillator is independent of its frequency.

5.1.5. DISCUSSION OF THE RESULTS

The above results are in agreement with quantum theory, although they have been derived here with no explicit use of any discrete property or quantum rule. For instance, equation (5.43) represents the SED version of the quantum description of the system that is normally made in terms of the canonical density matrix $\hat{\rho} = Z^{-1} \exp(-\beta \hat{H})$, or in terms of the discrete eigenstates of the field Hamiltonian, $\hat{\rho} = \sum p_n |n\rangle \langle n|$. Thus the average energy is $\bar{E} = \mathrm{tr} \hat{\rho} \hat{H} = \sum p_n \langle n| \hat{H} |n\rangle$, which shows that $E_n = \langle n| \hat{H} |n\rangle$ is to be treated as the average (and thus, fixed) energy over the corresponding state.

To get a better feeling of how it becomes at all possible that properties of the equilibrium radiation field generally held to be key evidence of its quantum nature reappear as key evidence of the presence of the zeropoint field, let us combine equation (5.35) with the Einstein formula (5.9) to write

$$-\frac{d\bar{E}_T}{d\beta} = \sigma^2_{E_T} = \overline{E}^2_T + 2\mathcal{E}_0 \overline{E_T}. \tag{5.44}$$

As seen above, an integration of this equation leads to the Planck distribution; if the zeropoint energy term is removed, the integration of the remaining equation gives the Rayleigh law $\overline{E}_T = 1/\beta = k_B T$.[8] The zeropoint energy makes therefore all the difference between the classical and the quantum laws, the reason being that the thermal fluctuations of the field are increased by the additional interferences generated by the zeropoint field, as clearly shown by equation (5.44). This change is enough to transform the Rayleigh-Jeans distribution into a Planck distribution.

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[7] As is well known, it was Planck who discovered that a quantization of the energy exchanged between atom and field leads to the Planck distribution. That Planck's law implies quantization of the exchanged energy, was first advanced by Einstein (1905). Since then there have appeared several demonstrations similar to the one given above to show that the quantization rule (5.41) follows from Planck's law (5.32). Of particular interest for SED are the discussions in Santos (1975c), Theimer (1976) and Landsberg (1981).

[8] Similarly, if the quadratic (classical) term is neglected (which is allowable at very low temperatures), the remaining equation leads upon integration to the (approximate) Wien law $\bar{E} = C \exp(-2\mathcal{E}_0 \beta)$.

When Einstein (1905, 1909) discovered equation (5.44) he did not know about the zeropoint field, so he proposed to read it in the now traditional corpuscular language of quantum theory. In this language $\overline{E}_T$ is written in the form $\overline{E}_T = \hbar\omega\bar{n} = 2\mathcal{E}_0\bar{n}$, and $\bar{n}$ is interpreted as the mean number of photons of frequency ω in the thermal field; the Einstein formula acquires thus its simplest form

$$\sigma_n^2 = \bar{n}^2 + \bar{n}. \tag{5.45}$$

The first term, $\overline{E}_T^2$ (or $\bar{n}^2$) is the one predicted for a Gaussian field by Maxwell's theory, $\sigma_{E_T}^2 = \langle E_T \rangle^2$ (cf. equation (4.78)), as was explicitly demonstrated by Lorentz (1916, p.114) [Tomonaga 1962, p.296].[9] It is the term that survives when the zeropoint energy is set equal to zero, although this is not evident when the equation is written in the form (5.45). The novelty was in the linear term, and to explain its origin Einstein considered the field as composed of n independently moving pointlike quanta of energy $2\mathcal{E}_0$ $(= \hbar\omega)$ (which become photons in the modern parlance), corresponding to the average energy $2\mathcal{E}_0\bar{n}$. Indeed, if the number n of photons is assumed to follow a Poisson distribution, as corresponds to independent events, then $\left\langle (\Delta n)^2 \right\rangle = \bar{n}$ [see, e.g., Papoulis 1965, p. 145] and they contribute to the variance of the energy with an amount $\left\langle (2\mathcal{E}_0 \Delta n)^2 \right\rangle = 4\mathcal{E}_0^2\bar{n} = 2\mathcal{E}_0\overline{E}_T$, which is just the last term in equations (5.44) and (5.45). Hence, according to this quantum explanation the extra fluctuations in the energy of the thermal field are due to the fluctuations in the *number* of photons contained in the field. This contrasts with our previous account of the extra fluctuations as due to the *interference* of the thermal field with the zeropoint field.[10]

Certainly, at very low frequencies or high temperatures the 'classical' term in equation (5.44) dominates over the 'particle' contribution, whereas at high frequencies or low temperatures the opposite occurs. Nevertheless, in each case the two contributions coexist, as was clearly stated by Einstein (1909). Although this is quite evident, and has been recently confirmed by experiment, it is contrary to the prevailing notion of quantum complementarity or duality, according to which the undulatory and corpuscular aspects of the behaviour of matter and light mutually exclude one another.[11]

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[9] The contribution $\bar{n}^2$ to the fluctuations can be shown to correspond more generally to any chaotic source of light in the limit of infinite coherence time; a detailed discussion can be seen in Loudon (1973), section 6.7.

[10] That the 'particle' (linear) term in the Einstein fluctuations formula can be interpreted as due to the zeropoint fluctuations, as we do here, is explicitly accepted in the quantum literature by some authors. Particularly clear examples are Milonni (1980, 1994) and Milonni and Shih (1991). An interesting reinterpretation of expression (5.45) is offered by Mandel et al. (1964, 1965) within a classical theory of photodetection counts, in which the discrete and fluctuating number n represents the number of *absorptions* rather than photons.

5.1.6. THERMAL AND NONTHERMAL ENERGY FLUCTUATIONS

The fluctuations of the energy considered in the energy distribution (5.1) are of a *thermal* nature and thus vanish at $T = 0$, as was proved in §5.1.1 and 5.1.2. At zero temperature there remain only fluctuations of a *nonthermal* character due to the presence of the zeropoint field, the ones normally called vacuum or quantum fluctuations (a statistical analysis of them was done in section 4.3). We recall that in terms of the random amplitudes, the total energy associated with a single mode of this field is given by $(\alpha = \mathbf{k}\sigma, \omega_\alpha = \omega_k)$

$$H_\alpha = \frac{1}{2}\hbar\omega_\alpha \left|a_\alpha\right|^2 \tag{5.46}$$

with $< |a_\alpha|^2 >= 1$, so that the average value of the energy is $\mathcal{E}_{0\alpha} = \frac{1}{2}\hbar\omega_\alpha$. Whether the energy of the mode is a fluctuating quantity or else has a fixed value, depends on the statistical properties assigned to the a_α. In particular, in the Gaussian representation one has $\sigma^2_{E_\alpha} =< H^2_\alpha > -\mathcal{E}^2_{0\alpha} = \mathcal{E}^2_{0\alpha}$, whereas in the random-phase representation one has $\sigma^2_\alpha = 0$.

It should be recalled, however, that the free radiation field of a given frequency ω is constituted by an immense number of modes with different wave vectors $\mathbf{k}$ such that $k =| \mathbf{k} |= \omega/c$, whence, according to the conclusions of §4.3.3 (see equation (4.70)), the corresponding energy $H(\omega)$ is a nonfluctuating quantity with

$$\sigma^2_H = 0, \tag{5.47}$$

regardless of the statistical properties assigned to the a_α. From the point of view of quantum theory this is obvious, since in equilibrium at $T = 0$ only the ground state of the field is realized, and this is an eigenstate of the Hamiltonian. Also from the point of view of SED, this result is explainable: one may conceive of the zeropoint field contained in a small volume as undergoing energy fluctuations, but unless the principle of conservation of energy is drastically violated, it would be difficult to understand how the total energy of the zeropoint field of a certain frequency contained in a large volume of the size of the universe can fluctuate.

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[11] The experiments are reported in Mizobuchi and Ohtaké (1992) and were realized with light following a suggestion by Ghose et al (1991, 1992); for a discussion of them, see Ghose and Home (1992). The authors recall the fresh enthusiasm of the American physicist H.D. Huffman when he reports having rediscovered on his own the Einsteinian interpretation, in a manuscript of 1989 which apparently he did not get published.

However, as mentioned above (§5.1.2), the vanishing of the *total* of the energy fluctuations does not imply that all fluctuations disappear. For we should be aware that this refers to the energy contained in an infinite volume, and the integration over all space smooths out the local fluctuations. To see this we write schematically for the electric field of one mode, with $kx = \mathbf{k}\cdot\mathbf{x} - \omega t$ and $a_\alpha = r_\alpha e^{i\varphi_\alpha}$ (see equations (4.38) and (4.50)),

$$\mathbf{E}_\alpha = i\mathbf{E}_0\left(e^{ikx}a_\alpha - e^{-ikx}a_\alpha^*\right), \tag{5.48}$$

and observe that the energy density associated with this mode, namely,

$$U_\alpha = \frac{E_0^2 r_\alpha^2}{2\pi}[1 - \cos 2(kx + \varphi_\alpha)]$$

fluctuates with r_α and φ_α. Even if r_α is taken to be fixed (in the fixed-amplitude representation), there is still a randomness in the phase. This is of no relevance for a single mode; however, when all the modes of frequency ω are superposed, for each set of random phases a different pattern for the energy density is obtained. An average or integration over the whole of space wipes out these differences and leads to a fixed value for the energy; but before this average is taken, the energy $U(\omega)$ is still a fluctuating quantity. In fact, as was seen in §4.3.3, the electric and magnetic fields are distributed normally and $U(\omega)$ follows a Laplace distribution. Hence, for the local energy density, or for the energy content of the field in a small volume, the distribution is of the form

$$W(E) = \frac{1}{\mathcal{E}_0}e^{-E/\mathcal{E}_0}$$

and the moments are given by

$$\overline{E^r}(0) = r!\mathcal{E}_0^r. \tag{5.49}$$

In particular, $\sigma_E^2 = \mathcal{E}_0^2$.

On the other hand, since the thermal contribution to the radiation field is completely chaotic, it also has a Gaussian distribution [see, e.g., Bartlett 1966 or Papoulis 1965, section 8.6]. Hence at $T > 0$ the complete field distribution is Gaussian, and we may write

$$W(E) = \frac{1}{\overline{E}}e^{-E/\bar{E}}; \quad \overline{E} = \mathcal{E}_0 \coth \mathcal{E}_0\beta, \tag{5.50}$$

according to equation (5.33). A most natural assumption is that the zero-point and thermal parts are statistically uncorrelated, because they have their origins in independent sources [Boyer 1969d, Theimer 1971]; then their dispersions are additive, so that

$$\sigma_E^2 = \sigma_{E_T}^2 + \sigma_{E_0}^2 = \sigma_{E_T}^2 + \mathcal{E}_0^2. \tag{5.51}$$

Since on the other hand, from (5.49)

$$\sigma_E^2 = \overline{E^2} = \overline{E_T^2} + 2\mathcal{E}_0\overline{E_T} + \mathcal{E}_0^2,$$

it follows that $\sigma_{E_T}^2 = \overline{E_T^2} + 2\mathcal{E}_0\overline{E_T}$, which takes us back to equation (5.35). Note however, that here we are dealing with*statistical* averages instead of thermal canonical averages as in section 5.1. This last equation was used by Boyer (1969d) and Theimer (1971) in their derivations of Planck's formula for the blackbody radiation spectrum.

Fluctuations of the canonical coordinates
Equation (5.50) reexpressed in terms of the canonical coordinates of the field oscillators of frequency ω with Hamiltonian $H = \frac{1}{2}(p^2 + \omega^2 q^2)$, leads to the distribution

$$W_W(q,p) = Ne^{-\left(p^2+\omega^2q^2\right)\tanh(\hbar\omega\beta/2)/\hbar\omega}, \quad N = \frac{1}{\pi\hbar}\tanh\frac{\hbar\omega\beta}{2} = \frac{\omega}{2\pi\overline{E}}. \tag{5.52}$$

This is precisely the Wigner phase-space distribution for the quantum oscillators [see, e.g., Feynman 1972, Hillery et al. 1984, section 2]. The corresponding marginal distribution for the q coordinate is

$$W_q(q) = \left(\frac{\omega^2}{2\pi\overline{E}}\right)^{1/2} e^{-\omega^2q^2/2\overline{E}} = \left(\frac{\omega}{\pi\hbar}\tanh\frac{\hbar\omega}{2}\beta\right)^{1/2} e^{-q^2(\omega/\hbar)\tanh\hbar\omega\beta/2} \tag{5.53}$$

and similarly for the distribution of the momentum, with the substitution $\omega q \longrightarrow p$. The latter expression coincides with the distribution predicted by quantum statistical physics when one uses the Gibbs weights p_n given by (5.43) and the state densities $\varphi^*(q)\varphi(q)$ given by the Schrödinger theory [see, e.g., Landau and Lifshitz 1967, section 30]. The classical density $W_q \sim \exp(-\frac{1}{2}\beta\omega^2q^2)$ is of course recovered in the limit $T \longrightarrow \infty$.

Further, from (5.52) one has $\sigma_q^2\sigma_p^2 = \overline{E}^2/\omega^2$, which reads explicitly

$$\sigma_q^2\sigma_p^2 = \frac{\mathcal{E}_0^2}{\omega^2} + \frac{\overline{E_T}^2 + 2\mathcal{E}_0\overline{E_T}}{\omega^2} = \frac{\mathcal{E}_0^2}{\omega^2} + \frac{\sigma_{E_T}^2}{\omega^2}. \tag{5.54}$$

The term $\mathcal{E}_0^2/\omega^2 = \hbar^2/4 > 0$ represents the minimum value of $\sigma_q^2\sigma_p^2$, attained at $T = 0$. The remaining term represents an extrinsic temperature-dependent contribution that can have any value from zero to infinity. Hence the product of the dispersions satisfies the Heisenberg inequality

$$\sigma_q^2 \sigma_p^2 \geq \frac{\mathcal{E}_0^2}{\omega^2} = \frac{1}{4}\hbar^2, \tag{5.55}$$

which expresses the minimum possible fluctuations of the canonical coordinates associated with the radiation field at $T \geq 0$.

A different situation is obtained when the thermal chaotic field is added to a zeropoint field having a fixed energy as discussed at the beginning of this section (see equation (5.47)). Then the distribution of the purely thermal part of the energy, $E_T = E - \mathcal{E}_0$, is Laplacian, whence

$$W(E) = \frac{1}{\bar{E}_T} e^{-(E-\mathcal{E}_0)/\bar{E}_T}; \quad E \geq \mathcal{E}_0, \tag{5.56}$$

and only the thermal field contributes to the dispersion of the energy,

$$\sigma_E^2 = \sigma_{E_T}^2. \tag{5.57}$$

Equation (5.56) corresponds to the Glauber P distribution, which is obtained in quantum theory by using a normal ordering of the operators $\hat{a}^\dagger, \hat{a}$ and is appropriate for the description of processes involving absorptions.[12] By contrast, the Wigner distribution (5.50), which is used in quantum statistics to describe the thermal radiation field in equilibrium, is obtained by a symmetrical ordering of the photon creation and annihilation operators. Hence we see that quantum theory is not in a position to define unambigously the fluctuation properties of a radiation field. In fact, the description used depends on the kind of phenomena considered, the operator formalism appearing thus as an efficient mathematical tool to deal in a concise way with the various possibilities. We see that also in SED, the possibility must be left open to use a statistical description of the field that is appropriate to the specific problem or circumstances. Some of these points are discussed in Cetto and de la Peña (1989), and are a subject of much concern within stochastic optics, as we will have opportunity to see in chapter 13.

5.2. Planck's distribution and the momentum fluctuations

Planck's law was obtained above from an analysis of the energy fluctuations, but it can also be derived from a consideration of the momentum fluctuations, as was shown by Einstein and collaborators (1909, 1910, 1913). The method has the important bonus that it allows for a statistical study of the directional properties of the radiation interchanged with matter under equilibrium conditions, which makes it extremely valuable for further studies of the problem (we will take advantage of this possibility in §5.3.2).

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[12] See, e.g., Glauber (1964, 1968), Hillery et al. (1984). Some elements of the theory of the quantum distributions are summarized in §13.3.1.

Einstein and Hopf (1910a, 1910b) studied the motion of the molecules of a gas embedded in a radiation field in equilibrium, using a procedure devised by Einstein (1909) to study the fluctuations. The molecules are represented by massive particles to which a small vibrating dipole is attached to simulate their interaction with the field. Due to the Doppler effect, the interaction involves a whole frequency band. This is important, because in order to find the distribution of energy over the frequency it is necessary to consider an interaction involving more than one frequency. It is assumed that the molecules move along the x-axis and the dipoles vibrate along the z-axis, with frequency ω. If at time t the momentum of the *translational* motion of a molecule is p, then a short time δt afterwards it becomes $p+\Delta-Rp\delta t$, where Δ is the impulse transferred to the particle during δt due to absorptions and emissions, and Rp is the force of resistance to the motion due to the anisotropy of the field as seen by the moving molecule. The equilibrium condition is

$$\left\langle p^2\right\rangle = \left\langle (p+\Delta-Rp\delta t)^2\right\rangle, \tag{5.58}$$

the averages being taken over the equilibrium ensemble. The authors assumed that $<p\Delta>=0$, since Δ reverses its sign constantly, so that developing and neglecting the term $<(Rp\delta t)^2>$ which may be made arbitrarily small by selecting δt small enough, the equilibrium condition reduces to

$$\left\langle \Delta^2\right\rangle = 2R\left\langle p^2\right\rangle \delta t. \tag{5.59}$$

The *translational* motion is assumed to satisfy energy equipartition, so that $<p^2>/2m = k_BT/2$. This the authors considered a firmly established assumption, since by selecting a sufficiently massive particle the translational motion becomes classical. It constitutes the single statistical hypothesis of the theory, which thus bypasses the problems previously faced by Planck with the definition of entropy in his theory of the blackbody derivation. The equilibrium condition becomes

$$\frac{\left\langle \Delta^2\right\rangle}{\delta t} = 2mRk_BT. \tag{5.60}$$

The drag coefficient R is given by equation (4.26) and the impulse Δ on the dipole of natural frequency ω is determined with the help of (4.17); the calculation is straightforward and gives[13]

$$\left\langle \Delta^2 \right\rangle = \left\langle \left(\int_t^{t+\delta t} dte \frac{\partial E_z}{\partial x} z \right)^2 \right\rangle = \frac{4\tau\pi^4 c^4}{5\omega^2} \rho^2 \left(\omega, T\right) \delta t, \tag{5.61}$$

$$R = \frac{6}{5} \frac{\pi^2 c\tau}{m} \left(\rho - \frac{1}{3} \omega \frac{\partial \rho}{\partial \omega} \right), \tag{5.62}$$

where the average is taken over the ensemble of realizations of the field **E** with spectral density $\rho(\omega)$. By substituting in (5.59) one obtains the differential equation

$$\rho^2 = \frac{3\omega^2}{\pi^2 c^3} \left(\rho - \frac{1}{3} \omega \frac{\partial \rho}{\partial \omega} \right) k_B T, \tag{5.63}$$

whose solution satisfying Wien's law is the Rayleigh spectrum

$$\rho = \frac{\omega^2}{\pi^2 c^3} k_B T. \tag{5.64}$$

Results such as this motivated Einstein's assertion that classical arguments lead unequivocally to the Rayleigh law. From the point of view of SED, an obvious shortcoming of the theory is the absence of the zeropoint field; interestingly enough, a similar consideration was made some years later by Einstein and Stern (1913). In an attempt to take into account the extra zeropoint energy recently discovered by Planck, these authors added to equation (5.59) an extra term $<\Delta^2>_0$,[14] to obtain

$$\left\langle \Delta^2 \right\rangle + \left\langle \Delta^2 \right\rangle_0 = 2R \left\langle p^2 \right\rangle \delta t, \tag{5.65}$$

where $<\Delta^2>_0$ is to be calculated with the help of equation (5.61), but setting $\rho \longrightarrow \rho_0 = (\omega^2/\pi^2 c^3)(\frac{1}{2}\hbar\omega) \longrightarrow 2\rho_0$. Proceeding as before they arrived at the Planck distribution without the zeropoint term. The authors concluded that the consideration of the zeropoint energy seems to be sufficient to deduce Planck's law without the need of further hypotheses. Unfortunately, their derivation is unsatisfactory due to the above-mentioned shortcomings.

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[13] Recent versions of these calculations can be found in Boyer (1969b), Bergia et al. (1979, 1980), Jiménez et al. (1980), Marshall (1981), Milonni (1981, 1994), and Milonni and Shih (1991).

[14] Einstein and Stern considered a zeropoint energy due to molecular rotational motions and not to vibrations, and argued in detail about the experimental support for their hypothesis from a study of the specific heat of hydrogen. The argument is incorrect according to present quantum knowledge, which attributes no zeropoint energy to molecular rotations. Further, they had to write quite arbitrarily $\hbar\omega$ instead of one half this quantity to get the correct result. With the substitution $\rho_0 \longrightarrow 2\rho_0$ they compensated the absence of the zeropoint energy of the field oscillators, of which they were totally unaware. Detailed discussions can be found in Bergia et al. (1980) and Milonni (1994).

The Einstein-Hopf theory is consistent as a classical theory; indeed, with $<\Delta^2>$ referring exclusively to the thermal field and $\rho = \rho_T$, both sides of equation (5.59) reduce to zero for $T = 0$. However, when the zeropoint field is considered, we have $<\Delta^2>_0 > 0$ and $\rho = \rho_0$ at $T = 0$. But ρ_0 does not contribute a resistive force due to its Lorentz invariance ($R = 0$, see (5.62)), and the energy absorbed ceases to be balanced by the dissipative force, so that neither equation (5.60) nor (5.65) hold. It thus seems as if free particles are accelerated by the vacuum field, without there being any drag force to counteract this acceleration. Some time ago, in his noted paper Boyer (1969b) argued that the collisions of the molecules with the container walls are sufficient to restore equilibrium due to the radiation produced during the impacts, and showed that with the help of some additional collateral assumptions one is led to still another version of equation (5.59), namely

$$\left\langle \Delta^2 \right\rangle - \left\langle \Delta^2 \right\rangle_0 = 2R \left\langle p^2 \right\rangle \delta t. \tag{5.66}$$

This *Einstein-Stern-Boyer formula* is a correct expression, with $< \Delta^2 >_0$ given by (5.61) with $\rho_0(\omega)$ inserted instead of $\rho(\omega, T)$. Combining this with (5.61) and (5.62) and solving the ensuing equation, one gets the Planck distribution for $\rho(\omega, T)$, including the zeropoint term.

Boyer's proposal represented a big step toward the solution of the problem, by showing that there are additional effects which should be taken into account, and that this can be done without the need of an exhaustive and detailed analysis of the problem. However, several points remained obscure. Firstly, the theory implies that at $T = 0$ the particles moving through free space are accelerated by the zeropoint field. This prediction, sometimes referred to as the *Boyer effect*, has received attention from several authors, who saw in it a possible acceleration mechanism to explain the high-energy component of the cosmic-ray spectrum.[15] Further, the additional assumptions introduced by Boyer to arrive at equation (5.66) restrict considerably the range of applicability of the theory [details can be seen in Jiménez et al. 1980].

Let us now revise the Einstein-Stern argument, allowing for the zeropoint field but trying simultaneously to avoid the acceleration of free particles. Instead of introducing another source of fluctuations as proposed by Boyer and elsewhere,[16] we reconsider the problem from the beginning. We rewrite equation (5.58) without the term of second order in δt,

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[15] This and related problems have been studied in considerable detail mainly by A. Rueda, who has shown that in nonrelativistic QED the acceleration phenomenon occurs only in the time-symmetric version, but not in the usual form of the theory, expressed in terms of retarded potentials [Rueda 1986b]. Some additional pertinent references are Rueda and Cavalleri (1983), Rueda (1990c), Cavalleri and Spavieri (1986).

[16] Alternatives to the Boyer (1969d) formulation where additional terms are introduced, are proposed in Jiménez et al. (1980, 1983) and Marshall (1981). A more formal attempt is presented in Payen (1984).

$$\left\langle \Delta^2 \right\rangle + 2\left\langle p\Delta \right\rangle - 2R\left\langle p^2 \right\rangle \delta t - 2R\left\langle p\Delta \right\rangle \delta t = 0. \tag{5.67}$$

This equation is quite general, since the source or mechanism for the fluctuations Δ has not been specified. At $T = 0$ we assume that there is no drag force ($R_0 = 0$ because $\rho_0 \sim \omega^3$), so that it gives

$$\left\langle \Delta^2 \right\rangle_0 + 2\left\langle p\Delta \right\rangle_0 = 0. \tag{5.68}$$

Since $<\Delta^2>_0 \neq 0$ due to the presence of the zeropoint field, the momentum $p(t)$ attained by the molecule up to time t and the quantity Δ representing the fluctuation of p in the interval $(t, t+\delta t)$ must be correlated, contrary to the assumption $<p\Delta>= 0$ used by Einstein and collaborators. This means that the stochastic process $\delta p = p - \bar{p}$ is not Markovian, and that the fluctuations at time t are not totally independent of the past ones. Only in the absence of the zeropoint field should one take $<p\Delta>_0= 0$, as was done by Einstein and coworkers. Thus, we see that the system acquires a certain degree of memory in its interactions with the zeropoint field. Now the thermal component of the field is not expected to add a significant contribution to the correlation $<p\Delta>_0$, at least for not too high temperatures, and so one can safely make the approximation

$$\left\langle p\Delta \right\rangle \approx \left\langle p\Delta \right\rangle_0 . \tag{5.69}$$

By combining (5.67)-(5.69) one gets

$$\left\langle \Delta^2 \right\rangle - \left\langle \Delta^2 \right\rangle_0 = 2R\left\langle p^2 \right\rangle \delta t + R\left\langle \Delta^2 \right\rangle_0 \delta t,$$

and since the last term is of order $(\delta t)^2$, one is left with

$$\left\langle \Delta^2 \right\rangle - \left\langle \Delta^2 \right\rangle_0 = 2R\left\langle p^2 \right\rangle \delta t, \tag{5.70}$$

which is again the Einstein-Stern-Boyer formula (5.66). If it is assumed as before that the term Δ comes basically from the interaction with the field, by introducing (5.61) and (5.62) into (5.70) and integrating one gets once more the full Planck distribution. Since at $T = 0$ both sides of the equation (5.70) reduce to zero, the present formulation predicts no acceleration of a free particle from the vacuum, neither in the laboratory nor on a cosmic scale. Indeed, according to (5.68) the vacuum constantly impresses a momentum to the free particle, but it is statistically cancelled out by the correlation with the previous fluctuations. This is analogous to what happens in atomic systems, in which there is no 'spontaneous' increase of the energy (no spontaneous absorptions), due to the balancing effect of radiation reaction, as is discussed in chapter 11 (and is well known in QED).

Note that in the above calculations the field averages have been interpreted as statistical averages over a Gaussian distribution, whereas $<p^2>\sim k_BT$ is obviously a thermal average. It would be more consistent to take all averages in the thermodynamic sense; in this case the quantities $<\Delta^2>$ and $<\Delta^2>_0$ differ with respect to Boyer's calculations, for the following reason. According to equation (5.61), Δ^2 contains products of four field amplitudes (two from the derivatives of the field and two from the product z_iz_j, with z_i linear in the field component for the linear oscillator), and hence gives a result that is proportional to $\overline{E^2}$. Since (5.36) gives $\overline{E^2} = 2\overline{E}^2 - \mathcal{E}_0^2$, and $\rho(\omega) \propto \overline{E}(\omega)$ for each frequency, it is straightforward to see that instead of equation (5.61) one gets

$$\left\langle\Delta^2\right\rangle = \frac{2\tau\pi^4c^4}{5\omega^2}\left[2\rho^2\left(\omega,T\right) - \rho_0^2(\omega)\right]\delta t. \tag{5.71}$$

The required difference of terms becomes

$$\left\langle\Delta^2\right\rangle - \left\langle\Delta^2\right\rangle_0 = \frac{4\tau\pi^4c^4}{5\omega^2}\left[\rho^2\left(\omega,T\right) - \rho_0^2(\omega)\right]\delta t, \tag{5.72}$$

or

$$\left\langle\Delta^2\right\rangle - \left\langle\Delta^2\right\rangle_0 = \frac{4\tau\pi^4c^4}{5\omega^2}\left[\rho_T^2\left(\omega,T\right) + 2\rho_0\left(\omega\right)\rho_T\left(\omega,T\right)\right]\delta t, \tag{5.73}$$

which is exactly what would be obtained from equation (5.61) and leads once more to the Einstein-Stern-Boyer equation and the Planck distribution law. Thanks to this coincidence, the two theories become equivalent for the determination of the equilibrium distribution. Note that the right-hand side is proportional to the thermal energy fluctuations (cf. (5.35)); thus the 'particle' term $2\rho_0\rho_T$ is identified again as being responsible for the departure from the classical results.

5.3. Quantum effects of radiation

Planck's distribution formula was the definitive assault on the equipartition law for the oscillators. No place where classical physics uses this law or leads to it remained untouched after Planck's discovery. Since, as we have seen, Planck's law can be explained as a consequence of the action of the zeropoint field, it becomes clear that this field is able to deeply affect the behaviour of systems that have electromagnetic interaction, among them and most importantly, atoms.

As already discussed, two different views on the above results have been given. According to the SED view, the 'particle' term in the formula for the fluctuations merely accounts for the increased fluctuations of the thermal field resulting from its interference with the zeropoint field. However, despite that in this theory the field is described as continuous and expressed in a characteristically stochastic language, as soon as one makes a thermal decomposition of the equilibrium distribution in terms of Boltzmann factors, a discontinuous energy spectrum arises, with the consequent discrete processes of absorption and emission. Thus the notion of discrete energy eigenstates appears as tightly linked to the canonical analysis of the equilibrium field. In any case, until this point the analysis applies only to the radiation field in thermal equilibrium with matter, and it would be unjustified to extrapolate it without further evidence to other systems, in particular to the free field or to a radiation beam which is not in thermal equilibrium with matter. One might, for instance, conceive of producing a radiation beam with a spectral distribution given exactly by Planck's law for a certain numerical value of β; but if this field does not represent a system in thermal equilibrium with matter, it is not liable to a description in terms of a canonical distribution and hence the decomposition (5.39) of the partition function, leading to the quantization rule (5.41), does not apply. Analogously, it is possible to artificially construct stationary chaotic fields that are not Planckian, but then the atomic populations would not be given by the Maxwell-Boltzmann law [see, e.g., Loudon 1973, section 7.2].

In quantum theory one goes much further, by interpreting the quantity $\hbar\omega$ appearing in Planck's law not just as the energy exchanged between the atom and the radiation field in each elementary interaction (as originally proposed by Planck and Einstein), but as a discrete entity which has an existence in itself, even in the absence of matter. Now, in what refers to the energy and momentum exchange between atoms and field, it happens that the line of reasoning used by Einstein in his 1917 paper[17] —which in the quantum language is formulated in terms of directed photons— can equally well be applied within SED, as we will see below. This suggests an alternative path to other quantum results, which makes no appeal to quantum rules but finds in the zeropoint field the explanation of the nonclassical behaviour of the system. In what follows we present such an adaptation of the Einstein model to the SED situation.

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[17] In his famous paper on the A–B coefficients, Einstein (1917) demonstrated by means of a statistical study of the atomic recoils that for each quantum of radiation $\hbar\omega$ emitted or absorbed in an atomic transition, a linear momentum $\hbar\omega/c$ in some well defined direction is exchanged. This considerably reinforced the Einstein notion of light quanta as radiation needles.

In earlier works, Einstein (1909) had referred to the quanta of radiation 'as if radiation is made of independently moving *pointlike* quanta', or '...(as if) radiation is made of independently moving small complexes with energy $\hbar\omega$'. His latter *radiation needles* [Einstein 1917] seem to be somewhat closer, though still not equivalent, to the notion used in modern quantum theory.

5.3.1. ENERGY EXCHANGE, BOHR'S FORMULA AND EINSTEIN'S A AND B COEFFICIENTS

Our first aim is to prove that Bohr's rule $E_n - E_m = \hbar\omega$ is an immediate outcome of the results presented up to now, and to establish its meaning from the point of view of SED. This important formula will be derived in chapter 10 as part of a more complete treatment of the atomic system, so here we present just a heuristic derivation, with the sole purpose of reinforcing the parallelism between the two descriptions. We start by briefly recalling Einstein's arguments adapted to our present needs [Jiménez et al. 1980; Cetto and de la Peña 1989]; a more detailed exposition of the original argument can be seen in Haken (1981) §2.5, or Milonni (1994). Einstein's analysis constitutes a more detailed, even if statistical, study of the process by which atoms maintain their equilibrium with the radiation field at a given temperature, while constantly interchanging energy and momentum with it. It goes much farther than a simple thermodynamical description, without however entering into the detailed dynamics, which was out of the question at the time when the paper was written, ten years before the advent of modern quantum theory.

In Einstein's original analysis it is assumed that an atom (or molecule) has discrete energy levels and that it undergoes transitions between pairs of them, with energies E_1 and E_2; this is his single explicit quantum assumption. However, in a barely cited later paper [Einstein and Ehrenfest 1923] it is shown that the demand of discrete energy levels is dispensable, since the levels may be immersed in an energy continuum without affecting the results. This point was important for the authors to allow for translational motions of the atoms, which add nonquantized contributions to the initial and final energies. They concluded that the analysis still holds when applied to the total energies. Thus, just for the sake of simplicity, in what follows we shall treat the levels as discrete, without however committing ourselves to this property; in other words, we consider only two energies E_1 and E_2 out of a continuum. Further, for simplicity one may ignore the possibility of degeneracies, without affecting the results of interest here.

Let N_1 and N_2 be the number of atoms with energies E_1 and E_2, respectively, and $E_2 > E_1$, say. Consider first the (stimulated) absorptions occurring in a small time interval dt that take the atom from state 1 to state 2. The rate of change of N_1 is assumed to be proportional to the population N_1 and to the spectral density of the thermal part of the field at the frequency of interest, so that

$$dN_1 = N_1 B_{12} \rho_T(\omega) dt. \tag{5.74}$$

The coefficient B_{12} characterizes the atomic transition and does not depend on the temperature, whereas of course the population N_1 depends on the temperature. That $\rho_T(\omega) = \rho(\omega) - \rho_0(\omega)$ should appear here rather than $\rho(\omega)$ follows by noting that in the presence of the pure zeropoint field no excited states are realized under equilibrium conditions. For the emissions the whole field contributes, however, and on top of that there is an additional contribution due to radiation reaction, the need of which is made apparent by noticing that if the whole radiation field were somehow reduced to zero, an excited state would still radiate according to Maxwell's laws. Thus we write

$$dN_2 = N_2 B_{21} [\rho_T(\omega) + \rho_0(\omega)] dt + N_2 C_{21} dt. \tag{5.75}$$

with the coefficients B_{21} and C_{21} independent of the temperature, since they characterize the atomic transitions. The condition for equilibrium is that the number of emissions of any type occurring during dt equals the number of absorptions during the same time, and it reads

$$B_{21} [\rho_T(\omega) + \rho_0(\omega)] + C_{21} = \frac{N_1}{N_2} B_{12} \rho_T(\omega). \tag{5.76}$$

In thermodynamic equilibrium the atomic populations obey Maxwell-Boltzmann statistics, so that $N_1/N_2 = \exp(E_2 - E_1)\beta$. In the high-temperature limit $N_1/N_2 = 1$ and ρ_T grows indefinitely, so that the last equation yields $B_{12} = B_{21}$; hence it can be recast in the form

$$\rho(\omega) = \rho_0(\omega) \left[1 + \frac{1 + \varsigma}{e^{(E_2 - E_1)\beta} - 1} \right], \quad \varsigma = \frac{C_{21}}{B_{21} \rho_0}. \tag{5.77}$$

In this expression ς is a dimensionless number; moreover, since nothing in this equation can depend on particular features of the system, its value must be a universal constant. A comparison with Planck's law (5.34) gives now the Bohr frequency condition

$$E_2 - E_1 = \hbar\omega \tag{5.78}$$

and $\varsigma = 1$, so that

$$C_{21} = B_{21} \rho_0(\omega). \tag{5.79}$$

Bohr's formula is a most important quantum law; it shows that the frequency of the radiation absorbed or emitted during atomic transitions is not directly related to any of the orbital frequencies of the motions as is the case in classical physics, but to the difference of the energies of the atomic states involved. As with other important results here derived, we will have occasion to arrive at it later on from other considerations. Introducing (5.79) in (5.75) one gets

$$\dot{N}_2 = N_2 B_{21}[\rho_T(\omega) + 2\rho_0(\omega)] \equiv N_2[B_{21}\rho_T(\omega) + A_{21}], \tag{5.80}$$

where the Einstein A probability for spontaneous emission is given by

$$A_{21} = 2\rho_0(\omega)B_{21} = \frac{\hbar\omega^3}{\pi^2 c^3} B_{21}. \tag{5.81}$$

This is the second well-known relation between Einstein's coefficients, the other one being the equality of the probability of stimulated absorptions and emissions, $B_{12} = B_{21}$. It must be stressed that the coefficient A contains equal contributions from the zeropoint field fluctuations and radiation damping, as is clearly seen by combining equations (5.76) and (5.79), or from (5.80), so that its name is somewhat misleading. Equation (5.80) explicitly describes the effects of both the *stimulated* and the *spontaneous emissions*, and equation (5.79) shows that the contributions of radiation reaction (called above Maxwell radiation) and of the zeropoint fluctuations to the spontaneous emissions are equal, which is a well established result in QED.[18] This explains the factor $2\rho_0(\omega)$ in (5.81); it says that, statistically speaking, half the spontaneous emissions are indeed stimulated by the zeropoint field, as was assumed in the early proposal by Park and Epstein (1949), but the other half is to be attributed to normal Maxwellian (or Larmor) radiation.

5.3.2. LINEAR MOMENTUM CONSERVATION AND THE DIRECTED SPONTANEOUS RADIATION

The second part of Einstein's 1917 paper is devoted to a statistical study of the exchange of momentum between atoms and the radiation field in equilibrium. In this part, which Einstein himself considered the most important one of his paper, the calculations prove that atomic radiation is directed, since along with an energy $\hbar\omega$ it carries a linear momentum $\hbar\omega/c$ in the direction of propagation (see equations (3.49) and (3.50)). From a classical point of view, spontaneous emissions cannot produce atomic recoil because the emitted radiation has inversion symmetry with respect to the atomic nucleus. But according to the notion of quanta of radiation, the whole of the emission, including the one associated with spontaneous processes, takes place in a given direction, so that the atom always recoils.[19]

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[18] See, e.g., Milonni 1994. In §§7.4.2 and 11.3.1 these matters are studied in more detail.

[19] Atomic recoil associated with spontaneous emission was reported as apparently observed for the first time by Frisch (1933). More recent observations are discussed in Picque

The classical result can be attained only in the average, since there is no *preferred* direction for the spontaneous emissions. Another difference with the classical prediction is that the fluctuations in the momentum transfer are also different from zero. Since the study showed that the Planck spectrum is consistent with the quantum viewpoint, Einstein found in it a strong argument in support of the notion of the directed quantum of radiation. The strength of the demonstration is that it involves only very general arguments about the statistics of atomic transitions and well-verified and simple effects such as the Doppler shift and the angular aberration of light. The calculations that follow are similar to those of Einstein, but with the required adaptations to take the zeropoint field into account.

Let us apply the balance of momentum described by equation (5.66), to the atom studied above. We need to calculate the momentum fluctuations and the force on the atom due to all transitions occurring during the small time interval dt, under the assumption that during each one —each *elementary interaction*, in Einstein's language— a momentum $\hat{\mathbf{n}}_i\pi = \hat{\mathbf{n}}_i\hbar\omega/c$ in some direction $\hat{\mathbf{n}}_i$ is transferred. First we calculate the momentum fluctuations. In each elementary interaction the atom receives or gives away a momentum $\pi\cos\theta_i$ in the x-direction, so that if n of them occur in a time interval dt, the net momentum transferred to the atom is $\sum_{i=1}^{n} \pi\cos\theta_i$. Considering these components as statistically independent random variables with zero mean, one gets

$$\left\langle\Delta^2\right\rangle - \left\langle\Delta^2\right\rangle_0 = \sum_{i=1}^{n}\left\langle(\pi\cos\theta_i)^2\right\rangle = n\left(\frac{\hbar\omega}{c}\right)^2\left\langle\cos\theta^2\right\rangle = \frac{n}{3}\left(\frac{\hbar\omega}{c}\right)^2. \tag{5.82}$$

The average number n of elementary processes occurring during dt is ($B = B_{12} = B_{21}$)

$$n = (N_1 B\rho_T + N_2 B\rho + N_2 B\rho_0)\, dt = 2N_1 B\rho_T dt, \tag{5.83}$$

where the equilibrium condition (5.76) has been used. Thus,

$$\left\langle\Delta^2\right\rangle - \left\langle\Delta^2\right\rangle_0 = \frac{2}{3}\left(\frac{\hbar\omega}{c}\right)^2 N_1 B\rho_T dt. \tag{5.84}$$

For the calculation of the force exerted on the atom during the transitions we use some results of chapter 4. The average momentum in the x-direction given to the particle during dt is

$$dp_x = (dN_1' - dN_2')\frac{\hbar\omega'}{c}\cos\theta',$$

and Vialle (1972) and Schieder et al. (1972). Alternative schemes have been proposed to account for the beaming of the radiation; an example can be seen in Beers (1973).

where the primes denote variables as seen from the moving frame. Thus the net average force on the atom is (cf. equation (4.25))

$$F=\left\langle\frac{dp_x}{dt}\right\rangle=\frac{1}{4\pi}\frac{\hbar\omega}{c}\int d\Omega'\cos\theta'\left(-2N_2B\rho_0\right)+$$

$$\frac{1}{4\pi}\frac{\hbar\omega}{c}\int d\Omega'\cos\theta'\left(N_1-N_2\right)B\left(1-3\frac{v}{c}\cos\theta'\right)\left(\rho_T+\frac{v}{c}\omega\frac{\partial\rho_T}{\partial\omega}\cos\theta'\right). \tag{5.85}$$

In writing this expression to first order in v/c we have dropped all the primes in the frequencies and have taken into account the Lorentz invariance of the zeropoint field, so that only ρ_T is affected by the Doppler shift and the angular aberration. Due to this invariance the integral proportional to ρ_0 vanishes and there remains, after performing the angular integration and to first order in the velocity,

$$F=-\frac{\hbar\omega}{c^2}\left(N_1-N_2\right)B_{12}\left(\rho_T-\frac{1}{3}\omega\frac{\partial\rho_T}{\partial\omega}\right)v\equiv-Rv. \tag{5.86}$$

Upon substitution of these results, the equilibrium condition (5.66) yields the equation

$$\rho_T-\frac{1}{3}\omega\frac{\partial\rho_T}{\partial\omega}=\frac{\hbar\omega}{3k_BT}\frac{N_1}{N_1-N_2}\rho_T, \tag{5.87}$$

which is satisfied indeed by the Planck distribution. From the point of view of SED this means that in each exchange of energy $\hbar\omega$ with the external field there is a transfer of directed momentum of magnitude $\hbar\omega/c$. Thus we conclude that in what refers to the Planck distribution, the quantum and the SED descriptions are equivalent. Further, the procedure discloses the fundamental role played by the Lorentz invariance of the zeropoint field in the momentum balance, by ensuring the vanishing of the first integral in (5.85), which is essential to recover Planck's law [Barranco and França 1992].

By re-writing the last equation with the help of (5.76) one obtains

$$\rho-\frac{1}{3}\omega\frac{\partial\rho}{\partial\omega}=\frac{\pi^2c^3}{3\omega^2k_BT}\left(\rho_T^2+\frac{A}{B}\rho_T\right). \tag{5.88}$$

Because of the relation $A=2\rho_0B$ the 'particle' term can be (and has been) interpreted as due to the spontaneous emissions although, as stressed above, it contains equal contributions from the zeropoint field fluctuations and radiation damping. It is now clear that the concept of spontaneous emission (properly reinterpreted) finds a full physical sense within SED, and that its suppression would be equivalent to the barring of the zeropoint field and return to equipartition. This shows that the true 'quantum postulate' in

Einstein's A–B derivation of the Planck law is not the assumption of transitions between states with quantized energies (which can be circumvented as explained above), but the introduction of the effects of the vacuum through the 'spontaneous emissions' term.

5.3.3. THE COMPTON EFFECT AND OTHER ZEROPOINT EFFECTS

Einstein and Ehrenfest (1923) generalized the previous results to the case in which several absorptions and emissions can take place within the time interval δt. If each event is considered as statistically independent of the others, the probability of their occurrence is the product of the probabilities of each one of them, or

$$dW_{1,2} = \prod_i^N \left[db_i \rho_T(\omega_i)\right] \prod_j^{N'} \left[da'_j + db'_j \rho_T(\omega'_j)\right] \delta t. \tag{5.89}$$

The first product describes N absorption processes and the second one N' emissions. In the particular case of dispersion of light by electrons, such as the Compton effect, there is only one absorption and one emission and the above expression reduces to

$$dW = db\rho_T(\omega)(da + db'\rho_T(\omega'))\delta t. \tag{5.90}$$

The coefficients da, db contain the corresponding A and B probabilities, as well as the volume elements of the solid angle of the incident and dispersed radiation, and the frequency intervals involved. Pauli (1923) had just shown the strange–looking expression (5.90) to be indispensable to guarantee consistency with Planck's law, and the explanation of its origin was one of the main tasks in Einstein and Ehrenfest's endeavour. In modern quantum theory this law is expressed in terms of the usual elements of dispersion theory applied to the case at hand.

The important point here is that the above equations hold under the conditions of energy and linear momentum conservation, which are expressed as

$$E_{\text{in}} + \sum_i^N \hbar\omega_i = E_{\text{fin}} + \sum_j^{N'} \hbar\omega'_j, \tag{5.91}$$

$$\mathbf{p}_{\text{in}} + \sum_i^N \frac{\hbar\omega_i}{c}\hat{\mathbf{k}}_i = \mathbf{p}_{\text{fin}} + \sum_j^{N'} \frac{\hbar\omega'_j}{c}\hat{\mathbf{k}}'_j. \tag{5.92}$$

The particular case of these equations with $N = N' = 1$, $\mathbf{p}_{\text{in}} = 0$ and $E^2_{\text{fin}} = m^2c^4 + c^2\mathbf{p}^2_{\text{fin}}$ was used by Compton (1923) and Debye (1923) to explain the reddening of X- and γ rays when scattered by atomic electrons and is now known as the Compton effect. This kinematics leads, as is well known, to Compton's formula for the frequency shift of the dispersed radiation, confirming the idea that both quanta of radiation involved have definite directions [Compton 1923]. This proposal, due to Barranco and França (1992), thus gives a satisfactory account of the Compton effect; once more the explanation of the nonclassical behavior of the system is directly linked to the zeropoint field. To show this fact we note that in the above equations the quantities $\hbar\omega$ come from the factor $2\mathcal{E}_0$ in the exponent in Planck's law (5.33), so that it is legitimate to write them as

$$E_{\text{in}} + \sum_i 2\mathcal{E}_0(\omega_i) = E_{\text{fin}} + \sum_j 2\mathcal{E}_0(\omega_j'), \tag{5.93}$$

and similarly for the momentum equation. To the list of those phenomena that can be explained as a direct consequence of the interaction with the zeropoint field, we may add besides the Planck law and the Compton effect, the photoelectric effect, whose explanation is traditionally associated with the 'particle' term in the Einstein fluctuations formula.[20]

As already remarked, this does not imply a reduction of these phenomena to classical physics, since the quantum behaviour of the field is there. What changes is the picture of the phenomena and with it, the language that conceptualizes it. Moreover, the previous results do not demonstrate that every quantum property of the field is correctly reproduced by the present theory, and further inquiry into these complex questions is obviously required. Some progress has been made within stochastic optics, a theory developed as an attempt to understand how far it is possible to advance into the optical domain with the stochastic description, which we will have occassion to study in chapter 13.

In this context, it is important to stress that many of the phenomena usually associated with the photon nature of light in conventional QED do not really require photons for their explanation. An immediate example is provided by the stimulated absorptions and emissions, whose probabilities can be calculated within quantum mechanics using a classical electromagnetic field. What really counts, to succeed with such calculations, is the

[20] Theimer (1971) has argued that in SED one should expect the photoelectrons to be released immediately after the incident light beam hits the photocathode, since as a result of the fluctuations produced by the zeropoint field, some electrons may by accident be in the conduction band at the time of illumination and can therefore be more easily released. Thus the incident beam does not by itself excite electrons to the conduction band, which would take some time, but merely affects the statistics of the fluctuations, which is an instantaneous effect, giving opportunity to the electrons that happen to be in the conduction band, to remain definitely there. This instantaneous response of photoelectrons is frequently considered as one of the proofs of the particle-like nature of photons. A similar argument had been given by Lamb and Scully (1969) within a semiclassical approach.

quantization of matter, as contained in the Schrödinger equation. Two further examples that can correctly be discussed without the need of resorting to a second quantized theory, namely, the photoelectric and Compton effects, are nevertheless frequently taken as paradigmatic for QED.[21] Thus, what the above results seem to say is that the zeropoint field is behind (and should explain) *matter* quantization.

5.3.4. FURTHER STATISTICAL AND THERMODYNAMIC EFFECTS

We have seen that the inclusion of the zeropoint field modifies substantially some thermodynamic expressions. A related observation is already present in Planck's first theory of the black body, where he proposed to modify the expression for the entropy of the cavity in equilibrium and used the new relation to derive his law. This is not surprising in the light of Einstein's fluctuations formula, since wherever the thermal fluctuations of the field appear in the form $\bar{E}^2$, they must be replaced by $\bar{E}^2 + 2\mathcal{E}_0 \bar{E}$; moreover, there are statistical fluctuations due to the randomness of the field, even at zero temperature. Observations of this kind led Boyer (1969d) to consider the need to distinguish between a *caloric* entropy and a *statistical* entropy, and thus to make it possible to discuss the differences, similar to those that exist between the thermodynamics of classical and quantum systems.

In his investigations, Boyer (1970a) arrived at a new understanding of the third law of thermodynamics as a result of the interaction of the system with the zeropoint field, which explains why this law does not pertain to classical physics; hence his central conclusion that the third law is obeyed by systems that have electromagnetic interactions.[22] The crux of his argument can be perceived by recalling that in §5.1.1 initially the specific heat and its derivatives were simply required to remain finite at any temperature, and in the end c_V turned out to go to zero as $T \to 0$. This result fits smoothly with the fact that $c_V \to_{T\to 0} 0$ is required to guarantee the validity of the third law of thermodynamics in the Nernst-Simon sense, which states that the entropy change associated with any isothermal reversible process in a condensed system should go to zero as $T \to 0$. It is again in the additional fluctuations —the ubiquitous 'particle' term— where one should look for the source of this other aspect of quantum behaviour. Once more, the terms

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[21] A detailed discussion can be seen, e.g., in Sakurai (1967), pp. 210-240. An entirely different problem are the spontaneous emissions, as they are excluded from a quantum mechanical description that predicts that all excited states are stationary.

[22] Boyer (1969d) also pointed out that the indistinguishability of identical particles, usually considered the essence of Bose-Einstein statistics, should be properly considered a classical concept, since its use in the classical context avoids the Gibbs paradox. [An introductory explanation of the Gibbs paradox can be seen in Reif 1965, pp. 243-246.] There are of course various other known proposals to solve the Gibbs paradox; a fine and illuminating discussion can be seen in Yourgrau et al. (1982).

'quantum' and 'due to the zeropoint field' become equivalent.

The thermodynamic investigations of Boyer have been continued in recent years by Cole, who extended them to the study of fundamental laws as those of Wien, Stefan, Boltzmann, etc. in the presence of the zeropoint field [see, e.g., Cole 1990b, c, 1993a]. As an example of the kind of extensions that can be made to enrich the present discussion as well as that of chapters 4 (particularly §4.2.1) and 6, let us sketch the derivation given in Cole (1992a) of the zeropoint spectrum from thermodynamic considerations, which can be seen as a modern version of arguments that go back to Nernst. Consider two parallel perfectly conducting plates placed a distance L apart, in equilibrium at temperature T. As is explained in chapter 6, only those modes of the zeropoint field that meet the boundary conditions on the metallic plates remain confined within the plates, and the consequent exclusion of modes brings about a change of the total energy content of the confined field, which becomes thus a function of the plate separation. Now, with the system being held at constant temperature, assume a quasistatic displacement δL of one of the plates. Since the new configuration confines a different set of modes, the energy of the system changes, so that a certain quantity of heat $\langle \delta Q\rangle_T$ flows into it (or out of it). Using the methods of chapter 6 and the first law of thermodynamics, it is not difficult to show that

$$\begin{aligned}\langle \delta Q\rangle_T &= A\sum_n \frac{\partial\omega_n}{\partial L}\left[\frac{\partial}{\partial\omega_n}\frac{\rho(\omega_n)}{\omega_n^2} - \frac{\rho(\omega_n)}{\omega_n^3}\right]\delta L \\ &= -3A\sum_n \frac{1}{\omega_n^3}\frac{\partial\omega_n}{\partial L}\left[\rho(\omega_n) - \frac{1}{3}\omega_n\frac{\partial\rho}{\partial\omega_n}\right]\delta L. \end{aligned} \tag{5.94}$$

The summation extends over all allowed modes of the initial configuration, globally denoted by the index n. The coefficient A is a geometrical factor that is of no particular relevance here; its value can be found in Cole (1992a). Now we define $T = 0$ as the temperature at which no heat can flow into or out of the system during a reversible isothermal process performed on it. Then one should have $< Q >_0 = 0$ for any value of L and δL, which means that the sum above must give zero. However, the equilibrium condition should apply separately for each frequency range, because we can suppress any desired narrow band by altering the distance between the plates. Thus each term in the sum must vanish separately; this gives a differential equation for $\rho(\omega)$ with the solution $\rho/\omega^2 = \lambda\omega$, where λ is an arbitrary constant, and one obtains finally $\rho(\omega) = \lambda\omega^3$, in agreement with the results of chapter 4. We verify that indeed the form of the spectrum at $T = 0$ can be determined by purely thermodynamic arguments applied to the complete electromagnetic field, with its zeropoint component included.

CHAPTER 6

ENVIRONMENTAL EFFECTS THROUGH THE ZEROPOINT FIELD

Most of this chapter is devoted to the study of retarded long-range forces and other similar effects on matter which arise as a direct consequence of the existence of the fluctuating electromagnetic zeropoint field. As will become clear, the properties of the vacuum that are used in the derivation of these effects can easily be framed in a classical language, and there is in principle no need to resort to the quantum formalism. In addition to providing at all times during the calculations a physically transparent image, the SED approach has the virtue of confirming that at least for the family of problems considered here, what is essential is the existence of the electromagnetic vacuum; not the quantization of it. Thus, phenomena that have been regarded as some of the "least intuitive consequences of QED" [Schwinger et al. 1978] appear here as some of the most intuitive consequences of SED.

There is at present a very vast and growing literature on Casimir forces and related phenomena; it is not our intention here to give a complete account of the subject, but rather to present and discuss the SED contribution to it, which is substantial indeed. Our analysis will have to do with electromagnetic interactions between microscopic or macroscopic neutral bodies, mediated by the background radiation field. This means that the static components of the electric and magnetic fields can be ignored and attention can be focussed on the radiation field alone; more specifically, on its zeropoint component, since, as will become evident, the inclusion of the thermal component of the radiation field normally does not present a major conceptual complication, though the numerical evaluation of its effects may be more cumbersome.

A very simple, schematic way of expressing the idea behind the calculations is the following. The field has an average energy of $\frac{1}{2}\hbar\omega$ per normal mode and is homogeneous and isotropic in free space. But the presence of material bodies affects the field; thus, the modes in a given region of space depend upon the geometry and on the electric and magnetic properties of the objects present, as required by the laws of electrodynamics. Therefore, changing for instance the position or orientation of objects changes the pattern of the underlying field, and hence also its overall energy content in that region of space. This change of energy is the potential function for the long-range forces on the objects.

There are of course other conceivable ways to affect the structure of the field, such as by changing the geometry of space by going over from an inertial to an accelerated frame of reference. Some of the phenomena which can arise as a result of the acceleration (or of gravitation, as follows from the principle of equivalence) of a body immersed in the background field are discussed in the last section of this chapter.

6.1. The Casimir effect

In the 1940's, when working on the quantum theory of the van der Waals forces between neutral molecules in an attempt to understand the experimental results on the stability of colloids, Casimir and Polder (1948) obtained a term for which the interaction energy varies as R^{-7} for large intermolecular separation R,

$$E_C(R) = -\frac{23\hbar c\alpha^2}{4\pi R^7}. \tag{6.1}$$

This is the so-called retarded term, that depends on the polarizability α of the molecules (see section 6.2). Intrigued by the simplicity of the result, Casimir mentioned it to Bohr, who in his response pointed out a possible relation with zeropoint energy [Milonni and Shih 1992]. This not only put Casimir on a new track, as he himself recalls; it also put zeropoint energy on a new track. Not so new, strictly speaking, but rather unexplored, since, as was mentioned in section 4.1, the hypothesis had been long before advanced by Nernst (1916, appendix) that the interatomic interactions arise as a result of the distortions of the zeropoint energy content of the atoms embedded in Planck's field at zero temperature. Indeed, it turned out from Casimir's calculations that the retarded part of the van der Waals force can be accounted for by considering the energy changes of the electromagnetic zeropoint field due to the presence of the interacting molecules. This represents a contribution that is not predicted by London's calculations, according to which the van der Waals (unretarded) forces arise directly from the Coulomb interaction between the molecules undergoing quantum fluctuations [London 1930].

Casimir went on to analyze the much simpler, ideal case of two macroscopic, uncharged, perfectly conducting parallel plates, which has since then been widely adopted as a model for the study of these phenomena. Considering the change in the pattern of the (quantum) vacuum fluctuations that is produced by the presence of the plates, he predicted a net attractive force between them, given by

$$F_C(R) = -\frac{\pi^2 \hbar c}{240 R^4} \tag{6.2}$$

per unit area [Casimir 1948]. The experiments performed to test Casimir's formula are in agreement with the theoretical prediction [e.g., Abrikosova and Deryagin 1953, Kitchener and Prosser 1957, Sparnaay 1958, Tabor and Winterton 1968], although it would not be quite correct to say that they offer conclusive evidence to support it, owing to the relatively large error margin in the fine measurements performed. Typical Casimir forces are very small; for example, for a plate area of 1 cm^2 and a separation of 1 μ, the force amounts to 0.013 dyn. Moreover, on this scale it is not easy to disentangle the Casimir contribution from the nonretarded van der Waals force due to charge fluctuations which, according to the London theory, is proportional to R^{-3} (see equation (6.28) below). To circumvent these difficulties, a method has been suggested [Iacopini 1993] that should allow the observation of the Casimir force with the help of a confocal optical resonator, for distances in the centimeter range; this method is expected to be fairly insensitive to the ordinary perturbative effects.

Today there is, in any case, an impressive list of further theoretical and experimental applications of the basic idea underlying the Casimir effect [see, e.g., Plunien et al. 1986], that leave no doubt of its validity. Some of these applications will be mentioned in due time. At this stage we turn the reader's attention to the SED approach to the Casimir effect.

6.1.1. CASIMIR FORCE BETWEEN TWO PARALLEL PLATES, ACCORDING TO SED

As mentioned earlier, there is in principle no need to resort to the use of a quantized electromagnetic field for the derivation of the Casimir effect, and one may instead consider the zeropoint radiation field as the source of this phenomenon. This observation was made for the first time by Marshall in 1965 and independently by Boyer three years later.[1] In fact, the

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[1] Marshall (1965b) studies the balance between the average energy lost through radiative reaction and the average energy gained from the background field for an oscillator in the space between two parallel conducting plates. A treatment based on the evaluation of the transverse component of the Maxwell stress tensor of the field at one of the plates reproduces the quantum result for the Casimir effect, including a (small) thermal contribution when $T > 0$.

In a similar spirit, Boyer (1968a) reproduces Casimir's result for the energy change by proposing that the force between the plates arises from the zeropoint field subject to boundary conditions. Further, following the work by Lifshitz (1955) on dispersion forces between dielectric bodies, Boyer calculates the force between the plates by evaluating the electromagnetic stress tensor, thus explicitly showing that there is a clearcut connection

Casimir effect constituted, along with the harmonic oscillator, one of the first problems successfully tackled by SED.

A schematic reproduction of the SED derivation based on the calculation of the average field energy between the plates provides a simple illustration of the basic ideas involved. There are by now numerous derivations of the Casimir effect; the one presented here follows a more recent work [Cetto and de la Peña 1993]. Assume a parallelepiped with conducting walls of length $L_x = L_y = L$ and $L_z = R$, as illustrated in figure 6.1. The vertical walls to the left and to the right of the box are the plates whose field-mediated interaction is to be evaluated; hence the length L is considered large in comparison with other relevant sizes. The field can be expanded in terms of normal modes, as was done in section 3.1. Taking into account the boundary conditions on the electric and magnetic components, it follows that the allowed frequencies for the radiation field within the box are given by the formula

$$\omega_{\mathbf{n}} = ck_{\mathbf{n}} = \pi c\sqrt{\frac{l^2+m^2}{L^2}+\frac{n^2}{R^2}}, \tag{6.3}$$

where $\mathbf{n} = \{n, l, m\}$ and l, m, n are positive integers or zero. The average energy of the zeropoint field trapped in the box is therefore given by the triple sum over modes

$$E(R) = 2\sum_{\mathbf{n}}{}' \frac{1}{2}\hbar\omega_{\mathbf{n}} = \pi\hbar c\sum_{\mathbf{n}}{}'\sqrt{\frac{l^2+m^2}{L^2}+\frac{n^2}{R^2}}; \tag{6.4}$$

the factor 2 arises from the two independent polarizations for $l, m, n \neq 0$. Only one of these numbers can be zero at a time; when this happens a factor of a half must be inserted because then there is only one polarization (this is the meaning of the prime on the summation sign).

For large values of L, the double sum over l, m transforms into a double integral over k_x, k_y

$$\sum_{l,m} \underset{L\to\infty}{\longrightarrow} \left(\frac{L}{2\pi}\right)^2 \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y; \tag{6.5}$$

hence

$$E(R) = \frac{\hbar c L^2}{\pi^2}\int_0^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \sum_{n}{}'\sqrt{k_x^2+k_y^2+(\frac{\pi n}{R})^2}. \tag{6.6}$$

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between the approaches of Casimir and of Lifshitz to the problem of dispersion forces between macroscopic objects.

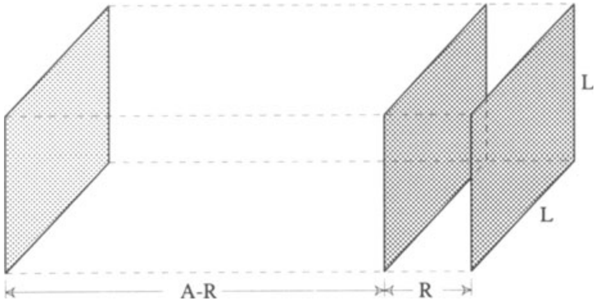

Figure 6.1. Model for the calculation of the Casimir force between two conducting plates of area L^2 separated a distance R. A third plate is situated a large distance A apart, to complete the calculations.

We now set $k_\rho^2 = k_x^2 + k_y^2$ and perform the angular integration on the $k_x k_y$-plane, to get

$$E(R) = \frac{\hbar c L^2}{4\pi} \int_0^\infty dk_\rho^2 \sum_n{}' \sqrt{k_\rho^2 + \left(\frac{\pi n}{R}\right)^2}\,. \tag{6.7}$$

However the field outside the box is also affected by the presence of the conducting walls. So assume a third vertical wall to be placed at a very long (eventually infinite) distance A, where A is defined to be the length of the entire space; the energy of the complete configuration of the three plates is then $E(R) + E(A - R)$. The Casimir energy, that is, the energy modification produced by the presence of the central plate at R, is obtained by subtracting from this quantity the energy in the absence of that plate and making the entire space infinitely large

$$E_C(R) = \lim_{A\to\infty} \left[E(R) + E(A - R) - E(A)\right]. \tag{6.8}$$

This expression represents the work performed to bring the central plate adiabatically to its position at R.

Now, since physical objects which are good conductors at long wave lengths normally become poor conductors at short wave lengths, a cutoff in the wave number $k = \sqrt{k_\rho^2 + (\pi n/R)^2}$ should be introduced when evaluating equation (6.7). The introduction of this cutoff has the advantage of giving a finite expression for the otherwise divergent integral involved in (6.7). It is actually for this computational reason that the cutoff is generally used in this context, both in SED and in QED, although the discussion on this point in §3.2.3 could be applied. An appropriate and convenient cutoff function is $\exp(-k/k_c)$, where k_c is assumed to be very large. Equation

(6.7) gives then after a change of variable

$$E(R;k_c) = \frac{\hbar c L^2}{2\pi} k_c^3 \sum_n{}' \int_{n/x}^{\infty} ds\ s^2 e^{-s}, \tag{6.9}$$

with $s = k/k_c$ and $x = Rk_c/\pi$. Integration gives

$$E(R;k_c) = \frac{\hbar c L^2}{2\pi} k_c^3 \frac{d^2}{dx^2} x^2 \sum_n{}' e^{-n/x}, \tag{6.10}$$

whence performing the summation one obtains

$$E(R;k_c) = \frac{\hbar c L^2 k_c^3}{2\pi} \left(-1 + 2\Delta + 2\frac{\Delta^2 \varsigma}{x} - \frac{\Delta^2 \varsigma}{x^2} + 2\frac{\Delta^3 \varsigma}{x^2} \right) \tag{6.11}$$

with $\varsigma \equiv \exp(-1/x)$ and $\Delta \equiv (1-\varsigma)^{-1}$. This is an expression for the energy between the two plates at 0 and R, valid for any value of R. The Casimir energy is then obtained by introducing (6.11) into (6.8), the exact result being

$$E_C(R;k_c) = \frac{\hbar c L^2 k_c^3}{2\pi} \left(-1 - 6x + 2\Delta + 2\frac{\Delta^2 \varsigma}{x} - \frac{\Delta^2 \varsigma}{x^2} + 2\frac{\Delta^3 \varsigma}{x^2} \right). \tag{6.12}$$

For macroscopic objects, $x = Rk_c/\pi \gg 1$[2] and the above expression acquires the very simple form

$$E_C(R;k_c) \doteq -\frac{\pi^2 \hbar c L^2}{720 R^3} \qquad (k_c \gg \pi/R), \tag{6.13}$$

which means that the force per unit area between the plates is given approximately by $F_C(R) = -\pi^2 \hbar c/240 R^4$, in accordance with the Casimir formula (6.2).

A pleasant feature of equation (6.13) is that it is independent of the value of k_c, and hence seems to have universal validity. Indeed, at some point it was proposed by Boyer (1969b, 1974b) to consider the cutoff-independent character of Casimir forces as being due to the vector character of this field. However, the above expressions show that this is only true in the limit of high k_c, when the dominant term inside the parentheses is $\sim x^{-3}$ and hence the prefactor k_c^3 is cancelled out. Before this limit is taken, the Casimir energy has a more complicated behaviour as a function of x, as is evident from figure 6.2, where E_C as given by (6.12) is plotted as a function of

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[2] For typical values, such as $k_c \sim 10^{10}\,\mathrm{m}^{-1}$ for metallic conductors that become transparent at X-ray frequencies and $R \sim 0.1\,\mathrm{mm}$ for the plate separation, $x = (Rk_c/\pi) \sim 10^6$.

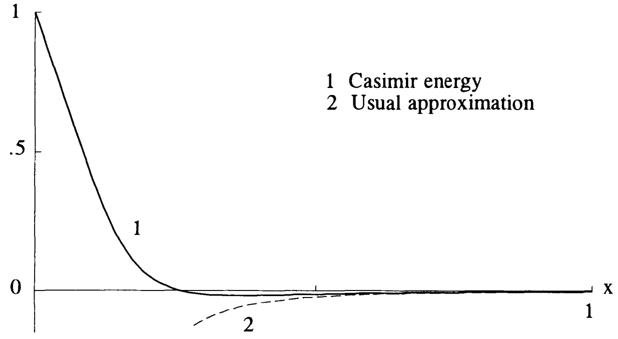

Figure 6.2. Casimir energy for the parallel plates, plotted as a function of the plate separation according to equation (6.12), in units of $\hbar c L^2 k_c^3/2\pi$. The dashed curve represents the (approximate) Casimir formula (6.13).

the plate separation in units of π/k_c. From this figure it is clear that the decrease in conductivity at high frequencies can have a serious effect on the behaviour of the Casimir energy at small distances.[3] In §6.1.3 an interesting consequence of such complicated behaviour for finite values of Rk_c will be discussed.

Alternative techniques frequently used for the derivation of the Casimir force involve calculating explicitly the electric and magnetic field components subject to the boundary conditions. Such a procedure has the advantage of being more generally applicable to bodies of different shapes and with arbitrary dielectric properties, as will be seen in §6.1.4. For the calculation of the field components it is convenient to start again with a representation of the field in terms of geometric factors that can be adjusted to the boundary conditions. Using equation (3.40) and $\mathcal{E}_\alpha = \frac{1}{2}\hbar\omega_\alpha$, we write the vector potential as

$$\mathbf{A}(\mathrm{x}, t) = \sum_\alpha \sqrt{\pi\hbar c^2/\omega_\alpha}\,\left[a_\alpha(t)\mathbf{G}_\alpha(\mathrm{x}) + a_\alpha^*(t)\mathbf{G}_\alpha^*(\mathrm{x})\right], \tag{6.14}$$

where $\alpha = \{\mathbf{k}, \sigma\}$, $\mathbf{k}$ is the wave vector and $\sigma = 1, 2$ the polarization index; the orthonormal functions $\mathbf{G}_\alpha(\mathrm{x})$ must satisfy the Helmholtz equation (3.25) and the transversality condition (3.26) in the Coulomb gauge, and must comply with the appropriate boundary conditions imposed on $\mathbf{E}$ and $\mathbf{B}$. In particular, in the presence of two conducting plates of area L^2 separated a distance R, the modes of the field between the plates can be represented in cylindrical coordinates by the functions

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[3] Further discussion on the role of the cutoff in the evaluation of Casimir forces can be found in Candelas (1982).

$$\mathbf{G}_{\mathbf{k}1} = \sqrt{2/L^2 R}\exp(i\mathbf{k}_\rho \cdot \boldsymbol{\rho})\hat{\mathbf{k}}_\rho \times \hat{\mathbf{z}}\sin\xi, \tag{6.15}$$

$$\mathbf{G}_{\mathbf{k}2} = \sqrt{2/L^2 R}\exp(i\mathbf{k}_\rho \cdot \boldsymbol{\rho}) \left[(n\pi/Rk_\alpha)\,\hat{\mathbf{k}}_\rho \sin\xi + i\,(k_\rho/k_\alpha)\,\hat{\mathbf{z}}\cos\xi\right], \tag{6.16}$$

where $\mathbf{k}_\alpha = \mathbf{k}_\rho + (n\pi/R)\mathbf{z}$, $\mathbf{k}_\rho \cdot \mathbf{z} = 0$, $\xi = (n\pi/R)z$ and z is the distance from the left plate.

With these expressions introduced in (6.14) one can calculate the various space-space components of the average electromagnetic stress tensor [see, e.g., Goldstein 1950, chapter 2]

$$\mathcal{T} = \frac{1}{4\pi}\left\langle \mathbf{EE} + \mathbf{BB} - \frac{1}{2}\mathbf{I}\left(\mathbf{E}^2 + \mathbf{B}^2\right)\right\rangle. \tag{6.17}$$

The force per unit area on the left plate due to the presence of the right plate at R becomes determined by the transverse component T_{zz} evaluated at $z = 0$, which gives [Marshall 1965b]

$$F_z(R) = T_{zz} = -\frac{\hbar c}{2\pi R}\int dk\, k^2 \sum_{n=0}^{\infty}\left(\frac{n\pi}{kR}\right)^2. \tag{6.18}$$

As before, for an evaluation of this integral it is necessary to introduce a cutoff in the frequency. Rearranging (6.18),

$$F_z(R) = -\frac{\pi\hbar c}{2R^3}\sum_{n=1}^{\infty} n^2 \int_{n\pi/R}^{k_c} dk = -\frac{\pi\hbar c}{2R^3}\sum_{n=0}^{\infty} f(n), \tag{6.19}$$

with $f(n) = \sum_{n=1}^{\infty} n^2 \int_{n\pi/R}^{k_c} dk$. To calculate the total Casimir force one must take into account the pressure exerted on the plate by the radiation in the region to the left ($z < 0$); this is obtained by letting R go to infinity in the above equation, whence

$$F_C(R) = -\frac{\pi\hbar c}{2R^3}\left[\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} dn\, f(n)\right]. \tag{6.20}$$

This expression can be simplified using the Euler-MacLaurin sum formula

$$\sum_{n=0}^{\infty}{}' f(n) - \int_0^{\infty} dn\, f(n) = \frac{1}{12}f'(0) - \frac{1}{30 \times 4!}f'''(0) + ..., \tag{6.21}$$

and thus one obtains again, in the limit $k_c \to \infty$, Casimir's result (6.2).

6.1.2. THERMAL CONTRIBUTION TO THE CASIMIR EFFECT

The extension of the above results to include thermal radiation is in principle an easy matter [Marshall 1965b, Boyer 1968a]. If the simple mode-summation procedure is used, it is necessary to take into account that the potential function is given in this case by the total Helmholtz free energy of the blackbody radiation field[4]

$$H = -\beta^{-1} \ln Z. \tag{6.22}$$

With the partition function given by equation (5.35) for every $\mathcal{E}_{0n} = \frac{1}{2}\hbar\omega_n$, namely

$$Z = \prod_{\mathbf{n}} \frac{e^{-\beta\hbar\omega_n/2}}{1 - e^{-\beta\hbar\omega_n}}, \tag{6.23}$$

the free energy becomes

$$H = 2\sum_{\mathbf{n}}{}' \left[\frac{1}{2}\hbar\omega_n + \beta^{-1} \ln\left(1 - e^{-\beta\hbar\omega_n}\right)\right]. \tag{6.24}$$

The Casimir energy is once again obtained as a difference of energies corresponding to different configurations (see figure 6.1 and equation (6.8))

$$H_C(R) = \lim_{A\to\infty} \left[H(R) + H(A-R) - H(A)\right]. \tag{6.25}$$

Following a similar calculational procedure, and introducing the exponential cutoff to make the integrals convergent, one obtains for the force per unit area on one of the plates

$$F_C(R) = -\frac{\partial H_C(R)}{\partial R} = -\frac{\pi^2\hbar c}{240R^4} - \frac{\pi}{\beta R^3}\sum_n n^2 \ln\left[1 - e^{-(\pi\beta\hbar c/R)n}\right], \tag{6.26}$$

provided $R \gg \pi/k_c$. This finite-temperature expression coincides with the result previously derived from a QED calculation [Sauer 1962].[5]

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[4] This is because for isothermal processes the change in the energy content of the field equals the change in the free energy. On the contrary, for adiabatic processes the former equals the change of the total energy of the system. See, e.g., Abraham and Becker (1933), §33.

[5] Alternatively, it is possible to calculate the stress tensor (6.17) corresponding to the complete thermal field by replacing the zeropoint energy in the coefficient of (6.14) with Planck's formula $E_\alpha(T) = \frac{1}{2}\hbar\omega_\alpha \coth(\beta\hbar\omega_\alpha/2)$. The force per unit area on the plate at $z = 0$ is then again given by equation (6.20), but with $f(n) = \sum_{n=1}^{\infty} n^2 \int_{n\pi/R}^{k_c} dk \cot(\beta\hbar c k/2)$. The calculation is performed using the Euler-Maclaurin sum formula, and it gives the same result for the thermal Casimir effect reported above, (6.26) [Marshall 1965b, Boyer 1968a].

For low temperatures the thermal contribution in equation (6.26) goes to zero, and the Casimir force reduces to (6.2). For high temperatures the thermal part of the force can be evaluated with the help of the Poisson sum formula [Whittaker and Watson 1958]

$$\sum_{n=1}^{\infty} b(n) = \sum_{n=0}^{\infty}{}' c(2\pi n), \quad c(\alpha) \equiv \int_{-\infty}^{\infty} dx\, b(x) e^{-i\alpha x}. \tag{6.27}$$

Introducing $b(n) = n^2 \ln\left[1 - e^{-(\pi\beta\hbar c/R)n}\right]$ one obtains for $(\pi\beta\hbar c/R) << 1$, retaining the first terms only in the exponential expansion,

$$F_C(R;T) = -\frac{\zeta(3)}{4\pi}\frac{kT}{R^3}, \tag{6.28}$$

where $\zeta(3)$ is the Riemann zeta function, $\zeta(3) = \sum_{n=1}^{\infty} n^{-3} \doteq 1.202$. This term decreases more slowly with R than the zeropoint term, so that for large plate separations it can represent an important contribution to the Casimir force [Mehra 1967]. Already for $R = 5 \times 10^{-4}$cm it makes the attraction between the plates increase by about 50% at room temperature.[6]

It is important to note that the force term (6.28), that has been obtained from the blackbody radiation spectrum in the limit of high temperatures, is completely independent of Planck's constant $\hbar$; it can actually be shown [Boyer 1975c] to coincide precisely with the Casimir force obtained by considering the plates to be immersed in a *classical* thermal radiation field with a Rayleigh-Jeans spectrum.

6.1.3. CASIMIR FORCE ON A SPHERICAL CONDUCTING SHELL

We have just seen how two conducting plates can change the allowed normal modes of the zeropoint field in a region of space, and hence modify the corresponding field energy, giving as a result an attractive force between the plates. In a similar way one can formulate the problem of the zeropoint energy of a conducting spherical shell, except that the calculations become much more complicated. By purely dimensional analysis, this energy is expected to be of the form

$$E_C(R) = -C\frac{\hbar c}{2R} \tag{6.29}$$

where C is some numerical factor and R is now the radius of the sphere. In fact, motivated by his result of an attractive interaction between parallel plates, Casimir (1953) conjectured that a similar effect could account for the stability of a charged particle. He proposed that the Abraham-Lorentz electron theory be modified by regarding the particle as a spherical conducting shell carrying a charge e; then the Poincaré stresses[7] should be provided by the zeropoint force on the conducting boundary. A balance between the Coulomb energy $E(R) = e^2/2R$ and the Casimir energy (6.29) would even provide a means for evaluating the fine-structure constant $\alpha = e^2/\hbar c$. This, he thought, could be a mousetrap to catch α [as quoted in Boyer 1970].

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[6] A slightly different approach to the subject, which also accepts a reformulation in the SED language, is given in González (1985).

Such fine-scale mice, however, are difficult to trap. The lengthy calculations performed first by Boyer (1968b, 1970) and later by other authors within the QED framework [Balian and Duplantier 1978, Milton et al. 1978] lead in different ways to an exact expression for the Casimir energy

$$E_C(R) = \frac{\hbar c}{2R} \sum_l (2l+1)$$

$$\int_0^\infty dz\,(s(z) - [s(z)] + \bar{s}(z) - [\bar{s}(z)] - 1)_l \frac{d}{dz} z\mathcal{F}(\frac{z}{Rk_c}), \tag{6.30}$$

with

$$s_l(z) = -\frac{1}{\pi} \arctan \frac{j_l(z)}{n_l(z)}, \tag{6.31}$$

$$\bar{s}_l(z) = 1 - \frac{1}{\pi} \arctan \frac{d[zj_l(z)]/dz}{d[zn_l(z)]/dz}, \tag{6.32}$$

where j_l, n_l are spherical Bessel and Neumann functions, respectively, and $[s(z)]$ is the step function defined as the integer part of $s(z)$. For the cutoff factor $\mathcal{F}(z/Rk_c)$, the exponential function is introduced as usual. In the limit $Rk_c \longrightarrow \infty$, numerical evaluation of (6.30) gives for the factor C the value $C \simeq -0.09$, which besides being much larger than the fine-structure constant, has the wrong sign! The repulsive force obtained in this case has been interpreted as an invalidation of the Casimir conjecture.

The numerical results obtained by Boyer have been confirmed in the usual range of approximations where the cutoff k_c is made infinite. However, the parameter that appears in equation (6.30) is the product Rk_c which for very small objects (such as electrons) cannot legitimately be taken as arbitrarily large (see the discussions in §3.2.3 and 3.3). Although a complete numerical evaluation for all values of Rk_c has not been carried out, it turns out that for $Rk_c \ll 1$ the Casimir energy increases with the third power of R [Cetto and de la Peña 1993],

$$E_C(R) \approx \frac{9\hbar c k_c^4}{\pi} R^3 \qquad (Rk_c \ll 1). \tag{6.33}$$

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[7] We recall that Poincaré postulated a covariant stress tensor that should be added to the electromagnetic stress tensor of the charged particle, in order to provide mechanical stability [see, e.g., Jackson 1975, chapter 17].

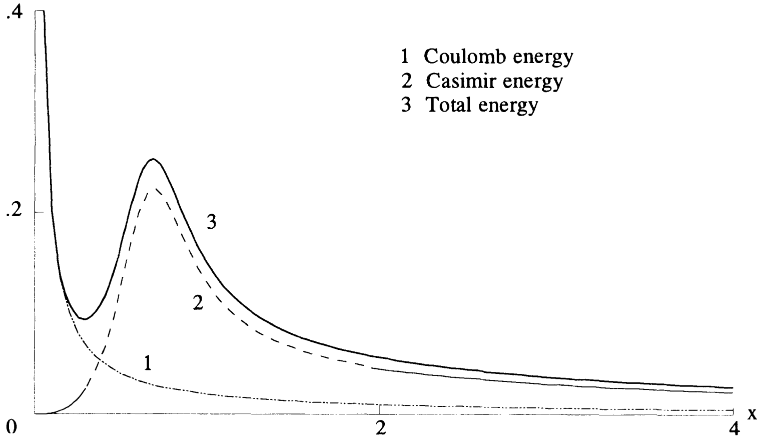

Figure 6.3. Energy for the electron modelled as a spherical conducting shell, in units of $9\hbar ck_c/\pi$. The solid parts of curve 2 are given by (6.29) with $C = -0.09$ for $x = Rk_c \gg 1$, and (6.33) for $x \ll 1$; the central part is an interpolation.

and hence, as is illustrated in figure 6.3, there is a whole range of values of Rk_c for which there may be an *inward* Casimir force on the conducting shell. The reason for this attractive force can be understood as follows. For a macroscopic shell, for which indeed $Rk_c \gg 1$, there is a great number of modes (those with k such that $k_c > k > 1/R$) that fit in the interior but are subject to boundary conditions and hence exert an outward pressure, opposing the pressure exerted on the shell by the external field; the net effect is a repulsive force. However, for a shell so small that $Rk_c < 1$, there is no field in the interior that is subject to boundary conditions and hence capable of counterbalancing the external pressure.

This result can be used to rescue the Casimir model for the electron by choosing appropriate values for the parameters such that $Rk_c \lesssim 1$. Even for a cutoff value as high as the Compton frequency $\omega_C = ck_C = mc^2/\hbar$, the result

$$R < \frac{\hbar}{mc} = \frac{1}{2\pi}\lambda_C \tag{6.34}$$

represents an entirely satisfactory condition on the size of the electron. As illustrated in figure 6.3, the sum of the Coulomb and the Casimir energies has a minimum, indicating that there is some value of R in agreement with (6.34) for which the particle is in stable equilibrium. Perhaps one should not throw away the mousetrap after all!

In the more general case of an arbitrary geometry, the net effect of the action of the internal and external, electric and magnetic field modes is difficult to evaluate and one cannot even anticipate the sign of the end result.[8]

6.1.4. CASIMIR FORCE BETWEEN DIELECTRIC PLATES

Casimir effect calculations in SED were extended by Henry and Marshall (1966) to study the attraction between two semi-infinite dielectric plates, using the stress-tensor procedure which Lifshitz had developed in connection with the molecular theory of dispersion forces. The plates are assumed to be characterized by a dielectric constant ϵ, and the electric and magnetic components of the vacuum field are again calculated assuming that they satisfy Maxwell's equations in their classical form. The expressions for the field become considerably more complicated as a result of applying the boundary conditions for dielectrics. The average stress tensor obtained leads to an attractive force per unit area between plates that is once more in agreement with the quantum-mechanical calculation [Lifshitz 1955],

$$F_C(R) =$$

$$-\frac{\hbar c}{2\pi^2}\int_0^\infty dq\, q^3 \int_1^\infty dp\, p^2 \left[\frac{1}{A_1^2 \exp(2pqR) - 1} + \frac{1}{A_2^2 \exp(2pqR) - 1}\right], \tag{6.35}$$

where

$$A_1 = \frac{p + \sqrt{p^2 - 1 + \epsilon}}{p - \sqrt{p^2 - 1 + \epsilon}}, \quad A_2 = \frac{\epsilon p + \sqrt{p^2 - 1 + \epsilon}}{\epsilon p - \sqrt{p^2 - 1 + \epsilon}}, \tag{6.36}$$

and q, ϵ are functions of the wave number,[9]

$$q = k/(p^2 - 1), \qquad \epsilon(k) = \epsilon(q(p^2 - 1)).$$

The expression (6.35) for $F_C(R)$ is a complicated function of the plate separation. However, one can obtain a rough idea of this function by recognizing that for $|A_1|$ and $|A_2|$ not too different from 1, the dominant contribution to the integral comes from $2pqR \sim 1$. Changing the integration variable to $x \equiv qR$ and assuming that the function $\epsilon(x(p^2 - 1)/R)$

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[8] A variety of examples are discussed, e.g., in Ambjørn and Wolfram (1983), where it is shown that the energy modification of the vacuum depends on the number of dimensions of space and the specific shape of the conductors. Although these authors refer to 'quantized' vacuum fields, their results are not at all dependent on the quantization of the fields and can equally well be phrased in the language of SED.

[9] We refer the reader to chapter 7 of Milonni (1994) for a more recent derivation of this result, based on the calculation of the zeropoint energy via the mode-summation method. Although the calculation is made there within QED, it can be reproduced step by step using the nonquantized zeropoint field.

can be replaced by a constant parameter ϵ, one gets once more the typical R^{-4}-dependence

$$F_C(R) = -\frac{\hbar c}{2\pi^2 R^4}\int_0^\infty dx\, x^3 \int_1^\infty dp\, p^2 \left[\frac{1}{A_1^2 \exp(2px) - 1} + \frac{1}{A_2^2 \exp(2px) - 1}\right]. \tag{6.37}$$

This result is a good approximation to the extent that the assumption of a frequency-independent dielectric constant ϵ is applicable. (Indeed, in the limit of perfect conductors ($\epsilon \to \infty$) integration of (6.37) gives Casimir's formula (6.2).) As in the case of perfect conductors, the calculation of each separate term of the total force between the dielectric plates involves a divergent integral, which is made convergent by introducing the usual cutoff function and taking the limit $k_c \to \infty$ at the end of the calculations. This mathematical trick is considered physically legitimate, for reasons similar to those pointed out in §6.1.2, namely, dielectric materials are known to have a frequency-dependent refractive index n that eventually goes to unity for high frequencies (see below). It should be mentioned, however, that the integral expressions for the force terms that contribute to (6.35) involve the product $k_c R$, rather than k_c alone; hence the limit taken is $k_c R \to \infty$. One can only be confident, therefore, that the results obtained from (6.37) represent a good approximation for large plate separations R.

Refractive index and effects of dispersion

To study the refractive index and the effects of dispersion on the Casimir force between dielectrics within the SED framework, we may use a simple model as follows. Assume the dielectric slabs in the system to have η atoms per unit volume, which we represent by small harmonic oscillators of natural frequency ω_0, subject to the random background force. The equation of motion for any of these atomic oscillators is (see section 7.1)

$$m\frac{d^2\mathbf{r}(\mathrm{x},t)}{dt^2} = -m\omega_0^2\mathbf{r}(\mathrm{x},t) + m\tau\frac{d^3\mathbf{r}(\mathrm{x},t)}{dt^3} + e\mathbf{E}(\mathrm{x},t). \tag{6.38}$$

$\mathbf{E}(\mathrm{x},t)$ is the electric part of the random field *inside* the dielectric, i.e., the zeropoint field modified by the atoms of the material. This field must in its turn satisfy the equation

$$\nabla^2\mathbf{A}(\mathrm{x},t) - \frac{1}{c^2}\frac{\partial^2\mathbf{A}(\mathrm{x},t)}{\partial t^2} = -\frac{4\pi}{c}\frac{\partial\mathbf{P}_\perp(\mathrm{x},t)}{\partial t}, \tag{6.39}$$

with $\mathbf{P}_\perp(\mathrm{x},t)$ representing the transverse part of the polarization of the medium,

$$\mathbf{P}(\mathrm{x},t) = \eta e\mathbf{r}(\mathrm{x},t). \tag{6.40}$$

In solving equation (6.38) one may safely take the dipole approximation, which implies assuming the size of the oscillators (and hence the variations of $\mathbf{r}$) to be much smaller than the relevant wavelengths, since, as will be seen below, the high-frequency modes do not contribute to the Casimir force.

As a result of the balance between the average effects of the fluctuating electric field and the dissipative radiation reaction, the oscillators described by equation (6.38) eventually reach a state of dynamical equilibrium. To find the corresponding stationary solution of this coupled system of equations, we write the vector potential again in the form of equation (6.14). To simplify the calculation of the functions $\mathbf{G}_\alpha(\mathrm{x})$, let us consider the transverse modes only, so that $\mathbf{P}_\perp = \mathbf{P}$. The corresponding solution for the atomic oscillators can then be expressed in terms of the same basis

$$\mathbf{r}(\mathrm{x}, t) = \sum_\alpha \sqrt{\pi\hbar c^2/\omega_\alpha}\, [\xi_\alpha a_\alpha(t)\mathbf{G}_\alpha(\mathrm{x}) + \xi_\alpha^* a_\alpha^*(t)\mathbf{G}_\alpha^*(\mathrm{x})], \tag{6.41}$$

and substituting this expression and its time derivatives (in the dipole approximation) in (6.38) one obtains

$$\xi_\alpha = \frac{ie\omega_\alpha}{mc\Delta(\omega_\alpha)}, \tag{6.42}$$

with $\Delta(\omega) = \omega_0^2 - \omega^2 - i\tau\omega^3$. Further, combining equations (6.39-6.41) one is left with the equation for the mode functions

$$\nabla^2\mathbf{G}_\alpha + \left(\frac{n_\alpha\omega_\alpha}{c}\right)^2 \mathbf{G}_\alpha = 0, \tag{6.43}$$

where n_α is the refractive index, given by

$$n_\alpha^2 = 1 + \frac{4\pi e^2\eta}{m\Delta(\omega_\alpha)}. \tag{6.44}$$

Hence dielectric materials are necessarily dispersive, their refractive index being a function of the frequency, with a complex term in the denominator as a result of the radiation-reaction contribution. With $\mu = 1$ for the magnetic permeability, the dielectric constant thus obtained, $\epsilon(\omega) = n^2(\omega) = \epsilon'(\omega) + i\epsilon''(\omega)$, satisfies correctly the Kramers-Kronig dispersion relations [see, e.g., Jackson 1975, chapter 7]

$$\epsilon'(\omega) = 1 + \frac{2}{\pi}\mathcal{P}\int_0^\infty dx\frac{x\epsilon''(x)}{x^2-\omega^2}, \quad \epsilon''(\omega) = -\frac{2\omega}{\pi}\mathcal{P}\int_0^\infty dx\frac{\epsilon'(x)-1}{x^2-\omega^2}. \tag{6.45}$$

Further, we see from (6.44) that for any dielectric the refractive index has the asymptotic value $n \rightarrow_{\omega\rightarrow\infty} 1$ corresponding to a transparent medium; this introduces a natural cutoff in all integrals over the frequency.

In the usual narrow-line-width approximation Δ is replaced by $\Delta_0 = \omega_0^2 - \omega^2 - i\tau\omega_0^2\omega$, and equation (6.44) reduces to

$$n^2(\omega) = 1 + \frac{4\pi e^2 \eta}{m(\omega_0^2 - \omega^2 - i\gamma\omega)} \tag{6.46}$$

with $\gamma = \tau\omega_0^2$. This coincides with the approximate result obtained from a simple, classical oscillator model calculation [e.g., Jackson 1975, chapter 7] or a QED calculation [e.g., Kupiszewska 1992] where the damping is introduced *ad hoc*.

To complete the exact calculation of the Casimir force it is necessary to solve equation (6.43) for the mode functions within the dielectric slabs, and to consider also the zeropoint field in the spaces outside the slabs (where $n = 1$), all subject to the appropriate boundary conditions. The lengthy calculations are performed in detail by Kupiszewska (1992) with only the transverse components of the field taken into account. Though they are made within the QED framework, again we may translate them to the SED language since they do not depend on the quantization of the field. The result is

$$F_C(R) = \frac{\hbar}{\pi}\int_0^\infty dk\,\omega_k\left[1 - \frac{1-|r|^4}{|1-r^2e^{2ikR}|^2}\right], \tag{6.47}$$

where r is the reflection coefficient of a plate of thickness d, given by

$$r = \frac{(n^2-1)(e^{2iknd}-1)}{(n+1)^2-(n-1)^2e^{2iknd}}. \tag{6.48}$$

Since $r \to 0$ as $\omega \to \infty$ by virtue of (6.44), the integral in equation (6.47) is already convergent without the need to introduce any further cutoff. The force given by (6.47) can be shown to be an increasing function of the thickness d, and to go over to Lifshitz's expression (6.35) with $n^2 = \epsilon$ and $q = \epsilon\omega/c$ in the asymptotic limit $d \to \infty$. Note that for large plate separations ($R\omega_0 >> c$) the dominant contribution to (6.35) comes from $q >> \omega_0/c$, i.e., $\omega >> \omega_0$, where indeed $\epsilon \simeq 1$ according to (6.44). This confirms the validity of the approximate result (6.37) for large values of R.

6.2. Long-range van der Waals forces

The study of the long-range interactions between neutral microscopic objects has acquired considerable importance, especially in relation to surface phenomena, such as adhesion, colloidal stability and foam formation. They have been even considered to give rise to the most fundamental physical forces controlling living beings and life processes [Elizalde and Romeo 1981]. It is well known that quantum theory has contributed in a fundamental way to the understanding of the deviations from the ideal-gas law due to these long-range interactions.

The quantum-mechanical work on van der Waals forces was initiated by London in the late '20s, with the successful derivation of a universal, long-range, temperature-independent attractive interaction term between neutral polarizable molecules. Modelling the molecules as harmonic oscillators of frequency $\omega = (E_1 - E_0)/\hbar$ corresponding to transitions between the first excited level and the ground state, he used quantum perturbation theory to calculate the interaction energy of the molecules as a function of their distance of separation R and thus obtained for $R << c/\omega$ (but R much larger than the molecular dimensions) [London 1930]

$$E(R) = -\frac{3\hbar\omega\alpha^2}{4R^6}, \tag{6.49}$$

where α is the polarizability of the molecules. Eighteen years later, Casimir and Polder (1948) performed the complete fourth-order QED calculation of the interaction between two molecules, modelled as neutral, polarizable particles, and thus they showed that for large intermolecular separation $(R >> c/\omega)$, the interaction energy varies as

$$E(R) = -\frac{23\hbar c\alpha^2}{4\pi R^7}, \tag{6.50}$$

as was mentioned at the beginning of section 6.1; this is the so-called retarded contribution to the van der Waals energy. The results of Casimir and Polder have been rederived along different procedures within perturbation theory up to fourth order, involving either dispersion theory [Feinberg and Sucher 1970] or canonical transformations [Power 1965]. Full QED calculations to all orders have also been provided [Renne 1971, Power and Thirunamachandran 1993]. Because of the relation of these long-range forces to the phenomenon of optical dispersion, they are normally termed *dispersion forces.*

6.2.1. VAN DER WAALS FORCES ACCORDING TO SED

From the point of view of SED, the two-body van der Waals force involves in principle a relatively simple system, that can be approached within classical electromagnetic theory with due account for the zeropoint field. This observation was made by Boyer, who in a series of papers [Boyer 1969a, 1972a, b, 1973] calculated the long-range forces between a polarizable particle and a wall and between two polarizable particles, as due to electromagnetic interactions when the particles are immersed in the random zeropoint field.

Before entering into more formal developments, and in order to give an insight into the nature of these forces, let us carry out a heuristic calculation of the unretarded term of the van der Waals force between two neutral polarizable particles (atoms or molecules). The basic idea is as follows. The two particles A, B attract each other because the fluctuations of the electric dipole moment of one of them induce an instantaneous dipole moment on the second particle. Since the dipole field of one particle falls as R^{-3}, the strength of the dipole induced on the other particle has this dependence, and the interaction energy has therefore an over-all dependence of R^{-6}. For a simple calculation it suffices to continue to model each atom as a small radiating dipole oscillator of frequency ω_0, the so-called Drude-Lorentz model,[10] and take the polarizability to be given by $\alpha = e^2/m\omega_0^2$. We further assume that the particles are far enough from each other, so that: i) the coupling between them is weak (i.e. $\alpha/R^3 << 1$) and can be represented by the (bilinear) electric dipole-dipole interaction potential, which implies an approximately linear coupling force on each particle; ii) the particles can be considered to be subject to uncorrelated random fields in the dipole approximation, $\mathbf{E}_A(t)$, $\mathbf{E}_B(t)$. Then their equations of motion are

$$m\frac{d^2\boldsymbol{\xi}_A}{dt^2} = -m\omega_0^2\boldsymbol{\xi}_A + m\tau\frac{d^3\boldsymbol{\xi}_A}{dt^3} + mK\boldsymbol{\xi}_B + e\mathbf{E}_A, \tag{6.51}$$

$$m\frac{d^2\boldsymbol{\xi}_B}{dt^2} = -m\omega_0^2\boldsymbol{\xi}_B + m\tau\frac{d^3\boldsymbol{\xi}_B}{dt^3} + mK\boldsymbol{\xi}_A + e\mathbf{E}_B, \tag{6.52}$$

where the coupling factor is $K = qe^2/mR^3$, q being the dipole-dipole orientation factor, $q = \langle \hat{\mathbf{p}}_A \cdot \hat{\mathbf{p}}_B - 3(\hat{\mathbf{p}}_A \cdot \hat{\mathbf{n}})(\hat{\mathbf{p}}_B \cdot \hat{\mathbf{n}})\rangle$, with $\hat{\mathbf{n}}$ the unit vector in the direction of $\mathbf{R}$ [see, e.g., Jackson 1975, chapter 4]. Since one dipole is induced by the fluctuations of the other, the two dipole moments are antiparallel and $q = 2$. This pair of equations can be uncoupled with the change of variables $\boldsymbol{\xi}_{A,B} = (\mathbf{v} \mp \mathbf{u})/\sqrt{2}$; the ensuing equations for the normal modes are

$$m\frac{d^2\mathbf{u}}{dt^2} = -m\omega_-^2\mathbf{u} + m\tau\frac{d^3\mathbf{u}}{dt^3} + e\mathbf{E}_1, \tag{6.53}$$

$$m\frac{d^2\mathbf{v}}{dt^2} = -m\omega_+^2\mathbf{v} + m\tau\frac{d^3\mathbf{v}}{dt^3} + e\mathbf{E}_2, \tag{6.54}$$

with $\omega_\pm^2 = \omega_0^2 \pm K$ and $\mathbf{E}_{1,2} = (\mathbf{E}_B \mp \mathbf{E}_A)/\sqrt{2}$; these normal modes correspond to the relative motion and to the center-of-mass motion, respectively. Since $\mathbf{E}_A$, $\mathbf{E}_B$ are not correlated, neither are $\mathbf{E}_1$, $\mathbf{E}_2$,

$$\left\langle \mathbf{E}_1^2 \right\rangle = \left\langle \mathbf{E}_2^2 \right\rangle = \left\langle \mathbf{E}^2 \right\rangle; \quad \langle \mathbf{E}_1\mathbf{E}_2 \rangle = 0. \tag{6.55}$$

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[10] See §7.2.1 for a brief discussion on the use of the Drude-Lorentz model for the atom both in SED and in quantum mechanics.

We are interested in the stationary solutions of equations (6.53) and (6.54) for the two independent harmonic oscillators with frequencies ω_+ and ω_-; the average energy of the system is therefore given by the sum of the respective average energies, $E = 3\hbar(\omega_+ + \omega_-)/2$. Since $K = (2\alpha/R^3)\omega_0^2 << \omega_0^2$, one has in a first approximation

$$E = 3\hbar\omega_0 - \frac{3\hbar K^2}{8\omega_0^3} = 3\hbar\omega_0\left(1 - \frac{\alpha^2}{4R^6}\right). \tag{6.56}$$

The total energy of interaction between the dipoles is therefore

$$E(R) = -\frac{3\hbar\omega_0\alpha^2}{4R^6}, \tag{6.57}$$

in agreement with London's formula. This calculation shows that the heuristic model gives a fair representation of the system, at least in what concerns its most basic properties. What makes it, however, more interesting is that it helps us to understand the reason for the success of the elementary quantum-mechanical calculation. The fact that the background fields $\mathbf{E}_A$, $\mathbf{E}_B$ acting on the two particles can be considered uncorrelated explains why London's derivation, in which the system is modelled simply as two weakly coupled interacting (quantum) oscillators that satisfy the Heisenberg equations

$$m\frac{d^2\hat{\xi}_A}{dt^2} = -m\omega_0^2\hat{\xi}_A + mK\hat{\xi}_B, \quad m\frac{d^2\hat{\xi}_B}{dt^2} = -m\omega_0^2\hat{\xi}_B + mK\hat{\xi}_A, \tag{6.58}$$

gives the correct result. Simply, there is no effective coupling except through the direct dipole-dipole interaction.

However, the interaction between the polarizable atoms is mediated by the electromagnetic field, and it is therefore necessary to consider the effects of retardation as the field propagates from one atom to the other. These effects should become important for interatomic separations $R \geq cT_0$, where T_0 is a characteristic time of the atomic electron (of the order of $2\pi/\omega_0$), because at such distances the motion in one atom cannot maintain a one-to-one phase relation with the motions in the other atom [Spruch 1986] and the simple linear dipole-dipole coupling breaks down. For the complete calculation of the van der Waals force between two electrically polarizable atoms, including the retarded contribution, one can again use the Drude-Lorentz model of the atom, but now taking into account the field radiated by each oscillator and acting on the other, in addition to the zeropoint field. The equations of motion for the displacements $\boldsymbol{\xi}_{A,B}$ of the oscillators are therefore [Boyer 1973]

$$m\frac{d^2\boldsymbol{\xi}_A}{dt^2} = -m\omega_0^2\boldsymbol{\xi}_A + m\tau\frac{d^3\boldsymbol{\xi}_A}{dt^3} + e\mathbf{E}_{DB}(\mathrm{x}_A, t) + e\mathbf{E}_{ZP}(\mathrm{x}_A, t), \tag{6.59}$$

$$m\frac{d^2\boldsymbol{\xi}_B}{dt^2} = -m\omega_0^2\boldsymbol{\xi}_B + m\tau\frac{d^3\boldsymbol{\xi}_B}{dt^3} + e\mathbf{E}_{DA}(\mathbf{x}_B, t) + e\mathbf{E}_{ZP}(\mathbf{x}_B, t). \tag{6.60}$$

As seen in equation (6.66) below, the high frequencies do not contribute to the interaction and the use of the dipole approximation is therefore legitimate, which means that the electric fields can be evaluated at the equilibrium positions of the oscillators, $\mathbf{x}_A$ and $\mathbf{x}_B$. In (6.59) and (6.60), $\mathbf{E}_{DB}(\mathbf{x}_A, t)$ and $\mathbf{E}_{DA}(\mathbf{x}_B, t)$ are the dipole fields at the position of each particle due to the other particle and $\mathbf{E}_{ZP}$ is of course the zeropoint electric field.

A Fourier transformation leads to a system of algebraic equations for the Cartesian components of the Fourier transform of the dipole moments $\tilde{\mathbf{p}}_A(\omega) = e\int dt\, \boldsymbol{\xi}_A(t)e^{i\omega t}$,

$$\Delta\tilde{p}_{Ai} + \eta_i\tilde{p}_{Bi} = \frac{3\tau c^3}{2}\tilde{E}_{Ai}, \quad \eta_i\tilde{p}_{Ai} + \Delta\tilde{p}_{Bi} = \frac{3\tau c^3}{2}\tilde{E}_{Bi}, \tag{6.61}$$

where $\tilde{\mathbf{E}}_{A,B}$ is the Fourier transform of the zeropoint field at A, B,

$$\eta_x = \eta_y = -\frac{3}{2}\tau\omega^3\left[\frac{1}{kR} + \frac{i}{(kR)^2} - \frac{1}{(kR)^3}\right]e^{ikR}, \tag{6.62}$$

$$\eta_z = -3\tau\omega^3\left[\frac{-i}{(kR)^2} + \frac{1}{(kR)^3}\right]e^{ikR}, \tag{6.63}$$

and $\Delta = \omega_0^2 - \omega^2 - i\tau\omega^3$. Equations (6.61) give

$$\tilde{p}_{Ai} = \frac{3\tau c^3}{4}\left(\frac{\tilde{E}_{Ai} + \tilde{E}_{Bi}}{\Delta + \eta_i} + \frac{\tilde{E}_{Ai} - \tilde{E}_{Bi}}{\Delta - \eta_i}\right) \tag{6.64}$$

and an analogous result for $\tilde{p}_{Bi}$. This allows to calculate the average force upon each dipole, using the general formula

$$\mathbf{F} = \langle(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E} + \mathbf{p}\times(\boldsymbol{\nabla}\times\mathbf{E})\rangle, \tag{6.65}$$

where the electric field at A is the sum of the zeropoint field and the contribution from the dipole at B, and reciprocally. A somewhat lengthy but straightforward calculation leads to the following expression for the force of attraction between two particles situated on the z axis a distance R apart [see Boyer 1973 for the details], namely,

$$F = -\frac{\partial E}{\partial R} = -\frac{\hbar c}{2\pi}\sum_{i=1}^{3}\int_0^\infty dk\frac{\partial}{\partial R}\arccos\operatorname{Im}\ln\left[1 - (\eta_i/\Delta)^2\right]. \tag{6.66}$$

This result is exact to all orders of R, and is in agreement with the exact QED formula [Renne 1971]. For small particle separations ($R\omega_0 << c$, but still $\alpha/R^3 << 1$) it gives the unretarded London force, (6.57). For large particle separation ($R\omega_0 >> c$), a series expansion of the logarithm term in (6.66) gives in the radiationless limit ($\tau \to 0$), with $s = kR$,

$$E(R) = -\frac{\hbar c\alpha^2}{\pi R^7}\int_0^\infty ds\left(s^4 + 2s^3 + 5s^2 + 6s + 3\right)e^{-2s} = -\frac{23\hbar c\alpha^2}{4\pi R^7}, \quad (6.67)$$

which coincides with the asymptotic retarded term of Casimir and Polder, equation (6.1).

A slightly different procedure, based on the calculation of the frequency shift produced by one particle on the vacuum field that acts on the other particle, has been used to derive the long-range force between particles that are both electrically and magnetically polarizable, assuming again constant polarizabilities [Boyer 1969a]. A generalization of equation (6.67) is thus obtained, which coincides with the result derived from dispersion-theory techniques [Feinberg and Sucher 1968]

$$E(R) = -\frac{\hbar c}{4\pi R^7}\left[23\left(\alpha_A\alpha_B + \beta_A\beta_B\right) - 7\left(\alpha_A\beta_B + \alpha_B\beta_A\right)\right]. \quad (6.68)$$

An entirely analogous procedure serves to calculate the force between a polarizable particle and a conducting wall [Boyer 1969a, 1973]. The particle is again modelled as a small dipole, subject to the zeropoint field and its own radiation reaction, both of which must satisfy boundary conditions imposed by the conducting wall. The calculations lead in the asymptotic limit to the retarded term,

$$E(R) = -\frac{\hbar c\alpha}{2\pi R^4}\int_0^\infty ds\left(s^2 + s + \frac{1}{2}\right)e^{-2s} = -\frac{3\hbar c\alpha}{8\pi R^4}. \quad (6.69)$$

When a magnetic polarizability β is included, the interaction energy becomes

$$E(R) = -\frac{3\hbar c(\alpha - \beta)}{8\pi R^4}. \quad (6.70)$$

Thus, if α and β are both positive, the electric polarizability gives an attraction towars the wall, but the magnetic polarizability causes repulsion

6.2.2. VAN DER WAALS FORCES AT ANY TEMPERATURE

As with the Casimir effect, in this case it is also possible to extend the treatment to arbitrary temperatures. For this purpose, the complete Planck formula for the average energy density of the blackbody radiation field must be inserted in the calculations, which means that the potential expansions will all have an extra factor $\coth(\hbar\omega/2kT)$. For instance, to obtain the energy between two dipoles at large separation, one must multiply the integrand of equation (6.67) by $\coth(\hbar c/2kTR)s$. The calculations carried out in SED for two polarizable particles and for a particle in the vicinity of a conducting wall are considerably easier and more transparent than those of QED perturbation theory, and have produced a variety of results, some of which had not been previously obtained in explicit form [see Boyer 1975c]. In particular, it is interesting to observe once more (as in the case of two parallel conducting plates, see equation (6.26)) that for high temperatures, the results obtained coincide with those of classical statistical mechanics using the Rayleigh-Jeans distribution, which emphasizes the fact that these dispersion forces do not depend on the quantum nature of the field. The energy of interaction between a pair of electrically polarizable particles modelled as dipoles of frequency ω_0 is given in this limit approximately by

$$E_C = -\frac{3\alpha^2 kT}{R^6} \qquad (kT \gg \hbar\omega_0) \tag{6.71}$$

instead of (6.67), and the energy of the particle-wall system is given approximately by

$$E_C = -\frac{\alpha kT}{4R^3} \qquad (kT \gg \hbar\omega_0) \tag{6.72}$$

instead of (6.69). These results coincide with the expressions obtained for the unretarded part of the dispersion force for high temperatures, but they hold for arbitrary separations R. Hence, for high temperatures the long-range asymptotic form of the van der Waals force coincides with the unretarded form, as is to be expected, since at these temperatures the effect of the zeropoint fluctuations on the interaction is surely negligible compared with the effect of the thermal field.

The Casimir effects discussed in section 6.1 can in principle be regarded as macroscopic manifestations of the van der Waals forces treated in this section. In practice, however, the equivalence between the microscopic many-body problem and the macroscopic boundary-value problem is not easily established. The reason for this is that the van der Waals forces are in general non-additive, because the interaction between two bodies is affected by the presence of a third one [Axilrod and Teller 1943; a closed expression for the general N-body dispersion energy is derived in Power and Thirunamachandran 1985]. The dispersion theory of Lifshitz (1955), which allows us to express the overall effect of the molecules of a dielectric body via the refraction index of the material, is a practical, though approximate means of establishing the connection between microscopic and macroscopic dispersion effects.[11]

All the above results lead to the same conclusion, namely that a classical electrodynamic theory which changes the homogeneous boundary condition on Maxwell's equations so as to include the random zeropoint field of average energy $\frac{1}{2}\hbar\omega$ per normal mode (or $\frac{1}{2}\hbar\omega\coth(\hbar\omega/2kT)$ at $T > 0$), is sufficient to explain the retarded dispersion forces. In all cases the results obtained are in agreement with the QED calculations and, as already mentioned, what is important for an understanding of this kind of effect is not the quantization of the electromagnetic field, but the very fact that there is this fluctuating radiation field, even at $T = 0$. It is difficult at this point not to share Boyer's optimistic conclusion "that much of what is now regarded as a physics of quanta may allow an explanation in classical terms using electromagnetic zeropoint radiation" [Boyer 1970].

6.3. A model for the Lamb shift

When two or more bodies are present, the changes in the energy content of the background field produced by them can be interpreted in terms of effective interactions between the bodies, mediated by the field. This is the conventional way of looking at the Casimir and van der Waals forces, for instance. However, it is clear that for perturbations of the field to take place, just one body is sufficient; one such instance is the spherical shell studied in §6.1.3. Another illustrative example is a polarizable atom, which modifies the energy of the vacuum by changing the dielectric properties of the medium; looking at the atom alone, one is led to interpret this as an *atomic* energy shift.

Such a simple consideration led Feynman (1961) to suggest that the atomic Lamb shift should be obtainable as the change caused by the polarizable atom on the vacuum energy. This suggestion was carried out in an elegant form by Power (1966b), who thus demonstrated the equivalence of Welton's heuristic derivation based on the zeropoint fluctuations [Welton 1948] and Bethe's QED result for the Lamb shift [Bethe 1947]. Barton (1972) extended the calculation to include thermal blackbody radiation, thus obtaining the temperature-dependent correction to the Lamb shift. In this section the same basic idea will be used to illustrate as simply as possible how the Lamb shift can be interpreted within the SED context. More complete treatments of this and other radiative corrections in SED, as well as calculations of the environmental effects on these radiative corrections, are presented in sections 7.5, 11.3 and 11.4.

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[11] Incidentally, a particularly interesting application of Lifshitz' technique is to the case of liquid helium [Dzyaloshinskii et al. 1961]. Owing to the small dielectric constant of this material ($\epsilon \simeq 1.057$), the Casimir force across an adsorbed liquid helium film turns out to be repulsive and thus gives rise to its remarkable climbing and wetting properties. Lifshitz's theory has also been applied successfully to a detailed study of the dispersion forces across the phospholipid layers that build up the biological membranes [Parsegian and Ninham 1970].

We recall from §6.1.4 that the oscillations of the atomic electrons in a dielectric material perturb the background field within the material. To a rough approximation, when the atoms can be modelled à la Drude-Lorentz, the bulk effect of this perturbation is described in terms of the refractive index $n(\omega)$ given by equation (6.44). Now consider N atoms of this dielectric to be contained within a bounded volume V; the refracting medium modifies the frequencies of the field modes contained in this volume, bringing about a change in the total energy content (at $T = 0$)

$$E_{FT} = \frac{1}{2}\hbar \sum_{\mathbf{n},\sigma} \omega \left(\frac{1}{n} - 1\right), \tag{6.73}$$

as suggested by Feynman. Assume the dielectric is a gas of very low density $\eta = N/V$; then $n - 1 << 1$, and the energy shift per atom is given approximately by

$$E_F = -\frac{1}{2N}\hbar \sum_{\mathbf{k},\sigma} \omega(n-1), \tag{6.74}$$

or using equation (6.44) to first order in e^2,

$$E_F = -\frac{\pi\hbar e^2}{mV} \sum_{\mathbf{k},\sigma} \frac{\omega_k}{\omega_0^2 - \omega_k^2}. \tag{6.75}$$

For a large volume the sum over **k** can be replaced by a triple integral, and after integrating over the solid angle one is left with

$$E_F = -\frac{\hbar e^2}{\pi m c^3} \int_0^\infty d\omega \frac{\omega^3}{\omega_0^2 - \omega^2}. \tag{6.76}$$

Now this energy shift contains two parts:

i) There is a free-particle contribution E_{FP}, independent of ω_0 and existing even for $\omega_0 = 0$

$$E_{FP} = \frac{\hbar e^2}{\pi m c^3} \int_0^\infty d\omega\, \omega; \tag{6.77}$$

this term comes from the quadratic contribution of the zeropoint field to the total energy of the particle, $(e^2/2mc^2)\langle \mathbf{A}^2 \rangle$, as we will have occasion to see in §7.2.3;

ii) the remaining contribution is the observable part of the energy shift and should therefore be identified with the Lamb shift

$$E_L = E_F - E_{FP} = -\frac{\hbar e^2 \omega_0^2}{\pi m c^3} \int_0^\infty d\omega \frac{\omega}{\omega_0^2 - \omega^2}. \tag{6.78}$$

The problems connected with the evaluation of this integral arise from the approximation made in writing down $n(\omega)$ only to first order in e^2. However, since at this point we are not interested in performing a more rigorous evaluation, we use the conventional procedure of determining it by replacing the infinite upper limit by a cutoff frequency $\omega_c >> \omega_0$, whence

$$E_L = \frac{\hbar e^2 \omega_0^2}{\pi m c^3} \ln \frac{\omega_c}{\omega_0}. \tag{6.79}$$

To compare with Bethe's result [Bethe 1947]

$$E_B = \frac{2e^2}{3\pi m^2 c^3} \sum_m \left|\mathbf{p}_{mn}\right|^2 \omega_{mn} \ln \left|\frac{\omega_c}{\omega_{mn}}\right|, \tag{6.80}$$

we write the latter assuming a single transition frequency $\omega_{mn} = \omega_0$, as we have assumed above; the Thomas-Reiche-Kuhn sum rule gives in this case $\left|\mathbf{p}_0\right|^2 \omega_0 = 3m\hbar\omega_0^2/2$, whence E_B indeed coincides with equation (6.79).

6.4. Effects of acceleration through the zeropoint field

As was seen in chapter 4, some of the well-established features of the zeropoint field in free space are its isotropy, its homogeneity and the Lorentz-invariance of its spectral density $\rho \sim \omega^3$. It is thanks to these properties that the field does not produce any friction on a rigid object moving with constant velocity, as shown in §4.2.2. Having said this, a most natural question that arises is: what happens if the object moves with a varying velocity, what will be the effect of the vacuum field on its motion? Will the object no longer see a true vacuum? One might in principle think that the zeropoint field for an accelerated observer would no longer be isotropic, and in particular that some difference would be detected between the forward and the backward directions. One might also expect the spectral properties of the field to change when being observed from an accelerated system, and this spectral change should then give rise to a frictional force. These and related questions have been analysed in SED and they will be the subject of the present section.

As an introduction to the subject, it is convenient to recall here related work carried out initially by Davies (1975) and Unruh (1976) in quantum field theory. In their relativistic analysis, motivated in large part by a desire to understand the phenomenon of black-hole radiance, they obtained a strikingly simple but remarkable result, namely, that a system undergoing a uniform acceleration of magnitude a relative to an inertial frame behaves as though it were immersed in a thermal bath at a temperature given by

$$k_B T = \frac{\hbar a}{2\pi c}. \tag{6.81}$$

It is interesting to note that this equivalent temperature turns out to be the same for all field modes, regardless of their frequency, i.e., it characterizes the field as a whole. Also, note that the value of T depends on Planck's constant, clearly indicating that the effect has its origin in the zeropoint field (or vacuum fluctuations, in the quantum language). Incidentally, to obtain a temperature reading as a result of this effect does not seem to be an easy matter; for instance, for temperatures of the order of 1°K one would need $a > 10^{20}\,\mathrm{m/s^2}$.

6.4.1. THE UNRUH-DAVIES EFFECT IN SED

In a series of papers on the subject, Boyer (1980b, 1984a, 1984e) has shown that the random zeropoint electromagnetic field also exhibits thermal-like effects when viewed by an accelerated observer. The derivation of these effects goes basically as follows [Boyer 1984a].

Assume a small harmonic electric-dipole oscillator of frequency ω_0. We are interested in the situation where this oscillator experiences a uniform acceleration through the zeropoint field; therefore we suppose that an external force F_x exists which causes a uniform acceleration $a = a_x = F_x/m$ of the equilibrium point of the spring. Even if a relativistic calculation is not required because high velocities are not being considered, there are effects of relativistic origin that will play a role. One must therefore start from the Lorentz-Dirac equation, which is the relavistic version of the Abraham-Lorentz equation of motion [see, e.g., Rohrlich 1965]. Thus, for the dipole in an inertial frame (with $\tau_0 \equiv 2e^2/3mc^3$, to avoid confusion)

$$m\ddot{x}^\mu = -F^\mu + m\tau_0\left[\dddot{x}^\mu + \frac{1}{c^2}\dot{x}^\mu\ddot{x}^\nu\ddot{x}_\nu\right] + \frac{e}{c}\mathcal{F}^{\nu\mu}\dot{x}_\nu, \tag{6.82}$$

where F^μ is the four-force due to the spring, $\mathcal{F}^{\nu\mu}$ is the electromagnetic field tensor, and the overdots refer to differentiation of the displacement four-vector x^μ with respect to the particle proper time. For simplicity of calculation, the oscillations are assumed to be restricted to the yz plane. After a transformation to the coordinate frame S carried along by the accelerating dipole one obtains

$$m\frac{d^2\mathrm{x}}{d\tau^2} = -m\omega_0^2\mathrm{x} + m\tau_0\left[\frac{d^3\mathrm{x}}{d\tau^3} - \frac{a^2}{c^2}\frac{d\mathrm{x}}{d\tau}\right] + e\mathbf{E}_\tau(0,\tau), \tag{6.83}$$

where $\mathbf{x} = y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is the relative displacement of the particle in the S frame and τ is the time at the origin of this frame. In the small-dipole approximation one can ignore the distinction between τ and the particle proper time used in equation (6.82). Note that there is a relativistic correction to the radiation reaction force, proportional to the velocity of the particle, with a relative coefficient $(\ddot{x}^{\nu}\ddot{x}_{\nu})/c^2 \simeq -(a^2/c^2)$. The driving electric force $e\mathbf{E}_{\tau}(0,\tau)$ comes from the zeropoint field seen in the inertial frame I_{τ} that moves with the oscillator equilibrium point at time τ. This field may be found by Lorentz transformation from the random field seen in an inertial frame I_* fixed at $t = 0$, given the relations between coordinates for a motion with constant acceleration; thus,

$$X_*(\tau) = \frac{c^2}{a}(\cosh\frac{a\tau}{c} - 1), \quad Y_*(\tau) = Z_*(\tau) = 0, \tag{6.84}$$

$$t_*(\tau) = \frac{c}{a}\sinh\frac{a\tau}{c}. \tag{6.85}$$

As a result of this transformation, the electric field changes into

$$\mathbf{E}_{\tau}(0,\tau) = \frac{i}{2\pi}\sum_{\sigma=1}^{2}\int d^3k\sqrt{\frac{\hbar\omega}{2}}$$

$$\left\{\hat{\imath}\epsilon_x + \hat{\jmath}[\cosh\frac{a\tau}{c}\epsilon_y - \sinh\frac{a\tau}{c}(\hat{k}\times\hat{\epsilon})_z] + \hat{k}[\cosh\frac{a\tau}{c}\epsilon_z + \sinh\frac{a\tau}{c}(\hat{k}\times\hat{\epsilon})_y]\right\}$$

$$\times \exp\left(-i[\frac{\omega c}{a}\sinh\frac{a\tau}{c} - \frac{k_x c^2}{a}\cosh\frac{a\tau}{c} + \theta(\mathbf{k},\sigma)]\right) \tag{6.86}$$

and its two-time correlation becomes

$$\langle E_{\tau i}(0,\tau)E_{\tau' j}(0,\tau')\rangle =$$

$$\delta_{ij}\frac{2\hbar}{3\pi c^3}\int_0^{\infty} d\omega\,\omega^3\left[1 + \left(\frac{a}{c\omega}\right)^2\right]\coth\left(\frac{\pi c\omega}{a}\right)\cos\omega(\tau - \tau'). \tag{6.87}$$

This correlation function depends only upon the time difference $\tau - \tau'$, which means that the stochastic driving term in equation (6.83) is still a stationary random process. In addition, note that in the mean the homogeneity and the isotropy of the field are preserved and only its spectrum has changed.[12]

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[12] Before the integration over the solid angle in $\mathbf{k}$–space is carried out, however, the correlations still depend on the angle between $\mathbf{k}$ and $\mathbf{a}$, which means that there is an anisotropy in the distribution of the radiation as seen by the accelerated detector, although it does of course not show up in the formula for the energy density; see Hinton et al. (1983). It is thanks to this anisotropy that the accelerated particle can be expected to meet a resistive force, as mentioned at the end of §6.4.2.

The stationary solution of the equation (6.83) can be readily obtained by means of a Fourier transformation, giving for the square average value of y and z

$$\left\langle y^2(\tau)\right\rangle = \left\langle z^2(\tau)\right\rangle = \frac{\hbar}{\pi m}\int_{-\infty}^{\infty} d\omega \frac{\tau_0\omega^3\left[1+(a/c\omega)^2\right]}{|\Delta_a|^2}\coth\left(\frac{\pi c\omega}{a}\right), \quad (6.88)$$

with

$$\Delta_a = \omega_0^2 - \omega^2 - i\tau_0\omega^3\left[1+(a/c\omega)^2\right]. \quad (6.89)$$

Owing to the smallness of the imaginary term in Δ_a, the integrand of (6.88) is sharply peaked at $\omega = \omega_0$. Using this to evaluate approximately the integral, the result is

$$\left\langle y^2(\tau)\right\rangle = \left\langle z^2(\tau)\right\rangle = \frac{\hbar}{2m\omega_0}\coth\left(\frac{\pi c\omega_0}{a}\right); \quad (6.90)$$

similarly one obtains $\langle (dy/d\tau)^2\rangle = \langle (dz/d\tau)^2\rangle = (\hbar\omega_0/2m)\coth(\pi c\omega_0/a)$. The resulting average energy of the oscillator

$$\mathcal{E}_y(\omega,a) = \mathcal{E}_z(\omega,a) = \frac{1}{2}\hbar\omega_0\coth\left(\frac{\pi c\omega_0}{a}\right) \quad (6.91)$$

coincides with the average energy of an oscillator in an inertial frame embedded in a thermal radiation field, equation (5.33), with T given by the Unruh-Davies formula (6.81).

It is clear from these calculations that the 'thermal' factor appearing in (6.91) comes about from a modification of the spectrum of the Lorentz-transformed field represented by $\mathbf{E}_\tau(0,\tau)$. Actually, according to equation (6.87) the complete formula for the corrected spectral energy density is

$$\rho(\omega;a) = \frac{\hbar\omega^3}{2\pi^2c^3}\left[1+\left(\frac{a}{c\omega}\right)^2\right]\coth\left(\frac{\pi c\omega}{a}\right); \quad (6.92)$$

thus, by comparing with equations (4.10) and (4.11), we see that it contains a further multiplying factor $[1+(a/c\omega)^2]$ which is due to a modification of the density of states

$$\rho(\omega;a) = \frac{\omega^2}{\pi^2c^3}\left[1+\left(\frac{a}{c\omega}\right)^2\right]\mathcal{E}(\omega,a) = \mathcal{N}(\omega,a)\mathcal{E}(\omega,a), \quad (6.93)$$

with

$$\mathcal{N}(\omega,a) = \mathcal{N}(\omega)\left[1+\left(\frac{a}{c\omega}\right)^2\right]. \quad (6.94)$$

However, this very same factor appears in the denominator Δ_a, as a result of the additional radiation-reaction term in equation (6.83); hence it cancels out upon integration of equation (6.88). Here we have an instance of the action of the fluctuation-dissipation theorem in SED:[13] the same geometrical correction factor due to the acceleration of the system appears in both the dissipative term and the mode density of the field, and therefore cancels out in the formula for the energy per mode.[14]

One can lift the constraint of perpendicular oscillations on the yz-plane, by using a coordinate system that is Fermi-Walker transported[15] along the path of equilibrium point of the string [Cole 1985]. The equation obtained for the oscillations along the direction of acceleration (i.e., along the x-axis) is again of the form (6.83), except that the oscillation frequency in this direction is shifted to a slightly lower value,

$$\omega_x = \omega_0\sqrt{1-(a/\omega_0 c)^2}, \tag{6.95}$$

due to the fact that the proper time of the oscillating particle does not coincide any more with the proper time of the equilibrium point. Once again, though, the effective background field is homogeneous, isotropic, and its energy distribution is Planckian with an equivalent temperature given by $T = \hbar a/2\pi ck$.

An analysis similar to the above one has been made for a spinning magnetic dipole subject to a constant magnetic field and undergoing uniform acceleration in the zeropoint field [Boyer 1984e]. The basic theory of the magnetic dipole is set forth in section 8.2; just for reference purposes let us mention here the effect of acceleration on the dipole energy. Starting from Bhabha's relativistic equation of motion for the spinning magnetic dipole [Bhabha 1940, Corben 1968], and assuming that the particle has no electric dipole moment in its own instantaneous rest frame, one obtains the following equation for the accelerated dipole,

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[13] The fluctuation-dissipation theorem is touched upon in sections 7.2 and 11.2. For a general discussion of it we refer the reader to Reichl (1980).

[14] It has been suggested by Boyer (1984b, 1985) that the thermal-like effect of acceleration on the energy of an oscillator as expressed in equation (6.91) can be understood by resorting to the equivalence principle. Using $\rho_a(\omega) = (\omega^2/\pi^2c^3)\left[\frac{1}{2}\hbar\omega\coth(\pi c\omega/a)\right]$ for the spectrum of the field that sustains the motion of the oscillator (either in a local gravitational field, or in an accelerated frame with $\mathbf{a} = -\mathbf{g}$), and calculating the excess radiation energy density produced by the acceleration $u_T = \int_0^\infty d\omega\,[\rho_a(\omega) - \rho_0(\omega)]$, Stefan's law $u_T = \sigma T^4$ is obtained with T given by the Unruh-Davies formula. Arguing further that if ρ_g is the spectrum seen by the oscillator in a gravitational field, it would become possible to construct a perpetual-motion machine based on ρ_g unless it corresponds to the equilibrium spectrum, Boyer considers the above as a derivation of the blackbody radiation spectrum from the equivalence principle. Note, however, that $\rho_a(\omega)$ differs from the spectral energy density given in (6.92). Other aspects of the link between the Unruh-Davies effect and the equivalence principle are discussed in Candelas and Sciama (1983).

Further, it has been suggested to consider gravitation itself as a residual effect of the vacuum fluctuations in curved space [Sakharov 1967]; for a calculation within SED based on this suggestion see Puthoff (1989a) and comments in Carlip (1993).

[15] See, e.g., Misner et al. (1973), pp. 169-174.

$$\frac{d\mathbf{S}}{d\tau} = \boldsymbol{\mu} \times \mathbf{B}_0 - \frac{2}{3c^3}\boldsymbol{\mu} \times \left[\frac{d^3\boldsymbol{\mu}}{d\tau^3} - \left(\frac{a}{c}\right)^2 \left(\frac{d\boldsymbol{\mu}}{d\tau} - \hat{\mathbf{n}}\hat{\mathbf{n}} \cdot \frac{d\boldsymbol{\mu}}{d\tau}\right)\right] + \boldsymbol{\mu} \times \mathbf{B}(0, \tau), \tag{6.96}$$

where $\boldsymbol{\mu} = g\mathbf{S}$, $\mathbf{B}_0 = B_0\hat{\mathbf{n}}$ is a fixed magnetic field and $\hat{\mathbf{n}}$ is an arbitrary unit vector. Observe that also here, a relativistic term proportional to $(a/c)^2$ appears contributing to the radiation reaction force. The magnetic component of the random field $\mathbf{B}(0, \tau)$ in the inertial frame I_τ is again found by a Lorentz transformation; the corresponding field spectrum is once more given by equation (6.92). An approximate form for the stationary solution to the nonlinear equation (6.96) is obtained by assuming $(d\boldsymbol{\mu}/d\tau)$ and $(d^3\boldsymbol{\mu}/d\tau^3)$ to be given by the unperturbed precession. As a result, it turns out that the accelerated spinning magnetic dipole is also subject to a thermal-like background field, with an average energy per normal mode given precisely by equation (6.91).

The effects of acceleration described above for pointlike electromagnetic systems have been shown to hold for a particular spatially extended situation [Cole 1987], consisting of two small dipole oscillators with their equilibrium positions lying on a plane perpendicular to the direction of acceleration. The reader might recall that in the absence of acceleration, a complete calculation of the interaction energy of this system gives the van der Waals force equation (6.66). For an accelerated system the calculation is made using once more a Fermi-Walker transported coordinate system, and introducing the 'small-laboratory' restriction $Ra << c^2$ to ensure that a light ray that goes from one dipole to the other travels a distance along the direction of acceleration, $aR^2/2c^2$, much smaller than the distance c^2/a to the event horizon. Under the unretarded van der Waals condition, $\omega_0 R << c$, the expectation value of the Lorentz force along the axis separating the two dipoles is found to agree with the corresponding van der Waals force of an inertial pair of oscillators subject to a thermal field at the Unruh-Davies temperature.

Phenomena as those described above can alternatively be seen as produced by changes in the space-time geometry of the system, and from this perspective they fall under the heading of generalized Casimir effects. Using this point of view, the study of the effect of acceleration on zeropoint energy has been extended to other systems, in particular to systems undergoing uniform rotation. The spectral density obtained in this case is [Hacyan and

Sarmiento 1986]

$$\rho(\omega; a) = \frac{\hbar\omega^3}{2\pi^2 c^3}\left[1 + \left(\frac{\gamma\beta\Omega}{\omega}\right)^2\right] + \frac{\hbar\Omega^3}{16\pi^3 c^3}\int_0^\infty dx\, F_\beta(x)\cos\left(\frac{2\omega x}{\Omega}\right) \quad (6.97)$$

where Ω is the angular velocity, $\beta = v/c$, $\gamma = (1-\beta^2)^{-1/2}$ and

$$F_\beta(x) = (1-\beta^2)\frac{(3+\beta^2)x^2 + (1+3\beta^2)\beta^2\sin^2 x - 8\beta^2 x\sin x}{(x^2-\beta^2\sin^2 x)^3} - \frac{3}{x^4} + \frac{2\gamma^2\beta^2}{x^2}. \quad (6.98)$$

The factor in square brackets in (6.97) modifies once more the density of states, with $\gamma\beta\Omega = a/c$; however, the modification of the spectrum in this case is clearly not Planckian. A numerical evaluation of the additional term in equation (6.97) shows that it is an increasing function of β, but it becomes significant only for ultrarelativistic velocities.

We should mention that more general studies have been made of the effect of acceleration on massless fields of arbitrary spin. In the case of linear aceleration, the ‘thermal’ factor appearing in the vacuum energy has the form of a Fermi-Dirac function or a Bose-Einstein function, depending on whether the field spin is a half-integer or an integer [Hacyan 1985]; for circular motion, however, the energy spectrum is not of the thermal form [Kim et al. 1987a, b]. These results, as all the previous ones, do not depend on whether the fields are classical or quantized, since a formal equivalence exists between them for any spin. This equivalence is based on the fact that the Wightman functions used for the quantized fields coincide with the corresponding autocorrelation functions of the random classical fields.

6.4.2. IS THE ACCELERATED OBSERVER EXCITED BY THE VACUUM FLUCTUATIONS?

According to the previous results, the zeropoint field as seen by an accelerated observer has an energy per normal mode different from that seen by an inertial observer. To what extent does this affect the state of the accelerated observer and what kind of response can this effect produce?

In quantum theory two intuitively natural ways to define a vacuum state are frequently used: as the state of lowest energy, or as the state devoid of field quanta. Normally the two criteria coincide and there is no need to distinguish between them; however, this is not the case in an accelerated frame. Let us take the second criterion. In the usual quantum analysis, the vacuum is considered as containing fluctuations but no photons; in contrast, a thermal spectrum (or, for that matter, any spectrum different from that of a pure vacuum) does indeed involve photons. In this context, the Unruh-Davies effect poses a dilemma, namely, how is it possible by acceleration to go from a situation of no photons to a situation where there are photons, or at least there seem to be photons? An (atomic) observer subject to a thermal background field is excited by it; one might therefore conclude that induced absorptions and emissions will take place in the accelerated frame, which can then be recorded with a detector. This would mean that the so-called virtual photons of the quantum vacuum are made real by acceleration, i.e., that the quantum fluctuations are 'promoted' to the level of real thermal fluctuations [Milonni 1988, Sciama 1991].

Let us look into this question from the perspective of SED. According to this theory, not only the transitions between atomic states originate in the interaction between matter and field, but the atomic states themselves are a product of this interaction. In particular, the ground state of a system corresponds to the state in equilibrium with the random field in its ground state (see section 7.1). Naturally, neither (discrete) absorptions nor emissions occur as long as both matter and field remain in their respective ground states.

Now from the point of view of a uniformly accelerated detector, the zeropoint field with its energy per normal mode locally transformed according to equation (6.91) is the new zeropoint field; this is the field of lowest possible energy content, obtained in the limit as the (thermodynamic) temperature $T \rightarrow 0$. It is of the same nature as the original zeropoint field; it is still a stationary stochastic process with an overall homogeneous and isotropic distribution, but with a spectrum that is of course not Lorentz-invariant any more. The fact that the ground level of this modified field has a higher average energy per normal mode in the accelerated frame, means that the ground-state energy of the accelerated oscillator in equilibrium with it is correspondingly increased to the value given by equation (6.91). This energy shift can be expected to produce in its turn a shift of the spectral lines, but it should not induce atomic absorptions from the zeropoint field.

According to the heuristic calculation of §4.1.3 the Bohr radius for the H atom also becomes an increasing function of the acceleration, and all atoms are subject to a similar scaling phenomenon. This seems to suggest that in general, the accelerated detectors are *not* robust systems, contrary to what is normally assumed in quantum theory [Sciama et al. 1981, Bell and Leinaas 1983]: when they are immersed in a locally distorted zeropoint field (i.e., distorted only from their point of view) they are distorted by it. However, they do not undergo radiative transitions, as long as the balance between the power lost by radiation reaction and the power gained from the background field is maintained.[16]

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[16] Since the frictional force of the background field on a (nonaccelerated) dipole of frequency ω vanishes only if $\rho \sim \omega^3$, a dipole which is accelerated through the zeropoint field is expected to meet a resistance due to the distorted spectrum of equation (6.92). Haisch

This balance is broken when the detector is suddenly transferred to a reference system with a different acceleration, or more generally when it moves with a variable acceleration. Under these circumstances one would indeed expect radiative processes to occur, until the detector reaches a new state of equilibrium with the corresponding background field. Such complicated situations —which are the physically realizable ones, as clearly pointed out in Barton and Eberlein (1993)— have not been specifically addressed in SED.

6.5. Casimir energy in other fields of physics

The investigation of the Casimir effect in conducting and dielectric media has stimulated renewed interest because of its possible application in other fields of physics, particularly in elementary–particle theory. A first attempt in this direction was made by Wentzel (1941) who, already before Casimir, used the method of zeropoint energy to calculate the forces in the meson-pair theory of fixed sources; more recently, a possible application to quantum chromodynamics has been analyzed [see Plunien et al. 1986 and references therein]. The idea that hadrons can be described as confined fermion and boson fields, the fields of quarks and gluons, respectively, has become generally accepted, although the precise nature of the confinement mechanism remains unknown. In the framework of the phenomenological bag model, which represents a widely used approximation, one has exactly the situation where the various vacuum fields are forced to satisfy certain boundary conditions on the bag surface, in order to guarantee confinement. In this context, the result of a Casimir binding force for spherical shells of very small dimension, equation (6.33), constitutes an interesting possibility.

Using a more general concept of Casimir effect within quantum field theory, systems have been considered where external fields created by an arbitrary source configuration play the role of external constraints to the vacuum. One example of such a situation is the vacuum energy of the Dirac electron-positron field in the presence of an external electromagnetic field [Ambjørn and Wolfram 1983, Plunien et al. 1986, Jáuregui et al. 1991, Grib et al. 1994]. Further, the role of vacuum fluctuations in spacetimes with a nontrivial topology has been studied as a mechanism leading to com-

et al. (1994a, b) have recently applied this idea in an appealing attempt to explain the origin of the inertia of particles, under the hypothesis that matter is ultimately constituted by primary charged entities or 'partons' bound in the manner of oscillators. According to their assumptions, the resistive Lorentz force comes out precisely proportional to the acceleration, and they interpret the factor of proportionality as the inertial mass m of the particle, which turns out to be inversely proportional to the partons' bare mass. Their frequency of oscillation, however, must be extremely high (of the order of the Planck frequency $\omega_P = (c^5/\hbar G)^{1/2}$, where G is the Newtonian gravitational constant) for m to acquire the correct value. For some informal comments on this attempt see Powell (1994).

pactification of extra dimensions in Kaluza-Klein theories; in these models, Casimir forces shrink the extra dimensions to a null value [Candelas and Weinberg 1984, Grib et al. 1994].

6.5.1. DYNAMICAL CASIMIR EFFECT

A problem that has recently aroused some interest is the so-called dynamical Casimir effect produced between moving boundaries. The problem can be stated in simple terms using the parallel-plate example, as follows. Assume that the distance between plates is made to vary as a function of time, $R = R(t)$; then also the frequencies of the allowed modes between the plates become time-dependent, as follows from equation (6.3),

$$\omega_{\mathbf{n}}(t) = \pi c\sqrt{\frac{l^2 + m^2}{L^2} + \frac{n^2}{[R(t)]^2}}. \tag{6.99}$$

In the adiabatic approximation, i.e., assuming that the motion of the walls is slow compared with the field oscillations, the average energy of every field mode can be shown to be still given by (see §3.1.3) [Calucci 1992]

$$E_{\mathbf{n}} = \langle H_n \rangle = \frac{1}{2}\left\langle p_{\mathbf{n}}^2 + \omega_{\mathbf{n}}^2 q_{\mathbf{n}}^2 \right\rangle = \frac{1}{2}\hbar\omega_{\mathbf{n}}\left\langle |a_{\mathbf{n}}|^2 \right\rangle = \frac{1}{2}\hbar\omega_{\mathbf{n}}; \tag{6.100}$$

the average energy of the zeropoint field trapped in the box is therefore given by the triple sum over modes

$$E(R(t)) = 2\sum_{\mathbf{n}}{}' \frac{1}{2}\hbar\omega_{\mathbf{n}}(t). \tag{6.101}$$

By carrying out the calculations of §6.1.1 one obtains of course the (now time-dependent) results reported there for the Casimir energy.

However, a nonadiabatic motion of the plates gives rise to extra contributions to the Casimir energy. For example, in the particular case of two parallel plates which move inwards or outwards with constant speed $v \ll c$, the result obtained in the usual approximation ($k_c R \gg 1$) is [Villarreal et al. 1995]

$$E_C(R(t)) = -\frac{\pi^2\hbar c L^2}{720[R(t)]^3}[1 - \frac{40v^2}{\pi^2 c^2}]; \qquad k_c R \gg 1,\ |v/c| \ll 1. \tag{6.102}$$

The first term reproduces the static Casimir result (6.13); the second term represents a correction due to the motion of the plates. It is clear from this formula that energy must be injected into the system to separate the plates and energy is liberated when the plates approach each other. What is not clear, however, is exactly in which form this energy liberation comes about. A conventional quantum calculation in the Heisenberg picture, using time-dependent creation and annihilation operators acting on a fixed vacuum, predicts that photons are produced [Man'ko 1992, Sassaroli et al. 1994]. In the case of a periodic modulation of the plates, the estimated intensity per pulse can be considerable for small plate separations. It has been suggested by Schwinger (1992) that the intense blue light produced in the sonoluminescence phenomenon[17] might be due to this dynamical Casimir effect [see also Eberlein 1995], although other, more mundane explanations to the phenomenon have been advanced [Barber and Putterman 1992]. We face here once more the problem of an ambiguous description of the time evolution of the field, i.e., of the choice of the creation and annihilation operators and the (vacuum) state vector on which they act. As observed by Moore in his pioneering quantum work on the subject [Moore 1970], in the vicinity of a moving mirror 'the very concept of photons becomes muddy, just because the absence of photons (namely, a time-translationally invariant vacuum state) cannot be defined'.

Within SED, the possibility of extracting useful energy from zeropoint fluctuations, taking advantage of the Casimir effect, has been considered by several authors. A simple model to achieve this was studied by Forward (1984) and consists of a charged foliated conductor. Since the attractive force between plates is proportional to R^{-4}, whereas the Coulomb repulsion is proportional to R^{-1}, the Casimir force is expected to dominate at very small plate separations, producing a concentration of the overall charge distribution as the plates of the foliated conductor come together. The zeropoint energy would be thus transformed into stored electrical energy, which could be extracted by a variety of means, e.g., by charging a battery. An alternative method that is considered potentially more practical involves a charged plasma [Puthoff 1990b]. Recently, the problem has been studied more generally by Cole and Puthoff (1993), who conclude that the possibility of finding mechanisms to extract energy and generate heat from the zeropoint field is not excluded in principle, at least not from the point of view of thermodynamics. In their very simple gedankenexperiment, the heat arises from Casimir forces in an irreversible process involving the pairwise collisions between uncharged capacitor plates, which are afterwards discarded as waste. Likewise, by means of a quasistatic displacement of the plates, work can be done on a probe, which could be transformed into useful energy. Both processes are shown to be thermodynamically feasible.

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[17] Sonoluminescence is a non-equilibrium phenomenon in which the energy of a sound wave becomes highly concentrated so as to produce the emission of light by collapsing bubbles of gas trapped in a liquid. The light bursts can carry an energy corresponding to over 10^5 photons in less than 100 ps, representing an amplification of energy by 11-12 orders of magnitude [see Barber and Putterman 1992 and references therein].

In considering these possibilities, however, it is important to pay attention to the behaviour of the zeropoint energy as a function of the distance for all distances, particularly those small distances at which the Casimir effect is expected to overcome Coulomb repulsion. For example, as is illustrated in figure 6.2, at a separation smaller than k_c^{-1} the force between two parallel conducting plates becomes repulsive, and hence it can no longer be used to counterbalance the Coulomb force. The geometry of the energy-extracting device will surely be a matter of careful calculations. Moreover, at such small scales other phenomena probably occur that could prevent the possibility of extracting the energy in a useful way before it is radiated, as suggested by the above discussion.

CHAPTER 7

THE HARMONIC OSCILLATOR

The harmonic oscillator provides a unique opportunity to study with simple mathematics the properties acquired by a mechanical system in permanent contact with the zeropoint field, and to assess with a specific example the merit of the assumption that the classical particle becomes through this interaction a quantum object. Of course, due to its linearity the oscillator is far from being a fair representative of the general quantum system, but nevertheless the importance of getting a concrete expression of the relationship between quantum theory and SED can hardly be overestimated.

In this chapter we shall be concerned with some of the most important features of the SED oscillator in the time-asymptotic limit. The comparison with the corresponding quantum predictions reveals a surprising agreement in many properties, which extends so far as to include radiative corrections —Lamb's shift and natural linewidth, etc.—, even though the physical imagery underlying the two descriptions differs so radically. All through the chapter the treatment is nonrelativistic and in one dimension; the transition to more dimensions is usually (though not always) immediate, and it is explicitly made only when required by the physical situation. Other simple problems, including a few additional applications of the harmonic oscillator to various physical systems, are left for chapter 8. A review with references on the subject up to 1982 can be found in de la Peña (1983).

7.1. Elementary theory of the harmonic oscillator

7.1.1. THE BRAFFORT-MARSHALL EQUATION

For a study of the SED harmonic oscillator it suffices to insert into the total Hamiltonian of equation (3.85) the potential $V = \frac{1}{2}m\omega_0^2 x^2$ (instead of $e\Phi$) and use the zeropoint field discussed in chapter 4. However, a more practical and usual approach consists in starting directly from the Abraham-Lorentz equation (3.103), as will be done here (note that the mass, which we write simply as m, refers then to the renormalized mass m_T in that equation). Hence,

$$m\ddot{x} = -m\omega_0^2 x + m\tau \dddot{x} + eE_x(\mathbf{x},t) + e\left(\frac{\dot{\mathbf{x}}}{c}\times\mathbf{B}\right)_x \tag{7.1}$$

with $\tau = 2e^2/3mc^3$. This particular form of the Abraham-Lorentz equation applied to SED is usually called the *Braffort-Marshall equation* and is the analog of the *Langevin equation* in Brownian motion.[1] It is normally assumed that under the conditions in which SED applies to atomic systems one may approximate $\dddot{x}$ by $-\omega_0^2\dot{x}$ and write

$$m\ddot{x} = -m\omega_0^2 x - m\tau\omega_0^2\dot{x} + eE_x(\mathrm{x}, t) + e\left(\frac{\dot{\mathbf{x}}}{c} \times \mathbf{B}\right)_x . \tag{7.2}$$

One should beware that this approximation is not always legitimate and there are specific circumstances where the precise structure of the radiation force is essential.[2] When applicable, however, the simplification is very handy because it simultaneously eliminates the noncausal behaviour implied in (7.1) and reduces the order of the equation. An apparent drawback is that some integrals over the frequency, which will eventually appear, become (more) divergent with this approximation; but as discussed at length in section 3.2, the introduction of a cutoff is an unavoidable requirement, so that this consideration becomes irrelevant in practice. Most of the chapter will therefore be based on this approximate causal form of the equation of motion.

Now, equation (7.2) is nonlinear and cannot be solved analytically in a closed form. However, in the nonrelativistic description all terms of order ≥ 1 in $\dot{x}/c$ can be neglected, so that assuming that the most important contributions to the motion come from the optical region of the spectrum or not very far from it, one may use the dipolar approximation, as discussed in §3.2.1. That one cannot always neglect the spatial dependence of the field is shown clearly by our study of the Casimir and related forces in chapter 6 (as well as some examples discussed in the present chapter); but in the simpler cases of interest the long-wavelength approximation is applicable and very useful, because it leads to the linear equation of motion

$$\ddot{x} = -\omega_0^2 x - \gamma\dot{x} + \frac{e}{m}E_x(t) \tag{7.3}$$

with

$$\gamma = \tau\omega_0^2. \tag{7.4}$$

It is helpful to rewrite this second-order equation as a system of two coupled first-order equations; this can be done in different ways, the two most frequent ones being the following.

a) *Mechanical variables.* The linear momentum is defined as the usual mechanical momentum of the particle, $p = m\dot{x}$. From the equation of motion it follows that

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[1] See, e.g., Wang and Uhlenbeck (1954), Papoulis (1965) or van Kampen (1981).

[2] An important instance is studied in chapter 11.

$$\dot{x} = \frac{1}{m}p, \quad \dot{p} = -\gamma p - m\omega_0^2 x + eE_x(t). \tag{7.5}$$

b) *Canonical variables.* The linear momentum is defined through $\dot{p} = F = -m\omega_0^2 x$.[3] Substituting in the original equation of motion and noting that $E = -(\partial A/c\,\partial t)$ in the Coulomb gauge, after an elementary integration one is led to the system of equations

$$\dot{x} = \frac{1}{m}p - \gamma x - \frac{e}{mc}A_x(t), \quad \dot{p} = -m\omega_0^2 x. \tag{7.6}$$

This is the set of variables that will be used preferentially in what follows. In the above equations the field terms represent the zeropoint field plus any external radiation field (in the Coulomb gauge).

For applications in free space the fields are appropriately represented in terms of plane waves, as in equation (4.38). In particular, using (3.69) for the transition to the continuum we have for the vacuum field

$$E_x(t) = -\frac{1}{c}\frac{\partial A_x}{\partial t} = i\sum_\lambda \int d^3k\sqrt{\frac{\hbar\omega_k}{8\pi^2}}\mathrm{e}_x^\lambda(\mathrm{k})\left(a_\lambda(\mathrm{k})e^{-i\omega_k t} - a_\lambda^*(\mathrm{k})e^{i\omega_k t}\right), \tag{7.7}$$

or in a more concise notation:

$$E_x(t) = \sum_\lambda \int d^3k\tilde{E}_x(\mathrm{k},\lambda)a_\lambda(\mathrm{k})e^{-i\omega_k t} + \mathrm{cc}, \quad \tilde{E}_x(\mathrm{k},\lambda) = i\sqrt{\frac{\hbar\omega_k}{8\pi^2}}\mathrm{e}_x^\lambda(\mathrm{k}). \tag{7.8}$$

The solution of equation (7.3) can be written as the sum of two independent terms $x = x_t + x_s$, where x_t represents the transient part and x_s the stationary motion. The transient part depends on arbitrary initial conditions, oscillates with a frequency very close to ω_0, is independent of the driving field, and decays with time as $\exp(-\gamma t)$. The time constant of this decay, $T_{\mathrm{rr}} \equiv \gamma^{-1}$, though small on a macroscopic scale (of the order of $10^{-6} - 10^{-7}$s for optical frequencies), is very large compared with atomic characteristic times (by a factor of $\sim \alpha^{-3}$), so that it is necessary to wait a long time by atomic standards for the transient to fade away. Eventually the solution reduces to x_s and the system reaches a stationary state of motion fixed by the driving force, i.e., by the random field. It is to this stationary motion to which we shall direct our main attention, leaving aside for the moment the term x_t; thus, in what follows $x(t)$ refers to the stationary part x_s. As we will have ample opportunity to verify, this is the part of the solution that leads to results comparable to quantum mechanics, and particularly to the quantum stationary states; the initial transient, in contrast, can have a significantly different behaviour. This means that the quantumlike properties are gradually acquired by the oscillator through its interaction with the background field.

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[3] Strictly speaking, the canonical momentum associated with the coordinate x is defined as $p = \partial L/\partial x$, where L is the total Lagrangian. This leads to the same expression as that introduced in the text on more intuitive grounds.

For a stationary state $x(t)$ (the former x_s) admits a Fourier expansion, so that one can write

$$x(t) = \sum_{\lambda} \int d^3k \left(\tilde{x}_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) e^{-i\omega_k t} + \tilde{x}^*_\lambda(\mathbf{k}) a^*_\lambda(\mathbf{k}) e^{i\omega_k t} \right). \tag{7.9}$$

From this and equation (7.3) it follows that

$$\tilde{x}_\lambda(\mathbf{k}) = \frac{e}{m} \frac{\tilde{E}_x(\mathbf{k}, \lambda)}{\Delta(\omega_k)}, \quad \Delta(\omega_k) = \omega_0^2 - \omega^2 - i\gamma\omega. \tag{7.10}$$

The two roots of Δ, which will be frequently required, are $\omega_\beta, -\omega_\beta^*$, where

$$\omega_\beta = -i\beta + \omega_1, \quad \beta = \frac{\gamma}{2} = \frac{1}{2}\tau\omega_0^2, \quad \omega_1^2 = \omega_0^2 - \frac{\gamma^2}{4}. \tag{7.11}$$

For all practical purposes ω_1 and ω_0 can be considered to be equal, though we shall distinguish them in writing. Upon substitution the stationary solution takes the form

$$x(t) = \frac{e}{m} \sum_{\lambda} \int d^3k \left(\frac{\tilde{E}_x(\mathbf{k}, \lambda)}{\Delta} a_\lambda(\mathbf{k}) e^{-i\omega_k t} + \frac{\tilde{E}^*_x(\mathbf{k}, \lambda)}{\Delta^*} a^*_\lambda(\mathbf{k}) e^{i\omega_k t} \right). \tag{7.12}$$

7.1.2. STATISTICS OF THE STATIONARY SOLUTION

The statistical properties of the stationary solution are determined by the properties of the random amplitudes $\{a, a^*\}$ discussed in chapter 4. In particular, from equation (4.44) the covariance of the components of the electric field is given by

$$\Gamma_{E_x E_x}(s - t) = \langle E_x(t) E_x(s) \rangle = \sum_{\lambda} \int d^3k \left| \tilde{E}_x(\mathbf{k}, \lambda) \right|^2 e^{i\omega_k (s-t)} + \text{c.c.}$$

$$= \frac{\hbar}{3\pi c^3} \int_0^\infty d\omega\, \omega^3 e^{i\omega_k (s-t)} + \text{c.c.} = \frac{4\pi}{3} \int_0^\infty d\omega \rho_0(\omega) \cos\omega(s - t), \tag{7.13}$$

where $\rho_0(\omega) = \hbar\omega^3/2\pi^2 c^3$ is the spectral density of the zeropoint field. As mentioned in chapter 4, this result is frequently expressed in terms of the power spectrum in the form

$$\Gamma_{E_x E_x}(t) = \int_0^\infty d\omega S_0(\omega) \cos\omega t, \quad S_0(\omega) = \frac{4\pi}{3}\rho_0(\omega) = \frac{2\hbar\omega^3}{3\pi c^3}. \tag{7.14}$$

Note that since the amplitudes a_λ, a_λ^* average to zero, also $\langle x(t) \rangle = 0$. We are interested in calculating covariances of the type $\Gamma_{xx}(s+t, s) = \langle x(s+t)x(s) \rangle$; since the process is stationary, they depend only on the time difference t, so that $\Gamma_{xx}(s+t,s) = \Gamma_{xx}(t)$ [see, e.g., Papoulis 1965]. It is simplest to start by calculating the covariance of the linear momentum, because according to equation (7.6), $p(t)$ is given by the time integral of $x(t)$, which introduces an extra factor ω^{-1}, allowing for a faster convergence of the integrals. In contrast, when the mechanical variables are used, the factor in the integrand becomes ω instead of ω^{-1} due to the time derivative in the first of equations (7.5), thus making the corresponding integral divergent. This is an obvious advantage of the canonical variables over the mechanical ones; other advantages will become clear below. Hence, proceeding as above one finds

$$\Gamma_{pp}(t) = \frac{4\pi}{3} e^2 \omega_0^4 \int_0^\infty d\omega \frac{\rho_0(\omega)}{\omega^2 \left|\Delta(\omega)\right|^2} \cos \omega t. \tag{7.15}$$

The calculation of this integral is quite lengthy and is postponed until §7.2.4. Here we consider the limiting case $t = 0$, which gives for the variance of p, using equation (7.10) and the fact that $\langle p(t) \rangle = 0$,

$$\sigma_p^2 = \Gamma_{pp}(0) = \frac{4\pi}{3} e^2 \omega_0^4 \int_0^\infty d\omega \frac{\rho_0(\omega)}{\omega^2 \left[(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2 \right]}. \tag{7.16}$$

It is instructive to make first an approximate evaluation of this expression to lowest order in τ. Due to the very small value of this parameter, it happens that $\tau\omega \ll 1$ even for frequencies as high as the Compton frequency $\omega_C = mc^2/\hbar$, which is the highest frequency of interest here. Hence all important contributions to the integral come from the extremely sharp resonance at $\omega \approx \omega_0$, and the integral can be evaluated by approximating ω by ω_0 everywhere except in the difference $u = \omega - \omega_0$. Even if elementary, due to the importance of this point we discuss it in more detail. Consider the ratio R of $\left|\Delta(\omega)\right|^{-2}$ at resonance ($\omega \simeq \omega_0$) and for a free particle ($\omega_0 = 0$),

$$R \equiv \frac{\left|\Delta(0)\right|^2}{\left|\Delta(\omega_0)\right|^2} = \left(\frac{1}{\tau\omega_0}\right)^2. \tag{7.17}$$

Now $\tau \sim 10^{-23}$s, and for frequencies in the optical region, $\omega_0 \sim 10^{15}\text{s}^{-1}$, so that $R \sim 10^{16}$. This is, for instance, the ratio of the resonance scattering cross-section of radiation by elastically bound electrons to the Thomson (free-particle) scattering cross-section. We verify that the resonance is extremely narrow and produces an enormous enhancement of the cross section. The contribution to the integral comes indeed from very small values of u/ω_0, so that one can approximate the integrand even by a delta function, using the formula

$$\lim_{\beta \to 0} \frac{\beta}{\beta^2 + x^2} = \pi \delta(x). \tag{7.18}$$

This is the so-called narrow-linewidth approximation. All factors of the form ω^r within the integral can then be taken out as ω_0^r, and the differences between the descriptions in terms of the canonical and the mechanical variables disappear; only with regard to the radiative corrections (which in quantum theory belong to the domain of QED) will they become relevant. We thus recognize the crucial role played by the extremely sharp resonances of the mechanical system due to the smallness of the value of the radiation reaction time τ; this point will prove to be of great significance in chapters 10 and 11.

According to the preceding discussion, one may extend the lower limit of the integral to $-\infty$ without introducing any appreciable error. Carrying out the simplified calculation one gets (we distinguish the approximate values from the exact ones adding when appropriate a subindex 0)

$$\sigma_{0p}^2 = \frac{\pi}{3} e^2 \rho_0(\omega_0) \int_{-\infty}^{\infty} du \frac{1}{u^2 + \frac{1}{4}\gamma^2} = \frac{\pi}{3} e^2 \rho_0(\omega_0) \frac{2\pi}{\gamma} \tag{7.19}$$

or, with $\rho_0(\omega_0)$ for the vacuum field given by equation (4.13),

$$\sigma_{0p}^2 = \frac{1}{2} m\hbar\omega_0. \tag{7.20}$$

Within this approximation the variance of x differs from the above by the factor $(m\omega_0)^2$, as follows from (7.6); thus,

$$\sigma_{0x}^2 = \frac{\hbar}{2m\omega_0}. \tag{7.21}$$

The mean energy is therefore given by $\mathcal{E}_0 = < p^2/2m + m\omega_0^2 x^2/2 >= (\sigma_{0p}^2/2m) + (m\omega_0^2 \sigma_{0x}^2/2) = \frac{1}{2}\hbar\omega_0$, and the product of dispersions is $\sigma_{0x}\sigma_{0p} = \hbar/2$. All these results coincide with the corresponding quantum mechanical values for the ground state of the one-dimensional harmonic oscillator.

The higher moments of the variables x and p can be constructed by observing that the equations of motion are linear and the driving field is a stationary Gaussian random process; thus x and p are normally distributed with zero mean and are furthermore statistically independent (see equation (7.29); these and other results will be derived below by following another procedure, so here we simply quote them). Thus [Marshall 1963, Boyer 1975b]

$$\left\langle x^{2n} \right\rangle = (2n-1)!!\sigma_x^{2n}; \quad \left\langle p^{2n} \right\rangle = (2n-1)!!\sigma_p^{2n}; \quad \langle x^n p^m \rangle = \langle x^n \rangle \langle p^m \rangle ; \tag{7.22}$$

$$\left\langle x^{2n+1}\right\rangle = \left\langle p^{2n+1}\right\rangle = 0, \tag{7.23}$$

and the corresponding joint stationary distribution becomes

$$W(x,p) = \frac{1}{2\pi\sigma_x\sigma_p} e^{-\frac{p^2}{2\sigma_p^2}-\frac{x^2}{2\sigma_x^2}}. \tag{7.24}$$

The marginal distributions $P_x(x) = \int W(x,p)dp = (1/\sqrt{2\pi}\sigma_x)\exp(-x^2/2\sigma_x^2)$ and $P_p(p) = \int W(x,p)dx = (1/\sqrt{2\pi}\sigma_p)\exp(-p^2/2\sigma_p^2)$ reproduce the corresponding quantum mechanical results for the ground state when the narrow-linewidth approximation $\sigma_x \rightarrow \sigma_{0x}, \sigma_p \rightarrow \sigma_{0p}$ is made. In this approximation, equation (7.24) coincides also with the Wigner distribution for the ground state of the quantum oscillator [see, e.g., Feynman 1972] and it can be cast in the form

$$W_0(x,p) = \frac{\omega_0}{2\pi\mathcal{E}_0} e^{-\frac{1}{\mathcal{E}_0}\left(\frac{p^2}{2m}+\frac{1}{2}m\omega_0^2x^2\right)}. \tag{7.25}$$

The energy, given in this approximation by the quantity within parentheses, is thus a continuous variable with dispersion $\sigma_E^2 = \mathcal{E}_0^2$. Notice that the same happens within the quantum phase-space description in terms of the Wigner distribution, so that these two descriptions coincide. Thus we verify that the electron, without being a quantum corpuscle to start with, *acquires* quantumlike properties through its interaction with the background field.

7.1.3. SOME USEFUL STATISTICAL RELATIONS

Consider the equations of motion (7.6) in terms of the canonical variables. After some simple algebra it follows that

$$\frac{1}{2}\frac{d}{dt}\left\langle p^2\right\rangle = \langle p\dot{p}\rangle = -m\omega_0^2\langle xp\rangle = -m\omega_0^2\left\langle x(m\dot{x}+m\gamma x+\frac{e}{c}A_x(t))\right\rangle, \tag{7.26}$$

or rearranging,

$$\frac{1}{2m}\frac{d}{dt}\left\langle p^2+m^2\omega_0^2x^2\right\rangle = -m^2\omega_0^2\left\langle \gamma x^2+\frac{e}{mc}xA_x(t)\right\rangle. \tag{7.27}$$

In writing these results we have taken into account that the probability distribution of the stationary random field is time independent, so that the operations of averaging over the realizations of the field amplitudes and taking the time derivative can be interchanged. The left–hand side of the last equation is the time derivative of a quantity closely related to the energy of the oscillator; a brief inspection shows that the right-hand side terms are very small (proportional to τ), as will be verified below. Once the oscillator has reached a state of equilibrium with the background field, any term of the form $(d/dt)\langle f(x,p)\rangle$ with $f(x,p)$ not explicitly dependent on time, is zero, so that from the above equations one gets

$$-\frac{e}{mc}\langle xA_x(t)\rangle = \gamma\left\langle x^2\right\rangle = \gamma\sigma_x^2, \tag{7.28}$$

$$\langle xp\rangle = 0. \tag{7.29}$$

In §7.2.2 the left-hand side of the equilibrium condition (7.28) will be expressed in yet different terms. Further, note from equations (7.6) that

$$p\dot{x} = \frac{d}{dt}xp + m\omega_0^2x^2 \tag{7.30}$$

and

$$\frac{d}{dt}xp + \gamma xp = \frac{1}{m}\left(p^2 - m^2\omega_0^2x^2\right) - \frac{e}{mc}pA_x. \tag{7.31}$$

The average of the latter expression gives a time-dependent version of the virial theorem for the SED oscillator.[4] This equation and the former give for the equilibrium state:

$$\frac{e}{mc}\langle pA_x\rangle = \frac{1}{m}\left\langle\sigma_p^2 - m^2\omega_0^2\sigma_x^2\right\rangle, \tag{7.32}$$

$$\langle p\dot{x}\rangle = m\omega_0^2\sigma_x^2. \tag{7.33}$$

According to (7.32), in the *radiationless approximation*,[5] when the coupling term $\langle pA_x\rangle$ is neglected, the average kinetic and potential energies become equal, just as for a classical or a quantum mechanical oscillator. The corresponding average values are obtained by calculating the integrals in the narrow-linewidth approximation, as was done in equations (7.20), (7.21); one thus gets

$$\mathcal{E}_0 = \frac{1}{2m}\left(\sigma_{0p}^2 + m^2\omega_0^2\sigma_{0x}^2\right) = \frac{1}{m}\sigma_{0p}^2 = m\omega_0^2\sigma_{0x}^2 = \omega_0\sigma_{0x}\sigma_{0p}. \tag{7.34}$$

In terms of the mechanical variables, which are also frequently used, one obtains a similar set of relations, namely,

$$\langle xp\rangle = 0; \quad \langle x\dot{p}\rangle = -\frac{1}{m}\sigma_p^2; \tag{7.35}$$

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[4] A more detailed discussion can be seen in de la Peña (1980) and Marc and McMillan (1983).

[5] This approximation can normally be made by considering the 'mechanical limit' $e \to 0$, or equivalently, τ (or γ) $\to 0$ (of course, only after a state of equilibrium has been reached, to guarantee that the stochastic field has played its central role, and leaving the Coulomb forces untouched during the limiting process). The value of the coupling constant e thus determines the time required by the system to attain equilibrium, but the final 'mechanical' equations do not depend on e.

$$e\langle pE_x(t)\rangle = \gamma\sigma_p^2; \quad e\langle xE_x\rangle = -\frac{1}{m}\left\langle \sigma_p^2 - m^2\omega_0^2\sigma_x^2\right\rangle. \tag{7.36}$$

Care should be taken to avoid mixing the two sets of expressions, since they refer to different variables, although the same letters are used to denote them.

7.2. Phase-space description of the harmonic oscillator

7.2.1. FOKKER-PLANCK EQUATIONS IN PHASE SPACE

The approximate equation of motion (7.3) has precisely the form of the Langevin equation for the Brownian harmonic oscillator [see, e.g., Wang and Uhlenbeck 1954, Papoulis 1965], the main differences being the size of the parameters and, notably, the spectrum of the stochastic force, which is here a coloured noise as opposed to the white (flat) noise of the Brownian case. But in making the narrow-band approximation that leads from (7.16) to (7.19) and so on, one extracts from under the integral sign the factor $\rho_0(\omega)/\omega^2$, which amounts to treating the noise as effectively white. Thus, the previous results apparently correspond to taking a Markov approximation of the original non-Markovian process, by which the SED oscillator becomes very approximately a Brownian oscillator.

Though this similarity is true in a certain sense, one must be careful in using it because there is an essential difference that makes the analogy break down: the intensity of the noise is proportional to $\rho(\omega_0)$ and hence depends on the frequency of the oscillator. By contrast, in the Brownian problem this quantity is proportional to the absolute temperature and is the same for *all* oscillators. In other words, different classical oscillators in a thermal bath are all subject to the same Wiener random process, whereas in the SED case there corresponds a specific noise to each oscillator; no 'equipartition of noise', so to speak, holds when the source of noise is the zeropoint field. This is the physical basis for the disappearance of equipartition of energy in SED. The weakness of the Brownian analogy becomes even more manifest when the oscillator is used, as is frequently done, as a simplified model for an atom in interaction with the radiation field;[6] since to an atomic state there correspond different transition frequencies, this leads to a 'state-dependent noise', which is far away from a white noise. Hence, strictly speaking there is no Brownian analog for the harmonic oscillator nor for any bound SED particle, as was already concluded in section 2.1.

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[6] The oscillator model of the atom, frequently called the *Lorentz model* and sometimes the *Drude-Lorentz model*, can account for a wide range of optical and nonoptical effects. In quantum theory its use can be shown to be legitimate (with the proper oscillator strengths included), provided, e.g., that the probability for the atom to remain in its initial state is close to unity [Cray et al. 1982; see also Allen and Eberly 1975, Cohen-Tannoudji et al. 1977, vol. II, p. 1318].

The fact that the stochastic processes of interest in SED are in general not Markovian means that the evolution of the corresponding phase-space probabilities is described by very complicated integro-differential equations. The reduction of such equations to second-order differential equations to meet the Brownian approximation does not work properly in the general case. However, for the harmonic oscillator (or single-frequency systems) the situation is different because, as we have just seen, an appropriate (frequency-dependent) Markov approximation can be made in this case. We can apply it therefore in the construction of Fokker-Planck equations for the SED oscillator.

Consider a dynamical system described by the equations of motion

$$\dot{x}_\alpha = f_\alpha(\mathrm{x}, t) + g_{\alpha K}(\mathrm{x}, t)\xi_K(t), \tag{7.37}$$

which corresponds to a *linear multiplicative noise* problem. Equations (7.6) are clearly a particular instance of (7.37). Here x_α represent the components of a vector in n-dimensional phase-space and $f_\alpha(x, t)$ the generalized (nonrandom) forces; the $g_{\alpha K}(\mathrm{x}, t)$ describe the couplings of the stochastic forces to the particle and may depend arbitrarily on their arguments and, finally, the $\xi_K(t)$ are N_K Gaussian, not necessarily stationary, random functions with zero mean and covariance tensor

$$\langle \xi_K(t)\xi_{K'}(s)\rangle = \varphi_{KK'}(t, s). \tag{7.38}$$

The correlation functions $\varphi_{KK'}(t, s)$ are assumed to be known; whereas for a white noise they reduce to delta functions, the ones of interest in SED are functions such as those studied in §4.3.2. The problem of constructing an approximate Fokker-Planck equation for such a non-Markov problem is extensively dealt with in the specialized literature.[7] Of the different known procedures to approximate a non-Markov process by a Markovian one, some lead to a genuine Fokker-Planck equation —i.e., an approximate representation of the original stochastic process by a true Markovian one— and others lead to a pseudo-Fokker-Planck equation which may give rise to nonpositive definite solutions; the results quoted here belong to this second category.

The Fokker-Planck equation for the probability density in $x = \{x_\alpha\}$ space comes out to be [see, e.g., van Kampen 1981]

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_\alpha} B_\alpha W = \frac{\partial^2}{\partial x_\alpha \partial x_\beta} D_{\alpha\beta} W, \tag{7.39}$$

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[7] Some pertinent references are: Zwanzig (1960, 1961), Stratonovich (1963, 1968), Lax (1966), Brissaud and Frisch (1974), Terwiel (1974), Haken (1975), van Kampen (1976, 1981), Santos (1978, 1985b), Claverie (1979, 1980b), San Miguel and Sancho (1980), Cetto et al. (1984), Cetto (1984), etc. The quoted works by Claverie contain detailed comparisons between the different approximations.

with the drift vectors and diffusion coefficients given by[8]

$$B_\alpha(\mathrm{x},t) = f_\alpha(\mathrm{x},t) + \int_0^t ds\varphi_{KK'}(t,s)\frac{\partial g_{\alpha K}(\mathrm{x},t)}{\partial \tilde{x}_\lambda(s;t)} g_{\lambda K'}(\tilde{\mathrm{x}}(s;t),s), \quad (7.40)$$

$$D_{\alpha\beta}(\mathrm{x},t) = \int_0^t ds\varphi_{KK'}(t,s) g_{\alpha K}(\mathrm{x},t) g_{\lambda K'}(\tilde{\mathrm{x}}(s;t),s)\frac{\partial x_\beta(t)}{\partial \tilde{x}_\lambda(s;t)}. \quad (7.41)$$

The variables $\tilde{x}_\lambda(s;t)$ are the solutions of the deterministic equations (those obtained from (7.37) by setting $\xi_K = 0$) that having started at $t = s$ reach the phase-space point $x_\lambda(t)$ at the present time $t > s$. These results hold good for times much larger than the correlation time of the stochastic force; for smaller times the solution $W(\mathrm{x},t)$ is not necessarily positive.

In the case of interest here, the minimum stochastic source is the zero-point field. As discussed in §4.3.1, this field contains all frequencies, and its correlation time, if defined by common rules, becomes infinitely small. However, in practice, for a given state of a mechanical system only frequencies within a finite range are relevant; the inverse of these frequencies is of the same order of magnitude as the periods of the deterministic motions, hence the correlation time becomes of the order of these periods, i.e., of order 1 in atomic units. This fact is at the root of the generally non Markovian behaviour of atomic SED systems, and it means that Markov approximations can be at most of value for the description of the slow diffusions that take place in the system, once an average has been performed over times large in comparison with the deterministic periods.

7.2.2. FOKKER-PLANCK EQUATIONS FOR THE HARMONIC OSCILLATOR

For the specific case of interest here, direct application of the above equations does not present any particular difficulty. We shall nevertheless follow a simpler procedure, based on a set of relations that can be established between the diffusion coefficients $D_{\alpha\beta}$ and the covariances $\Gamma_{AB}(t,s) = \langle A(t)B(s)\rangle - \langle A(t)\rangle\langle B(s)\rangle$. Specifically, it can be shown that the above formula for the diffusion tensor can be recast in the forms [Cetto et al. 1984]

$$D_{\alpha\beta} = g_{\alpha K}\langle \xi_K(t)x_\beta(t)\rangle = \Gamma_{\dot{x}_\alpha x_\beta}(t,t)) - \Gamma_{f_\alpha x_\beta}(t,t) = \Gamma_{g_{\alpha K}\xi_K(t)x_\beta}(t,t), \quad (7.42)$$

where the second equality comes from the equations of motion (7.37) [the particular case for $f_\alpha = 0$ is studied in Stratonovich (1963), section 8]. Note that in the calculation of the correlations, the $g_{\alpha K}$ are to be taken at a fixed point.

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[8] The formulas for the diffusion coefficients, here merely quoted, will be justified in chapter 9.

These formulas can be applied directly to the harmonic oscillator. Using the relations (7.28)-(7.33), they give

$$D_{11} = D_{xx} = \Gamma_{\dot{x}x} - \Gamma_{f_1 x} = -\Gamma_{(-\gamma x + p/m)x} = \gamma \Gamma_{xx} = \gamma \sigma_x^2$$

and so on; a complete calculation leads in the time-asymptotic limit to

$$D_{xx} = \gamma \sigma_x^2 = -\frac{e}{mc} \langle x A_x \rangle , \tag{7.43}$$

$$D_{xp} = \frac{1}{m} \left(m^2 \omega_0^2 \sigma_x^2 - \sigma_p^2 \right) = -\frac{e}{mc} \langle p A_x \rangle , \tag{7.44}$$

$$D_{px} = 0; \quad D_{pp} = 0. \tag{7.45}$$

Equation (7.43) can be identified as a fluctuation-dissipation relation;[9] equation (7.44) shows that the xp-diffusion coefficient is related to the mean work performed on the particle by the random field, as becomes even clearer in (7.50).

The functions $g_{\alpha K}$ are constant in the present case, and in terms of the canonical variables the Fokker-Planck equation becomes [Claverie 1979][10]

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x} \left(-\gamma x + \frac{1}{m} p \right) W - m \omega_0^2 x \frac{\partial W}{\partial p} = D_{xp} \frac{\partial^2 W}{\partial x \partial p} + \gamma \sigma_x^2 \frac{\partial^2 W}{\partial x^2} . \tag{7.46}$$

Introducing the Liouvillian operator

$$L_0 = (p/m)(\partial/\partial x) - m \omega_0^2 x (\partial/\partial p) \tag{7.47}$$

one can write

$$\frac{\partial W}{\partial t} + L_0 W - \frac{\partial}{\partial x} \gamma x W = D_{xp} \frac{\partial^2 W}{\partial x \partial p} + \gamma \sigma_x^2 \frac{\partial^2 W}{\partial x^2} . \tag{7.48}$$

Note that whereas L_0 changes sign under time reversal, other operators in this equation —such as $(\partial/\partial x)\gamma x$— do not; the equation therefore describes an irreversible process and allows us to calculate $W(x,p;t)$ from $W(x,p;t_0)$ for $t > t_0$ only.

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[9] We will meet other examples of fluctuation-dissipation relations below. In general this kind of relationships —first discovered in the study of Brownian motion— connect coefficients that measure the dissipation in the system with coefficients related to the source of the fluctuations under equilibrium conditions. They show that fluctuations and dissipation are concomitant, so that it is not possible to have one without the other. For a general introduction to the fluctuation-dissipation theorem, see, e.g., Reichl (1980).

[10] The first Fokker-Planck type equation in SED was constructed by Marshall (1963) in his pioneering studies of the harmonic oscillator, using different approximations and variables from those discussed here. A version of this equation in terms of integrals of motion is given in Marshall (1980b).

When the description is made in terms of the mechanical variables the results are [Claverie and Diner 1976b, de la Peña and Cetto 1977a]

$$D_{xx} = 0; \quad D_{xp} = 0, \tag{7.49}$$

$$D_{px} = \frac{1}{m}\left(m^2\omega_0^2\sigma_x^2 - \sigma_p^2\right) = e\left\langle xE_x\right\rangle, \tag{7.50}$$

$$D_{pp} = \gamma\sigma_p^2 = e\left\langle pE_x\right\rangle, \tag{7.51}$$

$$\frac{\partial W}{\partial t} + L_0 W - \frac{\partial}{\partial p}\gamma p W = D_{px}\frac{\partial^2 W}{\partial x \partial p} + \gamma\sigma_p^2\frac{\partial^2 W}{\partial p^2}. \tag{7.52}$$

Although equations (7.48) and (7.52) have a different appearance, they describe the same system in their own terms. According to the first one, all the diffusion stems from the variable x and is generated by the noise $A(t)$ coupled to x, in full agreement with equations (7.6). In the second case it is the field $E(t)$ which impresses a diffusive behaviour on the variable p, just as one can read from equations (7.5). This shows that there is not one Fokker-Planck equation for a given problem, but a number of them, each one appropriate to the set of variables selected for the description. Very useful sets of variables that simplify the description at or near equilibrium may be constructed using the integrals of motion of the deterministic equations, as will be shown in the next section.

It is important to realize that, except for the calculation of the final values (7.20) and (7.21), so far nothing has been assumed with regard to the spectrum denoted by $\rho_0(\omega)$. In particular, one is free to assign to ρ_0 any functional form and to make it depend on external parameters, provided the statistical properties of the amplitudes a, a^* remain appropriately defined. Thus, for instance, one may insert instead of the $\rho_0(\omega)$ describing the zeropoint field, a Planckian distribution representing the equilibrium state at any temperature. This will be explicitly done in section 7.4, so that for the time being we simply write $\rho(\omega)$ instead of $\rho_0(\omega)$ in the general equations as a reminder of the general character of the description.

As can be easily checked, the Fokker-Planck equations for the harmonic oscillator admit general solutions of the form

$$W(x,p,t) = Ne^{-\alpha(x-\bar{x})^2-\beta(p-\bar{p})^2}, \tag{7.53}$$

where the functions $\bar{x}(t)$ and $\bar{p}(t)$ are solutions of the deterministic equations of motion

$$\ddot{\bar{x}}(t) + \gamma\,\dot{\bar{x}}(t) + \omega_0^2\bar{x}(t) = 0, \quad \ddot{\bar{p}}(t) + \gamma\,\dot{\bar{p}}(t) + \omega_0^2\bar{p}(t) = 0. \tag{7.54}$$

These solutions describe decaying oscillations with a frequency very close to ω_0; of course, by adding external fields it is possible to force oscillations with different frequencies [see, e.g., Goedecke 1983b, c; this point will be returned to in §7.5.2]. In the radiationless approximation the damping term is neglected and the oscillations become permanent; such states, corresponding to the coherent states of the quantum description, are studied by Santos (1974a) and Goedecke (1983c). Here they are represented by distributions around a center of mass that oscillates according to classical rules; in quantum mechanics they are normally represented by time-dependent minimum uncertainty wave packets. Still more complicated states can be constructed, which will not be dealt with here.

7.2.3. ACTION AND ANGLE VARIABLES

The Fokker-Planck equation can of course be written in terms of any complete set of canonical variables. Especially convenient sets are those that include the integrals of motion of the deterministic (radiationless) problem. Since L_0 applied to any function of the integrals of motion gives zero, from (7.48) we see that in the radiationless approximation W_0 must be a function of the integrals of motion only (cf. equation(7.65)).

Among such sets of canonical variables a particularly important one is provided by the action and angle variables, so we rewrite the Fokker-Planck equation of the one-dimensional oscillator in terms of them. Thus [see, e.g., Goldstein 1980],

$$x = \sqrt{\frac{2J}{m\omega_0}}\cos\theta, \quad p = \sqrt{2m\omega_0 J}\sin\theta, \tag{7.55}$$

$$\frac{\partial W}{\partial x} = \sqrt{2m\omega_0 J}\cos\theta\frac{\partial W}{\partial J} - \sqrt{\frac{m\omega_0}{2J}}\sin\theta\frac{\partial W}{\partial \theta}, \tag{7.56}$$

$$\frac{\partial W}{\partial p} = \sqrt{\frac{2J}{m\omega_0}}\sin\theta\frac{\partial W}{\partial J} + \sqrt{\frac{1}{2m\omega_0 J}}\cos\theta\frac{\partial W}{\partial \theta} \tag{7.57}$$

and

$$L_0 W(J,\theta) = -\omega_0\frac{\partial W}{\partial \theta}. \tag{7.58}$$

By calculating the second-order derivatives and substituting in equation (7.48) one gets a complicated Fokker-Planck equation in terms of the new variables, namely

$$\frac{\partial W}{\partial t} - \gamma\left(W + m\omega_0\sigma_x^2\frac{\partial W}{\partial J}\right) - 2\gamma J\cos^2\theta\frac{\partial}{\partial J}\left(W + m\omega_0\sigma_x^2\frac{\partial W}{\partial J}\right)$$

$$-\omega_0 \frac{\partial W}{\partial \theta} + \gamma \frac{\partial}{\partial \theta} \mathcal{F}_\theta W = 0, \tag{7.59}$$

where $\left(\partial \mathcal{F}_\theta W\right) / \partial \theta$ contains a linear combination of $\partial W / \partial \theta$ and higher derivatives with respect to θ. In writing (7.59) we have neglected the cross-derivative terms proportional to D_{xp} that lead to a small (radiative) correction of the average energy, as will be seen in §7.3.2.

7.3. Equilibrium properties of the oscillator in the ground state

7.3.1. APPROXIMATE STATIONARY SOLUTION OF THE FOKKER-PLANCK EQUATION

Let us consider the case $\partial W / \partial t = 0$, when the solution of the Fokker-Planck equation becomes the invariant measure $W_0(J)$ of the problem. Since this equilibrium distribution should be independent of the angle θ, integrating over this variable one obtains

$$-\gamma\left(W_0 + m\omega_0\sigma_x^2 \frac{\partial W_0}{\partial J}\right) - \gamma J \frac{\partial}{\partial J}\left(W_0 + m\omega_0\sigma_x^2 \frac{\partial W_0}{\partial J}\right) = 0, \tag{7.60}$$

whence it follows that the stationary distribution satisfies the first order equation [Boyer 1978a][11]

$$W_0 + m\omega_0\sigma_x^2 \frac{\partial W_0}{\partial J} = 0. \tag{7.61}$$

Multiplying by J and integrating from 0 to ∞ one gets $\bar{J} \equiv \langle J \rangle = m\omega_0\sigma_x^2$; this gives for the ground state, with σ_{0x}^2 given by (7.21), the correct value $\bar{J} = \hbar/2$. Similarly, it follows that

$$\langle J^n \rangle = n! \langle J \rangle^n \tag{7.62}$$

and

$$W_0 = \frac{1}{\bar{J}} e^{-J/\bar{J}}. \tag{7.63}$$

Since W_0 depends only on the adiabatic invariant J of the mechanical part of the system, the average value $\bar{J}$ remains an adiabatic invariant in the presence of the zeropoint radiation [Boyer 1978a]. However, note that according to equation (7.63) the action variable is also continuously distributed, as is the case with the energy; only its average value coincides with the quantum prediction in terms of eigenstates. In the presence of blackbody radiation at temperature $T = 1/k_B\beta$, one gets instead of the above results

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[11] Actually, the invariant solution to equation (7.60) satisfies the more general equation $W_0 + m\omega_0\sigma_x^2(\partial W_0/\partial J)$ =const/J, as is easily verified; but the requirement that W_0 be normalizable in $[0, \infty)$ can be met only by setting the constant equal to zero, which leads to equation (7.61).

$$W_0(J;\beta) = W_0(0;\beta)e^{-(J/\bar{J}_o)\tanh(\beta\hbar\omega_0/2)}, \quad \bar{J} = \bar{J}_0 \coth(\beta\hbar\omega_0/2); \tag{7.64}$$

this more general solution is studied in section 7.4.

Let us now consider the stationary solution of the Fokker-Planck equation (7.48) directly in x,p phase space. $W_0(x,p)$ is an even function of p, since it must be invariant with respect to a time inversion, which inverts the sign of p. This allows us to separate the equation into its even and odd parts and thus obtain a pair of simultaneous differential equations [de la Peña and Cetto 1979][12]

$$L_0 W_0 = D_{xp}\frac{\partial^2 W_0}{\partial x \partial p}, \tag{7.65}$$

$$-\gamma\frac{\partial}{\partial x}xW_0 = D_{xx}\frac{\partial^2 W_0}{\partial x^2}; \quad D_{xx} = \gamma\sigma_x^2. \tag{7.66}$$

In the radiationless approximation, D_{xp} is set equal to zero (see (7.44)) and the first equation $L_0 W_0 = 0$ then merely says that the equilibrium solution is a function of the integrals of the (deterministic) motion (the energy or the action integral, in the present case). The diffusion term is thus expected to produce a (radiative) correction to the energy, to be calculated below. A first integration of (7.66) gives $xW_0 + \sigma_x^2 W_0' = h(p)$; assuming that W_0 vanishes at infinity because $\int_{-\infty}^{\infty} dx W_0 < \infty$, the integral $\int_{-\infty}^{\infty} xW_0 dx = h(p)\int_{-\infty}^{\infty} dx$ must also be bounded and hence $h(p) = 0$, so that finally

$$xW_0 + \sigma_x^2\frac{\partial W_0}{\partial x} = 0. \tag{7.67}$$

From this equation it follows that x has a normal distribution with variance σ_x^2. Equivalently, by multiplying by x^{2n-1} and integrating over the whole phase space (the average, still denoted by $\langle\cdot\rangle$, is now over the phase space distribution), one gets after an integration by parts

$$\left\langle x^{2n}\right\rangle = -\sigma_x^2\left\langle x^{2n-1}\frac{\partial W_0}{\partial x}\right\rangle = (2n-1)\sigma_x^2\left\langle x^{2(n-1)}\right\rangle. \tag{7.68}$$

By iteration one obtains equations (7.22), corresponding to a Gaussian distribution with zero mean. Eliminating $\partial W_0/\partial x$ from (7.65) with the help of (7.67) one is led to $pW_0 + (m^2\omega_0^2\sigma_x^2 - mD_{xp})(\partial W_0/\partial p) = 0$ or, using (7.44),

$$pW_0 + \sigma_p^2\frac{\partial W_0}{\partial p} = 0, \tag{7.69}$$

---

[12] General procedures to separate a Fokker-Planck equation into its reversible and irreversible parts in the time-dependent case are given, e.g., in van Kampen (1981).

whence also p is normally distributed. Note that equations (7.67) and (7.69) do not any longer contain the parameter γ, i.e., they are 'purely mechanical'. From these equations one readily gets

$$\langle xh_1(p)\rangle = 0, \quad \langle ph_2(x)\rangle = 0 \tag{7.70}$$

for arbitrary functions h_1 and h_2; more generally, one verifies that x and p are statistically independent joint normal variables, in agreement with (7.24). A treatment in terms of the mechanical variables leads of course to equivalent results.

7.3.2. THE OSCILLATOR ENERGY AND ITS CORRECTIONS. THE LAMB SHIFT

Present-day physics does not offer a precise rule to define the energy of the particle. The assignment of one part of the total Hamiltonian to each of the subsystems (field and particle) is somewhat arbitrary, if at all meaningful. This ambiguity plagues the treatment of the problem of the SED harmonic oscillator, as can be verified by comparing the discussions of different authors [cf. Braffort and Tzara 1954, Sokolov and Tumanov 1956, Santos 1974a , de la Peña and Cetto 1979, Goedecke 1983b, c, etc.]. Here we propose, following Santos (1974a), to define the energy of the particle in terms of observable quantities, so that theory and experiment can be compared. This gives preeminence to the canonical variables over the mechanical ones, because the change of momentum is then due to controllable forces acting on the particle (cf. equations (7.5) and (7.6)). In the radiationless approximation, when the calculations are made to lowest order in τ, one can take out of the integrals all functions of ω^s (except of course for $\omega - \omega_0$), which become merely ω_0^s; this means that the ambiguity only affects the radiative corrections, whose definition will demand careful attention. Having said this, we define the mean energy of the oscillator as the sum of its mean kinetic and potential energies:[13]

$$\mathcal{E} = \frac{m}{2}\left\langle \dot{x}^2\right\rangle + \frac{1}{2}m\omega_0^2\left\langle x^2\right\rangle, \tag{7.71}$$

where $m\dot{x} = p - (e/c)A_T$ and $A_T = A + A_{\text{rr}}$, with A representing the electromagnetic potential of the zeropoint field and $A_{\text{rr}} = mc\gamma e^{-1}x$ the potential associated to radiation reaction. Substituting and using equations (7.28) to (7.33) and (7.44) one gets the following expression, where the terms $\frac{1}{2}m\gamma^2\sigma_x^2$ and $\dfrac{e}{c}\gamma\left\langle xA_x\right\rangle$ have been neglected because they are of second order in γ:

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[13] This definition for the energy is appropriate when the approximation $m\tau\dddot{x} \to \tau\dot{F}$ (or its equivalent in canonical variables) is made; otherwise the definition should be modified to take into account the Schott term, as discussed in §8.1.3.

$$\mathcal{E} = \frac{1}{2m}\sigma_p^2 + \frac{1}{2}m\omega_0^2\sigma_x^2 + \kappa\left\langle A_x^2\right\rangle + D_{xp}. \tag{7.72}$$

For clarity we have set $\kappa = e^2/(2mc^2)$. To separate the corrections from the main contributions we introduce the definitions

$$\mathcal{E} = \mathcal{E}_0 + \delta\mathcal{E}, \quad \sigma_x^2 = \sigma_{0x}^2 + \delta\sigma_x^2, \quad \sigma_p^2 = \sigma_{0p}^2 + \delta\sigma_p^2, \tag{7.73}$$

where

$$\mathcal{E}_0 = \frac{1}{2m}\sigma_{0p}^2 + \frac{1}{2}m\omega_0^2\sigma_{0x}^2. \tag{7.74}$$

Equation (7.44) then leads to

$$D_{xp} = -\frac{e}{mc}\left\langle pA_x\right\rangle = \frac{1}{m}\left(m^2\omega_0^2\delta\sigma_x^2 - \delta\sigma_p^2\right), \tag{7.75}$$

whence

$$\delta\mathcal{E} = \kappa\left\langle A_x^2\right\rangle + \tfrac{1}{2}D_{xp} + m\omega_0^2\delta\sigma_x^2. \tag{7.76}$$

Let us proceed to the analysis of the various contributions.

1) The term $\mathcal{E}_0$ is the mean energy in the radiationless narrow-linewidth approximation; for the ground state it reduces to $\frac{1}{2}\hbar\omega_0$.

2) The terms $\kappa\left\langle A_x^2\right\rangle + \frac{1}{2}D_{xp}$ are due to the interaction of the oscillator with the zeropoint field and are to be considered responsible for the Lamb shift of the energy levels. They are further analyzed as follows:

2a) The term $\kappa\left\langle A_x^2\right\rangle$ is infinite, since all field modes are present; but since this contribution is the same even for a free particle ($\omega_0 = 0$) and for all values of the momentum, it is usually interpreted as the Lamb shift of the free particle, which adds to its proper energy and is thus unobservable. Therefore one gets usually rid of it by a re-renormalization of the mass. Of course, one could equally well subtract it from the definition of $\mathcal{E}$, which is equivalent to assigning it to the field instead of the particle. However, as will be discussed in §7.4.4, the value of this term can be changed by means of boundary conditions due to macroscopic bodies, and such changes, which affect the energy of the oscillator, can be observed. Thus, even if the term itself is irrelevant and may be normally swept under the carpet, it should be borne in mind when studying systems enclosed in small cavities and similar systems sensitive to the geometry of the surroundings.

2b) The contribution $\frac{1}{2}D_{xp}$ represents the usual Lamb shift; from (7.75) we see that it is given by

$$\delta E_L = \frac{1}{2}D_{xp} = -\frac{e}{2mc}\left\langle pA_x\right\rangle. \tag{7.77}$$

In the atomic case this energy shift leads to observable effects; in the linear oscillator case it is the same for all levels and hence produces no observable effect in the spectrum. Nevertheless, as a matter of principle it is important to study it and to compare it with the predictions of QED, as will be done in the next section.

We would like to recall that in QED the Lamb shift is a second-order effect arising from the interaction Hamiltonian $-(e/mc)\mathbf{p}\cdot\mathbf{A}$ in the Coulomb gauge, which has almost the form of the above expression; the extra factor of $\frac{1}{2}$ appears later on in the quantum calculation, as a result of the treatment of this perturbation to second order. This point will be discussed in detail in section 11.4, where the formula $\frac{1}{2}D_{xp}$ for the Lamb shift will be recovered from a perturbative treatment to second order within SED.

3) Finally, the remaining term $m\omega_0^2\delta\sigma_x^2$ is of the form $\mathrm{const}(\langle V(x)\rangle - \langle V(x)\rangle_0)$. This term is related to the fluctuations of the potential due to the fluctuations of the instantaneous position around the radiationless value, and obviously it does not exist for a free particle. In quantum theory a similar correction is known under the name of *Darwin term*; however, the Darwin term is a correction of relativistic nature, predicted by the Dirac equation [see, e.g., Cohen-Tannoudji et al. 1977, section XIIB; Milonni 1994, section 9.7]. The present theory furnishes a nonrelativistic version of it, which by this very fact should be expected to be a poor prediction. Indeed, its calculated value is about one order of magnitude below the relativistic Darwin term appearing in the theory of the fine structure of atomic spectra. Thus, the nonrelativistic theory shows the need of such a correction, but is intrinsically unable to give a correct prediction of its value.[14]

Some additional comments seem to be in place. Normally the Darwin term is attributed to the zitterbewegung of the electron [see, e.g., Milonni 1994]. Here we are dealing rather with a jittering motion that is related not only with the Lamb shift and the Darwin term, but with the quantum fluctuations in general. In analogy with the zitter, this jitter comes about from the different behaviour of the velocity and the momentum, as follows from $m\dot{x} = p + \tau f - \frac{e}{c}A_x$ and $\dot{p} = f$ (cf. (7.6)). For instance, a free particle has a constant p, but its velocity $\dot{x}$ fluctuates around the classical trajectory. From this point of view the theory contains an element that is reminiscent of the zitter, although the very high frequency oscillations of noticeable amplitude have been suppressed in the treatment. Recall, however, from section 3.3 that the radiation reaction gives rise to very high frequency oscillations of the zitterbewegung type; a more complete treatment of the problem (preferably a relativistic one) should therefore be expected to give rise to both the zitterbewegung and the Darwin term. These matters will be returned to in chapter 12.

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[14] The Darwin term for the harmonic oscillator has the value $\hbar^2\omega_0^2/8mc^2$; the present theory gives this same value multiplied by the numerical coefficient $(16\alpha/3\pi)\ln(\omega_c/\omega_0)$, where α is the fine structure constant; see equation (7.103) below.

The mean energy (7.72) contains contributions that depend on $\delta\sigma_p^2$; with the decomposition of the energy given above this term contributes to the Lamb shift only. However, such contribution becomes eventually irrelevant, because any kinetic term of the form $\delta \langle T \rangle \equiv (\eta/2m)\delta\sigma_p^2$ can be considered to merely renormalize the mass. Indeed, in writing the renormalized mass as $m = m_0 + \delta m$ one has $p^2/2m \approx p^2/2m_0 - (\delta m/2m^2)p^2 = T_0 + \delta T$, so that equating both average corrections one gets

$$\frac{\delta m}{m} = -\eta \frac{\delta\sigma_p^2}{\sigma_{0p}^2}. \tag{7.78}$$

For the harmonic oscillator the value of this additional mass correction is very small, of order α^3 for frequencies in the optical region.

Now we are in a position to compare with other results obtained in the literature using an alternative but non-equivalent definition for the Lamb shift. Sokolov and Tumanov (1956) define the energy of the oscillator as $\langle p^2/2m + m\omega_0^2 x^2/2 \rangle$, which according to our previous discussion is only satisfactory in the radiationless approximation; its use has been justified by noting that it works, in the sense that it gives the correct result for the Lamb shift. Also, since it is a mechanical definition of the energy, the predicted correction corresponds, at least in spirit, to the interpretation of the Lamb shift proposed by Welton (1948) as due to fluctuations of the position of the electron generated by the vacuum field.[15] Distinguishing this new definition with a prime we write

$$\mathcal{E}' = \mathcal{E}_0 + \delta\mathcal{E}' = \frac{1}{2m}\sigma_p^2 + \frac{1}{2}m\omega_0^2\sigma_x^2 = \mathcal{E}_0 + \frac{1}{2m}\delta\sigma_p^2 + \frac{1}{2}m\omega_0^2\delta\sigma_x^2 \tag{7.79}$$

and obtain for the energy correction, using (7.75),

$$\delta\mathcal{E}' = \frac{1}{2}D_{xp} + \frac{1}{m}\delta\sigma_p^2. \tag{7.80}$$

This gives indeed the Lamb shift plus a mass correction; the free-particle contribution and the Darwin term are absent. This is fortunate, but the accidental coincidence has of course added to the confusion concerning the appropriate definition of the energy.

7.3.3. CALCULATION OF THE SECOND-ORDER MOMENTS

We have arrived at a point where a more precise calculation of the second-order moments and correlations becomes necessary. Let us go back to equation (7.15) and rewrite it with the help of (7.11) in the form[16]

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[15] For a discussion of Welton's model see, e.g., Milonni 1994, section 3.6.

$$\Gamma_{pp}(t) = \langle p(t_0+t)p(t_0)\rangle = \frac{m\tau}{\pi}\omega_0^4 \int_0^{\omega_c} d\omega \frac{\hbar\omega}{|\Delta(\omega)|^2}\cos\omega t$$

$$= \frac{m\hbar\omega_0^2}{2\pi\omega_1}\cdot\beta\int_0^{\omega_c} d\omega\left(\frac{1}{|\omega+\omega_\beta|^2} - \frac{1}{|\omega-\omega_\beta|^2}\right)\cos\omega t \equiv \frac{m\hbar\omega_0^2}{2\pi\omega_1}\mathsf{S}(t). \quad (7.81)$$

With suitable changes of variables one can write

$$\mathsf{S}(t) = -\beta\int_{\omega_1}^{\omega_c+\omega_1} du\frac{\cos(u-\omega_1)t}{u^2+\beta^2} + \beta\int_{-\omega_1}^{\omega_c-\omega_1} du\frac{\cos(u+\omega_1)t}{u^2+\beta^2}$$

$$= [-I_c(\omega_c+\omega_1) + I_c(\omega_1) + I_c(\omega_c-\omega_1) - I_c(-\omega_1)]\cos\omega_1 t$$

$$+ [-I_s(\omega_c+\omega_1) + I_s(\omega_1) - I_s(\omega_c-\omega_1) + I_s(-\omega_1)]\sin\omega_1 t, \quad (7.82)$$

with

$$I_c(b,t) \equiv I_c(b) = \beta\int_0^b du\frac{\cos ut}{u^2+\beta^2}, \quad I_s(b,t) \equiv I_s(b) = \beta\int_0^b du\frac{\sin ut}{u^2+\beta^2}. \quad (7.83)$$

Note that the integrals are convergent; the cutoff is introduced to eliminate the incorrect description at very high frequencies associated with the radiation reaction, as discussed in §3.2.3, and not to regularize the theory. To evaluate the function I_c, note that it satisfies the differential equation

$$\ddot{I}_c - \beta^2 I_c = -\frac{\beta}{t}\sin bt, \quad (7.84)$$

so that I_c takes the form

$$I_c(b,t) = -\beta e^{-\beta t}\int_{t_1}^t dr e^{2\beta r}\int_{t_0}^r ds e^{-\beta s}\frac{\sin bs}{s} \quad (7.85)$$

subject to the initial conditions $I_c(b,0) = \arctan(b/\beta)$, $\dot{I}_c(b,0) = 0$, as follows from the definition (7.83). Hence,

$$I_c(b,t) = e^{-\beta t}\left[\arctan\frac{b}{\beta} + \beta\int_0^t dr e^{2\beta r}\int_r^\infty ds e^{-\beta s}\frac{\sin bs}{s}\right]. \quad (7.86)$$

An analogous procedure leads to

$$I_s(b,t) = -e^{-\beta t}\beta\int_0^t dr e^{2\beta r}\int_r^\infty ds e^{-\beta s}\frac{\cos bs - 1}{s}. \quad (7.87)$$

---

[16] Recall that the factor $\hbar\omega$ in the integrand is the energy $\mathcal{E}(\omega)$; it is precisely this linear dependence of $\mathcal{E}$ on ω that gives rise to results which differ qualitatively from the Brownian ones.

With the help of the condensed notation

$$I_c(b,t) = e^{-\beta t}\arctan\frac{b}{\beta} + e^{-\beta t}k_s(b),\ \ k_s(b) = \beta\int_0^t dre^{2\beta r}\int_r^\infty dse^{-\beta s}\frac{\sin bs}{s}, \tag{7.88}$$

$$I_s(b,t) = -e^{-\beta t}k_c(b), \quad k_c(b) = \beta\int_0^t dre^{2\beta r}\int_r^\infty dse^{-\beta s}\frac{\cos bs - 1}{s} \tag{7.89}$$

and

$$\mathsf{K}_s(t) = k_s(\omega_1) - k_s(-\omega_1) + k_s(\omega_c - \omega_1) - k_s(\omega_c + \omega_1), \tag{7.90}$$

$$\mathsf{K}_c(t) = k_c(\omega_1) + k_c(-\omega_1) - k_c(\omega_c - \omega_1) - k_c(\omega_c + \omega_1), \tag{7.91}$$

equation (7.81) becomes

$$\Gamma_{pp}(t) = \frac{m\hbar\omega_0^2}{2\pi\omega_1}e^{-\beta t}\left[\mathsf{S}_0\cos\omega_1 t + \mathsf{K}_s(t)\cos\omega_1 t - \mathsf{K}_c(t)\sin\omega_1 t\right], \tag{7.92}$$

where

$$\mathsf{S}_0 = 2\arctan\frac{\omega_1}{\beta} + \arctan\frac{\omega_c - \omega_1}{\beta} - \arctan\frac{\omega_c + \omega_1}{\beta}. \tag{7.93}$$

The functions $\mathsf{K}_s(t)$ and $\mathsf{K}_c(t)$, as well as $k_s(b)$ and $k_c(b)$, are defined so that they take the value 0 at $t = 0$; their time derivatives are

$$\dot{k}_c(b) = \beta e^{2\beta t}\int_t^\infty dse^{-\beta s}\frac{\cos bs - 1}{s}, \quad \dot{k}_s(b) = \beta e^{2\beta t}\int_t^\infty dse^{-\beta s}\frac{\sin bs}{s}; \tag{7.94}$$

$$\dot{k}_c(b,0) = \frac{\beta}{2}\ln\frac{\beta^2}{\beta^2 + b^2}, \quad \dot{k}_s(b,0) = \beta\arctan\frac{b}{\beta}; \quad \dot{\mathsf{K}}_s(0) = \beta\mathsf{S}_0. \tag{7.95}$$

We have in particular the important result

$$\sigma_p^2 = \Gamma_{pp}(0) = \frac{m\hbar\omega_0^2}{2\pi\omega_1}\mathsf{S}_0. \tag{7.96}$$

The cross-correlation Γ_{xp} can be easily calculated using the equation of motion (7.6), from which it follows that

$$\Gamma_{xp} = \frac{1}{m\omega_0^2}\dot{\Gamma}_{pp}(t), \tag{7.97}$$

whence

$$\Gamma_{xp}(t) = -\frac{\hbar e^{-\beta t}}{2\pi}\left\{\left[\mathsf{S}_0 + \mathsf{K}_s + \frac{\dot{\mathsf{K}}_c - \beta\mathsf{K}_c}{\omega_1}\right]\sin\omega_1 t\right.$$

$$+\left[\mathsf{K}_c-\frac{\dot{\mathsf{K}}_s-\beta\mathsf{S}_0-\beta\mathsf{K}_s}{\omega_1}\right]\cos\omega_1 t\Bigg\}\tag{7.98}$$

so that indeed $\Gamma_{xp}(0)=0$. With $\Gamma_{xx}=-\mathring{\Gamma}_{xp}(t)/m\omega_0^2$ one gets further

$$\Gamma_{xx}(t)=\frac{\hbar\omega_1 e^{-\beta t}}{2\pi m\omega_0^2}\Bigg\{\left[\left(1-\frac{\beta^2}{\omega_1^2}\right)(\mathsf{S}_0+\mathsf{K}_s)+2\frac{\mathring{\mathsf{K}}_c-\beta\mathsf{K}_c}{\omega_1}\right]\cos\omega_1 t-$$

$$-\left[\left(1-\frac{\beta^2}{\omega_1^2}\right)\mathsf{K}_c-2\frac{\dot{\mathsf{K}}_s-\beta\mathsf{S}_0-\beta\mathsf{K}_s}{\omega_1}\right]\sin\omega_1 t\Bigg\}.\tag{7.99}$$

Then, in particular,

$$\sigma_x^2=\Gamma_{xx}(0)=$$

$$=\frac{\hbar\omega_1}{2\pi m\omega_0^2}\left[\left(1-\frac{\beta^2}{\omega_1^2}\right)\mathsf{S}_0+\frac{\beta}{\omega_1}\ln\frac{\left[\beta^2+(\omega_c-\omega_1)^2\right]\left[\beta^2+(\omega_c+\omega_1)^2\right]}{\left[\beta^2+\omega_1^2\right]^2}\right].\tag{7.100}$$

The above results show that the correlations of the SED harmonic oscillator have a much more complicated time dependence than those of the Brownian oscillator, which do not contain functions such as $\mathsf{K}_s(t)$ and $\mathsf{K}_c(t)$ [see, e.g., Wang and Uhlenbeck 1954, or Papoulis 1965, chapter 15]; in contrast to the Brownian system, the SED oscillator keeps a record of its past through the correlations of the field. As already remarked, the difference comes about from the dependence on the frequency of the energy $\mathcal{E}(\omega)\sim\rho(\omega)/\omega^2$, as is seen from equation (7.81), compared with the equipartition value for the classical oscillators. A further, particularly interesting difference is the following. By approximating the quantity $k_s(\omega_c+\omega_1)-k_s(\omega_c-\omega_1)$ to first order in $\omega_1/\omega_c\ll 1$, one gets

$$e^{-\beta t}\mathsf{K}_s=e^{-\beta t}\left(2k_s(t,\omega_1)+\frac{2\beta\omega_1}{\omega_c^2}\right)-\frac{2\beta\omega_1}{\omega_c^2}\cos\omega_c t.\tag{7.101}$$

This result reveals the existence in $\Gamma_{xx}(t)$ of oscillatory terms that do not decay with time; hence, oscillations with frequency ω_c are always present in the system, even if of a very small amplitude, proportional to ω_c^{-2}. Although the details cannot be taken verbatim owing to the nonrelativistic character of the treatment, once more the theory reveals that associated with the radiation reaction mechanism there are zitterbewegung-like oscillations present, of the kind discussed in sections 3.3 and 3.4.

It is pointless to insist in on using the above exact expressions, due to their complexity and lack of transparency; let us only write explicit approximations for the quantities which are of direct interest, namely, the values of the equilibrium variances. To first order in $\beta=\gamma/2$ and neglecting terms of order β/ω_c, it can be shown that $\mathsf{S}(0) \approx \pi - (2\beta/\omega_0)$, so that from equations (7.96) and (7.100) one gets

$$\sigma_p^2 = \sigma_{0p}^2 \left(1 - \frac{\gamma}{\pi\omega_0}\right), \tag{7.102}$$

$$\sigma_x^2 = \sigma_{0x}^2 \left(1 - \frac{\gamma}{\pi\omega_0} + \frac{2\gamma}{\pi\omega_0} \ln \frac{\omega_c}{\omega_0}\right). \tag{7.103}$$

An immediate consequence of these results is

$$\sigma_x^2 \sigma_p^2 = \frac{\hbar^2}{4} \left[1 + \frac{2\gamma}{\pi\omega_0} \left(\ln \frac{\omega_c}{\omega_0} - 1\right)\right], \tag{7.104}$$

which shows the presence of small radiative corrections in the Heisenberg inequalities, even though the dispersions have their minimum value, determined by the spectral density of the zeropoint field; it is remarkable that these corrections come out to be the same in QED and in SED. Further, from equations (7.75), (7.102) and (7.103) it follows that

$$D_{xp} = \mathcal{E}_0 \frac{2\gamma}{\pi\omega_0} \ln \frac{\omega_c}{\omega_0}, \tag{7.105}$$

so that equation (7.77) gives for the Lamb shift of the ground state of the harmonic oscillator

$$\delta\mathcal{E}_L = \mathcal{E}_0 \frac{\tau\omega_0}{\pi} \ln \frac{\omega_c}{\omega_0}, \tag{7.106}$$

a result that agrees with the prediction of nonrelativistic QED.[17]

7.3.4. CONDITIONAL PROBABILITIES AND TIME EVOLUTION

The bilinear covariances can be used to calculate conditional probabilities, which allows us to follow the evolution of the distribution from an arbitrary initial condition. Without entering into details, let us recall a general procedure based on the orthogonality method applicable to normal distributions.[18]

For concreteness, assume that we are interested in the conditional density $f_{x(t)|x_0} \equiv f_{x(t)}(x|x_0)$ of $x(t)$ under the condition that $x(0) = x_0$. More generally, consider the conditional expectation $E\{y|x\}$, where x and y are jointly normal with zero mean. Any linear combination of x and y is also normal, so that the new variable $z = y - ax$ is normal with zero mean; then $\Gamma_{xz} = E\{xz\} = E\{xy\} - aE\{x^2\} = \Gamma_{xy} - a\sigma_x^2$. Now select the constant a such that x and z are orthogonal (which means that $\Gamma_{xz} = 0$),

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[17] The first nonrelativistic quantum calculation of the Lamb shift for the atomic case is due to Bethe (1947). A detailed discussion of the Lamb shift in QED can be found in Milonni (1994), chapters 3, 4 and 16. Explicit evaluations of the nonrelativistic QED prediction for the Lamb shift of the harmonic oscillator are given in appendix B of Santos (1974a) and in Goedecke (1984).

[18] A thorough exposition of the orthogonality principle and its applications can be found in Papoulis 1965, sections 7.4 and 7.5.

$$a = \frac{\Gamma_{xy}}{\sigma_x^2}. \tag{7.107}$$

Since $E\{x\} = E\{z\} = 0$, the variables x and z are uncorrelated, and being jointly normal, they are statistically independent. Thus, in particular $E\{z|x\} = E\{z\}$, from which it follows that

$$E\{z|x\} = E\{y|x\} - aE\{x|x\} = E\{y|x\} - ax = E\{z\} = 0. \tag{7.108}$$

Combining with the previous result one is led to

$$E\{y|x\} = ax = \frac{\Gamma_{xy}}{\sigma_x^2}x, \tag{7.109}$$

a relation which expresses the expectation of y conditioned by x in terms of unconditioned quantities. To calculate the conditional distribution one still needs the conditional variance; this is obtained immediately by observing that the independence of x and z allows us to write

$$\sigma_{y|x}^2 = E\{(y-ax)^2|x\} = E\{(y-ax)^2\} = \sigma_y^2 - 2a\Gamma_{xy} + a^2\sigma_x^2 = \sigma_y^2 - \frac{\Gamma_{xy}^2}{\sigma_x^2}. \tag{7.110}$$

With $\Gamma_{xy} = r\ \sigma_x\sigma_y$ where r is the correlation coefficient of x and y, the last equation reads simply $\sigma_{y|x}^2 = (1-r^2)\sigma_y^2$. With the above results the conditional probability becomes

$$f(y|x) = \frac{1}{\sqrt{2\pi}\sigma_{y|x}}e^{-(y-ax)^2/2\sigma_{y|x}^2}. \tag{7.111}$$

We come back now to the construction of the probability of x conditioned by an initial x_0 for the harmonic oscillator. We have, recalling that $\Gamma_{xx}(t) \equiv \Gamma_{xx}(s+t,s) = \Gamma_{xx}(t,0)$,

$$E\{x|x_0\} = \frac{\Gamma_{xx}(t)}{\sigma_x^2}x_0,\ \sigma_{x|x_o}^2 = \sigma_x^2 - \frac{\Gamma_{xx}^2(t)}{\sigma_x^2}, \tag{7.112}$$

with $\Gamma_{xx}(t)$ given by (7.99); hence

$$f(x|x_0) = \frac{1}{\sqrt{2\pi}\sigma_{x|x_0}}e^{-(x-ax_0)^2/2\sigma_{x|x_0}^2}. \tag{7.113}$$

We see that $\sigma^2_{x|x_o}$ evolves from its zero initial value to its final value given by σ^2_x plus the oscillatory contributions of very high frequency discussed in relation to equation (7.101). This evolution proceeds with a 'long' relaxation time γ^{-1}, of order $c^3 \sim \alpha^{-3}$atomic units, as was noticed above. Of course, the general problem can be solved directly by integrating the time-dependent Fokker-Planck equation (7.46).

A particularly interesting case occurs when the initial phase-space distribution is $W(x,p;t=0) = \delta(x-x_0)\delta(p-p_0)$, corresponding to fixed values of the position and momentum coordinates. This distribution falls entirely outside the quantum domain since it obviously violates the Heisenberg inequality; however, in SED it represents a system that is far from equilibrium and evolves according the equation (7.48). It is possible to establish an H-theorem [Rodríguez 1983] that guarantees that the SED harmonic oscillator always evolves toward thermodynamic equilibrium, for any temperature $T \geq 0$; but only when the system has interacted long enough with the fluctuating field as for its variables to have acquired dispersions that comply with the quantum rules, one may say that it has entered into the *quantum regime.* This is an irreversible process, that normally takes a time of the order of γ^{-1} and is not described by quantum mechanics; only when the system is close to equilibrium, well within the quantum regime, can we expect its time evolution to be given by the quantum laws (see §7.4.2).

7.4. Excited states of the harmonic oscillator

Let us now go over to the study of the equilibrium state of the oscillator with blackbody radiation at arbitrary temperature. As already discussed, to take the extra thermal component of the field into account it suffices to make everywhere the substitution $\rho_0(\omega) \to \rho(\omega) = \rho_0(\omega)[1 + \exp(-\beta\hbar\omega)]/[1-\exp(-\beta\hbar\omega)]$. The exact calculation of the covariances becomes very involved due to the extra dependence on the frequency; however, in the narrow-linewidth approximation this extra factor can be extracted from the integrals setting $\omega = \omega_0$, which gives for the dispersions

$$\sigma^2_{Tx} = \sigma^2_{0x}\Pi(\beta\hbar\omega_0), \quad \sigma^2_{Tp} = \sigma^2_{0p}\Pi(\beta\hbar\omega_0), \tag{7.114}$$

with

$$\Pi(x) \equiv \frac{1+e^{-x}}{1-e^{-x}} = 1 + \frac{2}{e^x - 1} = \coth \tfrac{1}{2}x. \tag{7.115}$$

The mean energy (to lowest order in γ) becomes

$$\mathcal{E} = \mathcal{E}_0\Pi(\beta\hbar\omega_0). \tag{7.116}$$

One thus obtains results which are similar to the previous ones, the main differences being factors of $\Pi(\beta\hbar\omega_0)$ here and there. However, it is possible to follow a different and more sophisticated path which allows us to make formal contact with the language of the usual quantum analysis and helps to point out the differences and similarities between the two approaches.

7.4.1. THE WIGNER AND SCHRÖDINGER DESCRIPTIONS

Ever since the first studies of the excited states of the SED harmonic oscillator by Marshall (1963, 1968), the problem has been approached in many ways. Here we follow closely the exposition given by de la Peña and Cetto (1979); a similar construction including the radiative terms is given in Goedecke (1983b, c, 1984). In França and Marshall (1988) an approach in terms of Wigner functions is developed.

As seen in section 7.1, to zero order in τ the Hamiltonian of the oscillator can be written $H = (1/2m)(p^2 + m^2\omega_0^2 x^2)$; its average is then $\langle H \rangle = \mathcal{E} = \mathcal{E}_0 \Pi(\beta\hbar\omega)$, and its phase-space equilibrium distribution is given by the Wigner function (cf. equation (5.52)):

$$W(x, p; \beta) \equiv W_\beta(H) = \frac{1}{\pi\hbar\Pi} e^{-H/\mathcal{E}}; \tag{7.117}$$

the equilibrium mixture for the harmonic oscillator predicted in quantum statistical mechanics [Feynman 1972] is thus reproduced in SED.

It is most interesting to analyze this solution in terms of canonical energy states, since, as was seen in chapter 5 in connection with the field oscillators, such analysis brings us close to the usual quantum description. Thus we assume that the state with energy E has a relative weight given by $e^{-\beta E}$ and simultaneously introduce the intrinsic probabilities $g(E)$, writing the partition function in the form

$$Z = \int_0^\infty dE g(E) e^{-\beta E}. \tag{7.118}$$

The average state energy $\mathcal{E}$ is then

$$\mathcal{E} = \frac{1}{Z} \int_0^\infty dE E g(E) e^{-\beta E} = -\frac{\partial}{\partial \beta} \ln Z(\beta). \tag{7.119}$$

The inverse of this expression is

$$Z(\beta) = e^{-\int^\beta d\beta' \mathcal{E}(\beta')} = \frac{e^{-\frac{1}{2}\beta\hbar\omega_0}}{1 - e^{-\beta\hbar\omega_0}} = \sum_{n=0}^\infty e^{-(n+1/2)\beta\hbar\omega_0}, \tag{7.120}$$

with the function $\mathcal{E}$ given by equation (7.116). These are of course the same functions obtained in §5.1.4 for the field oscillators. Equation (7.118) gives now the intrinsic probability function $g(E)$ as the inverse Laplace transform of the partition function:

$$g(E) = \mathcal{L}_E^{-1} Z(\beta) = \sum_{n=0}^{\infty} \mathcal{L}_E^{-1} e^{-(n+1/2)\beta\hbar\omega}, \tag{7.121}$$

or

$$g(E) = \sum_{n=0}^{\infty} \delta(E - E_n), \quad E_n = \hbar\omega(n + \frac{1}{2}), \tag{7.122}$$

since the Laplace transform of an exponential is a delta function. Just as happened with the field oscillators, the analysis of the equilibrium solution for the mechanical oscillator in terms of a canonical distribution leads to a discrete set of energy states. In other words and as was also the case before, the SED oscillator becomes characterized by a set of discrete properties. Such a result appears unnatural in terms of stochastic physics, and one is led to conclude that the analysis in terms of canonical states, though very valuable, merely provides a formal description of the situation. To keep this observation in mind, we shall speak of energy *quasistates* —or *q-states* for short— instead of energy states. Now let us use (7.122) to decompose the phase space density $W(x, p; \beta)$ into q-states of energy E_n:

$$W_\beta(H) = \frac{1}{Z} \sum_0^{\infty} W_n(H) e^{-\beta E_n}; \quad Z(\beta) = \sum_0^{\infty} e^{-\beta E_n}, \tag{7.123}$$

with $E_n = \int dH\, H W_n(H)$. The temperature is thus separated from the mechanical variables, allowing a purely mechanical description in terms of the q-states $W_n(H)$. Applying the normalization condition:

$$\int dxdpW(x, p; \beta) = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-\beta E_n} \int dxdpW_n(H(x, p)) = 1, \tag{7.124}$$

we see that the simplest selection consists in separately normalizing to unity each phase-space energy state, $\int dxdpW_n(H(x, p)) = 1$. From the differential equations (7.67) and (7.69) it follows that the Fokker-Planck equation for the oscillator can be rewritten in the form

$$W + \mathcal{E} \frac{\partial W}{\partial H} = 0. \tag{7.125}$$

This expression is equivalent to (7.61) because the frequency ω_0 is fixed for the oscillator (it is not a function of the dynamical variables), so that H is proportional to J. From (7.123) and (7.114) one obtains[19]

$$\sum_0^\infty W_n e^{-\beta E_n} + \mathcal{E}_0 \Pi(\beta) \sum_0^\infty e^{-\beta E_n} (\partial W_n / \partial H) = 0, \tag{7.126}$$

or in terms of the dimensionless variable $\xi = H/\mathcal{E}_0$,

$$\sum_0^\infty \left[W_n e^{-\beta E_n} + \Pi(\beta) e^{-\beta E_n} \frac{\partial W_n}{\partial \xi} \right] = 0. \tag{7.127}$$

With the expansion $\Pi(\beta) = \sum_{k=0}^\infty (e^{-k\beta\hbar\omega_0} + e^{-(k+1)\beta\hbar\omega_0})$ and after a rearrangement of terms, the above equation transforms into

$$\sum_{n=0}^\infty e^{-n\beta\hbar\omega_0} \left[W_n + \frac{\partial W_n}{\partial \xi} + 2 \sum_{k=0}^{n-1} \frac{\partial W_k}{\partial \xi} \right] = 0. \tag{7.128}$$

Since this result holds for any temperature, it follows that

$$W_n + \frac{\partial W_n}{\partial \xi} + 2 \sum_{k=0}^{n-1} \frac{\partial W_k}{\partial \xi} = 0 \tag{7.129}$$

with $W_k = 0$ for $k < 0$. Writing the same equation for $n - 1$ and taking the difference, one gets the simple recurrence relation

$$\frac{\partial W_n}{\partial \xi} + W_n = -\frac{\partial W_{n-1}}{\partial \xi} + W_{n-1}. \tag{7.130}$$

Note that by multiplying this equation by ξ and integrating one obtains $E_n = E_{n-1} + 2\mathcal{E}_0$, in agreement with (7.122). For $n = 0$ it follows that $W_0 = C_0 e^{-\xi}$, so that as a final step it is convenient to write the W_n in the form

$$W_n = C_n e^{-\xi} P_n(\xi), \tag{7.131}$$

leading with the further change of variable $P_n(\xi) = L_n(y)$, $y = 2\xi$ to

$$\frac{dL_n}{dy} = L_{n-1} - \frac{dL_{n-1}}{dy}. \tag{7.132}$$

This is a recurrence relation characteristic of Laguerre polynomials. Retracing our steps we write

$$W_n = C_n e^{-\xi} L_n(2\xi), \quad C_n = \frac{(-1)^n}{\pi\hbar}; \tag{7.133}$$

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[19] The function Π depends on the variable $\beta\hbar\omega$ as shown in equation (7.114); for clarity we will write merely $\Pi(\beta)$ or $\Pi(\omega)$, as convenient.

the C_n have been fixed by the normalization condition. From the differential equation satisfied by the Laguerre polynomials it follows that the W_n are solutions of the equation[20]

$$\xi W_n'' + W_n' + (2n + 1 - \xi)W_n = 0. \tag{7.134}$$

It follows that the W_n form a set of orthonormal functions in ξ space and therefore only one of them, namely the W_0 for the ground state, is everywhere positive. Holding fast to the notion that probabilities should be positive numbers, the W_n cannot be treated as genuine probabilities of physical states, even if they serve as a basis to represent the Wigner phase-space distributions;[21] only nonnegative combinations of them, such as the $W_\beta(\xi)$ above, can represent true probabilities. The fact that the analysis in terms of canonical weights, though very tempting, does not belong to the fundamental theory and that it leads to unphysical states, reinforce the observation made above that it is more formal than physical. It allows us to make contact with the usual tenets of quantum mechanics, but this is achieved at the expense of losing transparency in the physical interpretation. In §7.4.3 we come back to this fundamental dilemma.

We have learned that it is possible to decompose the equilibrium distribution in terms of energy q-states if one is ready to pay the price for it; let us apply this idea directly to the set of differential equations (7.67) and (7.69) that describe the equilibrium situation. Recall that in the radiationless approximation there is no need to distinguish between the mechanical and the canonical variables. It is convenient to introduce the dimensionless variables x', p' defined as $x' = (m\omega_0/\hbar)^{1/2}x$, $p' = (1/m\omega_0\hbar)^{1/2}p$; since there will be many equations in terms of the new variables we omit the primes, which can be restored in the final results. Then, using (7.114), one can write the starting equations in the form

$$p\frac{\partial W_\beta}{\partial x} - x\frac{\partial W_\beta}{\partial p} = 0; \quad pW_\beta + \frac{1}{2}\Pi(\beta)\frac{\partial W_\beta}{\partial p} = 0. \tag{7.135}$$

Applying the same procedure as above to eliminate the dependence on the temperature from the second equation, one arrives after a few transformations at

$$\left(x\frac{\partial}{\partial p} + p\frac{\partial}{\partial x} + 4xp\right)W_n = \left(4xp - x\frac{\partial}{\partial p} - p\frac{\partial}{\partial x}\right)W_{n-1}. \tag{7.136}$$

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[20] This is also the differential equation for the radial Schrödinger amplitude for the s states of the H atom; the generalization to other states can be achieved by considering the multidimensional case. This relationship between the oscillator and the Coulomb problem is well known [see, e.g., Hillery et al. 1984]. The functions W_n appear for the first time in Groenewold (1946).

[21] In França and Marshall (1988) it is shown that the functions W_n form a basis for the representation of phase-space Wigner states. For a discussion on the use of W_n in optical problems see section 13.3.

These equations can be combined into totally symmetric expressions:

$$\left(\frac{\partial}{\partial x}+2x\right)W_n=\left(2x-\frac{\partial}{\partial x}\right)W_{n-1},\quad \left(\frac{\partial}{\partial p}+2p\right)W_n=\left(2p-\frac{\partial}{\partial p}\right)W_{n-1}. \tag{7.137}$$

In quantum statistical mechanics it is more usual to describe the equilibrium system in terms of the density matrix than with the Wigner function. Canonical weights are then a normal assumption, and if one uses a representation in configuration space, the density matrix depends on two position variables x_1, x_2, say, so that $\rho = \langle x_1|\,\hat{\rho}_\beta\,|x_2\rangle = \rho(x_1, x_2; \beta)$. The transition from the Wigner distribution to the density matrix in configuration space is made by means of a *Weyl transformation*, which consists of two steps: first the variable p in $W(x,p)$ is replaced by a new position variable z by means of a Fourier transformation, $W(x,p) \to \tilde{W}(x,z)$; subsequently a linear transformation from x, z to x_1, x_2 yields the density matrix as $\rho(x_1,x_2) = \tilde{W}\left(\frac{1}{2}(x_2+x_1), \frac{1}{2}(x_2-x_1)\right)$ [see, e.g., de Groot and Suttorp 1972, Hillery et al. 1984; this transformation was used in the present context for the first time in Surdin 1971a]. To apply the procedure we introduce the Fourier transform

$$\tilde{W}_n(x,z)=\int_{-\infty}^{\infty}dpW_n(x,p)e^{2ipz}; \tag{7.138}$$

the factor 2 in the exponent is for convenience in the definition of the variable z. Note that $\tilde{W}_n(x,z)$ is the p-momentum generating function (for a given x), and for $z=0$ it gives the marginal density in configuration space for state n: $\tilde{W}_n(x,0)=\int_{-\infty}^{\infty}dpW_n(x,p)=\rho_n(x)$. It will be seen below that these quantities are nonnegative and normalized to unity, so that they can be legitimately interpreted as probabilities. Upon the transformation, the equations become

$$\left(\frac{\partial}{\partial x}+2x\right)\tilde{W}_n=\left(2x-\frac{\partial}{\partial x}\right)\tilde{W}_{n-1},\quad \left(\frac{\partial}{\partial z}+2z\right)\tilde{W}_n=-\left(2z-\frac{\partial}{\partial z}\right)\tilde{W}_{n-1}. \tag{7.139}$$

Now we perform the transition to (x_1,x_2) space by means of the linear transformation

$$x_1=x-z,\quad x_2=x+z, \tag{7.140}$$

which leads with $\rho_n(x_1,x_2)=\tilde{W}_n(x,z)$ to

$$\left(x_1+\frac{\partial}{\partial x_1}\right)\rho_n=\left(x_2-\frac{\partial}{\partial x_2}\right)\rho_{n-1},\quad \left(x_2+\frac{\partial}{\partial x_2}\right)\rho_n=\left(x_1-\frac{\partial}{\partial x_1}\right)\rho_{n-1}. \tag{7.141}$$

We observe from these equations that if ρ_{n-1} can be factorized in the form $\rho_{n-1}=\varphi_{n-1}(x_1)\varphi_{n-1}(x_2)$, then ρ_n can be factorized in a similar form; from the same equations with $n = 0$ it follows that $\rho_0 = \text{const} e^{-x_1^2/2} e^{-x_2^2/2}$, which means that indeed one can set in general

$$\rho_n = \tilde{W}_n \left(\frac{x_2 + x_1}{2}, \frac{x_2 - x_1}{2} \right) = \varphi_n(x_1)\varphi_n(x_2). \tag{7.142}$$

(The functions φ_n are real in the present problem, hence there is no need to distinguish them from their complex conjugate.) Now, $z = 0$ means $x_1 = x_2 = x$, so that $\rho_n(x) \equiv \tilde{W}_n(x, 0) = \varphi_n^2(x)$, whence indeed $\rho_n(x) \geq 0$. One thus obtains from the last pair of equations, applying the operators $\partial_2 + x_2$ to the first one and $\partial_1 + x_1$ to the second, the following separate equations for $\varphi_n(x)$:

$$-\frac{\partial^2 \varphi_n}{\partial x^2} + x^2 \varphi_n = C_n \varphi_n. \tag{7.143}$$

An algebraic way of determining the separation constants C_n is by observing that equations (7.141) imply

$$\left(-\frac{\partial}{\partial x} + x \right) \varphi_n = \alpha_{n+1} \varphi_{n+1}, \quad \left(\frac{\partial}{\partial x} + x \right) \varphi_n = \alpha_n \varphi_{n-1}, \tag{7.144}$$

with the parameters α_n to be fixed. Applying the operator $(\partial + x)$ to the first equation and $(\partial - x)$ to the second one and combining with (7.143) one gets $C_n = \alpha_n^2 + 1$ and the recurrence relation $\alpha_{n+1}^2 - \alpha_n^2 = 2$, from which it follows that $\alpha_n^2 = 2n, C_n = 2n + 1 = E_n/\mathcal{E}_0$.[22] Equation (7.143) becomes, again in terms of the physical variables,

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \varphi_n}{\partial x^2} + \frac{1}{2} m \omega_0^2 x^2 \varphi_n = E_n \varphi_n, \quad E_n = \frac{1}{2} \hbar \omega_0 (2n + 1), \tag{7.145}$$

which is the stationary Schrödinger equation, with the corresponding energy eigenvalues inserted (so that the boundary conditions have been incorporated).

The density matrix in x-space becomes now, from (7.123) and (7.142) (inserting the appropriate conjugation signs),

$$\rho(x_1, x_2; \beta) = \tilde{W}_\beta \left(\tfrac{1}{2}(x_2 + x_1), \tfrac{1}{2}(x_2 - x_1) \right) = \frac{1}{Z} \sum_n \varphi_n^*(x_1) \varphi_n(x_2) e^{-\beta E_n}, \tag{7.146}$$

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[22] We are here merely applying the factorization procedure to determine the eigenvalues. The interesting point is, however, that the idea that quantization follows as the result of algebraic overdetermination of the eigenvalues of a pair of operators that leads to consistency requirements, can be developed for all textbook problems; see de la Peña and Montemayor (1980) and Fernández and Castro (1984).

whereas the Wigner function for the q-state n is, from the inverse of (7.138),

$$W_n(x,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dz\varphi_n^*(x-z)\varphi_n(x+z)e^{-2ipz/\hbar}. \tag{7.147}$$

7.4.2. TIME DEPENDENCE NEAR EQUILIBRIUM

Though up to now we have considered only the state of equilibrium, the same mathematical apparatus can be used to study the solutions of the time-dependent Fokker-Planck equation in the quantum regime. Consider for this purpose any ρ of the form:

$$\rho(x_1,x_2) = \sum_{m,n} C_m^* C_n \varphi_m^*(x_1)\varphi_n(x_2); \tag{7.148}$$

if it does not coincide with equation (7.146), it will of course not be an equilibrium solution and hence the coefficients C_n will in general depend on time. To determine this time dependence, one performs a Weyl transformation of the complete Fokker-Planck equation (7.46); to lowest order in τ one gets

$$\frac{\partial\rho}{\partial t} = \frac{i\hbar}{2m}\left(\frac{\partial^2\rho}{\partial x_1^2} - \frac{\partial^2\rho}{\partial x_2^2}\right) - \frac{im\omega_0^2}{\hbar}\left(x_1^2 - x_2^2\right)\rho. \tag{7.149}$$

This is the *von Neumann equation* that gives the time evolution of the density matrix [see, e.g., de Groot and Suttorp 1972]. Introducing (7.148) into this equation and using (7.145), one obtains upon integration over x_1 and x_2:

$$i\hbar\left(\dot{C}_m^* C_n + C_m^*\dot{C}_n\right) = C_m^* C_n\left(E_n - E_m\right), \tag{7.150}$$

which has the solution $i\hbar\dot{C}_n = C_n E_n$, so that

$$C_n(t) = C_n(0)\exp(-iE_n t/\hbar). \tag{7.151}$$

One can therefore write

$$\rho(x_1,x_2;t) = \sum_{m,n} C_m^*(0)C_n(0)\psi_m^*(x_1,t)\psi_n(x_2,t), \tag{7.152}$$

where $\psi_n(x,t)$ satisfies the time-dependent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_n}{\partial x^2} + \frac{1}{2}m\omega_0^2 x^2\psi_n = i\hbar\frac{\partial\psi_n}{\partial t}. \tag{7.153}$$

In terms of the Wigner function, the time-dependent solution for the oscillator can be written in the general form [see, e.g., Hillery et al. 1984]

$$W(x,p) = \sum_{m,n} C_{mn}(t) W_{mn}(x,p),$$

$$W_{mn}(x,p) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} dz \varphi_m^*(x-z)\varphi_n(x+z)e^{-2ipz/\hbar}. \tag{7.154}$$

These expressions are immediate generalizations of (7.147).

7.4.3. A COMMENT ON PHYSICAL AND UNPHYSICAL STATES

The direct calculation within SED of the equilibrium phase-space density for the oscillator shows that the energy is a continuous variable with a Laplace distribution and with a mean value given by Planck's law, equation (7.117). These results are in agreement with the corresponding quantum statistical predictions when expressed in terms of the Wigner distribution. By adding the demand that the equilibrium distribution be analyzable in terms of Maxwell-Boltzmann weights, equations (7.123), one arrives at the usual quantum description in terms of a Schrödinger equation and energy eigenstates corresponding to the energy q-states previously defined. We have remarked already that the latter formulation is not merely SED, but SED plus the extra postulate on the statistical weights of the energy q-states, and that the W_n cannot be considered true phase-space distributions since they are not everywhere positive, so the q-states can hardly be considered physical states with an independent existence. Nevertheless, the q-states still have a distributed energy if calculated with standard statistical procedures, E_n being just the corresponding average value; in particular, for the ground state the energy has a Laplacian distribution with average $\mathcal{E}_0$. By way of contrast, E_n appears in equation (7.145) as a discrete eigenvalue of the operator $\hat{H} \equiv (-\hbar^2/2m)(\partial^2/\partial x^2) + (m\omega_0^2 x^2/2)$.

It seems that, at least for the description of the harmonic oscillator, the Wigner distribution is to be preferred from the standpoint of SED, since it is compatible with a consistent statistical description. One can still use the alternative Hilbert-space description in SED, but then the meaning of derived constructs such as the wave function or the energy eigenvalues should be ascertained from the primitive description. In quantum theory the inverse procedure is used, namely, the Hilbert-space formalism fixes the starting premises from which the rest follows; thus from the rule $\hat{O}^n = (\hat{O})^n$ applied to the solutions of (7.145) it is concluded that they describe dispersion-free energy states, a conclusion totally incompatible with the statistical interpretation in phase space. As was discussed in section 1.2, the reason for this contradiction lies in the different correspondence rules used in each description; specifically, the Wigner distribution goes hand in hand with the Weyl-Wigner correspondence rule, that establishes that the operators should be put in symmetrical form, which obviously contradicts the above rule.[23] As already mentioned, one advantage of this formulation in terms of correspondence rules is that the dilemma may find eventually an empiricalsolution, as has been repeatedly pointed out by Julg [see Julg 1993 and references therein].

On the other hand, it is true that the excited states play an essential role in quantum theory and cannot be discarded so easily. For example, the transition frequencies between these states are undoubtedly real and observable, so that at least in this sense, a reality should be ascribed to them. It seems that even if the excited states cannot be taken literally as independent physical states, certain physical structures correspond to them, which in one form or another are represented by the quantum excited states. It is illustrative to repeat the above analysis for the Brownian case [de la Peña and Cetto 1977a]. The corresponding equilibrium field has a Rayleigh-Jeans distribution, which gives $Z = \beta^{-1}$, $g(E) = 1$ and a continuum of values for E; no discrete energy levels are obtained of course. This clearly exhibits the essential role played by the Planck distribution in obtaining the discrete spectrum when analyzed in terms of Boltzmann statistical weights. Combining with the results of chapter 5, the quantization can be discerned as a consequence of the action of the zeropoint field.

As mentioned above, the formal character of the excited states is not detrimental to their mathematical utility and legitimacy; similar situations occur frequently in physics, and can be illustrated with the following example borrowed from França and Marshall (1988). Consider a diffusion described by the equation

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2}. \tag{7.155}$$

The physically accessible region for the diffusing particles is assumed to be $0 \leq x \leq a$, and the current $J = D\partial P/\partial x$ is taken to be zero at both end

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[23] Let us verify with a simple example that the algebra of the Wigner functions is not isomorphic to the algebra of the quantum observables. Consider the creation and anihilation operators $\hat{a}, \hat{a}^\dagger$ mapped by the Weyl transform into the complex variables z, z^*, respectively. Then, to the number operator $\hat{N}$ and its square $\hat{N}^2$ there correspond the numbers N_W and N_W^2, respectively, as follows:

$$\hat{N} = \hat{a}^\dagger \hat{a} \longrightarrow \frac{1}{2}\left(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger - 1\right) \longrightarrow |z|^2 - \frac{1}{2} \equiv N_W;$$

$$\hat{N}^2 = \hat{a}^\dagger \hat{a}\hat{a}^\dagger \hat{a} \longrightarrow |z|^4 - |z|^2 \equiv N_W^2 = (N_W)^2 - \frac{1}{4}.$$

This shows that $(F_W)^n$ is no more the function associated to $\hat{F}^n$.

points. Then the solution can be written in the form

$$P(x,t) = \frac{1}{a} + \sum_{n=1}^{\infty} B_n e^{-Dk_n^2 t} \cos k_n x, \quad k_n = \frac{\pi}{a} n. \tag{7.156}$$

The coefficients B_n are determined by the initial distribution of particles, but the condition $P(x,t) \geq 0$ must hold for all t, so that not all values of the constants B_n are allowed by the physical constraints. Even if the terms which describe the time evolution towards the final uniform distribution $W = 1/a$ =const are not positive definite and cannot be considered each one to be a probability in itself, their combined contribution is of course a probability. This shows that in the present instance the Fourier decomposition is deprived of a true probabilistic meaning, although it is mathematically meaningful, legitimate and useful. Quite the contrary happens, for example, with a vibrating string and similar periodic problems for which the Fourier representation is so physically meaningful, that it is even possible to separate any component by means of appropriate filters and manipulate it at will.

Cole (1993c) has recently presented a related discussion from a complementary point of view. By expressing the statistical distribution in terms of eigensolutions of an equation of the Schrödinger type and demanding that the thermal flux for each one of the respective 'states' be zero at equilibrium, he is able to show that the method in quantum statistical mechanics of counting the 'states' in thermal equilibrium by using the energy eigenstates, appears simply as a convenient counting scheme rather than actually representing averages over physically discrete energy states.

The linear combinations of the q-states that lead to nonnegative probabilities are of course allowed by the present theory, and correspond to coherent states or combinations of them, as shown by Santos (1974a), Goedecke (1984) and França and Marshall (1988) (see also §7.5.3). A first-order perturbative calculation proves that the evolution of a coherent state on a large time scale compared with the natural period of oscillation, is indeed well represented by a minimum-uncertainty wave packet whose centre moves according to the laws given by the above description [Santos 1974a]. Thus, the inherent physics seems to be equivalent in both descriptions up to a certain extent, even if the imageries differ widely.

7.5. Radiative effects on the harmonic oscillator

7.5.1. TRANSITION PROBABILITIES; EINSTEIN *A* AND *B* COEFFICIENTS

We shall now study to first order in γ the effects of the correction terms in the Fokker-Planck equation (7.48) due to both radiation reaction and diffusion. The cross diffusion term, proportional to D_{xp}, produces the Lamb shift of the levels, which will be calculated in §7.5.2; the remaining terms give rise to the evolution of the otherwise stationary states of the system. In quantum theory this evolution is commonly described in terms of transitions between the energy eigenstates; thus, in order to compare with QED we shall study the time evolution predicted by the Fokker-Planck equation in terms of transitions between q-states. We follow the exposition given in França and Marshall (1988). A different procedure consists in deriving a modified Schrödinger equation that includes the radiative terms, instead of studying them at the level of the Fokker-Planck (or Wigner) equation; such an approach can be seen in de la Peña and Cetto (1977a) and Goedecke (1984). It is of interest to note that the radiation reaction terms that appear in the modified Schrödinger equation coincide with those proposed by Crisp and Jaynes (1969) within the semiclassical approach.

Our starting point is the Fokker-Planck equation for $W(J)$, obtained by integrating (7.59) over θ. Since $m\omega_0\sigma_x^2 = (\hbar/2)\Pi(\omega_0)$, with $\Pi(\omega) = \mathcal{E}/\mathcal{E}_0$ according to (7.116), this equation can be recast into

$$\frac{\partial W}{\partial t} = \gamma \left[W + \Pi \frac{\partial W}{\partial \xi} + \xi \frac{\partial W}{\partial \xi} + \Pi \xi \frac{\partial^2 W}{\partial \xi^2} \right] \tag{7.157}$$

in terms of the dimensionless variable $\xi = (2/\hbar)J = H/\mathcal{E}_0$. Recall that in writing equation (7.59) the term with crossdiffusion coefficient D_{xp} was neglected, so that the study of its effects requires separate consideration. In the presence of an external field the spectral density has the form $\rho(\omega) = \rho_0(\omega) + \rho_e(\omega)$, where the first term refers to the zeropoint field and the second one to the external contribution; alternatively, $\Pi(\omega) = 1 + (\rho_e(\omega)/\rho_0(\omega))$. Now let us single out a q-state n as the initial state (without presupposition about its independent existence); with time, other q-states become populated. The evolving situation can be represented by a solution of the form

$$W(\xi, t) = \sum_{k=0}^{\infty} C_k(t) W_k(\xi), \quad \sum_{k=0}^{\infty} C_k(t) = 1, \quad C_k(0) = \delta_{nk}, \tag{7.158}$$

with the $W_k(\xi)$ given by (7.133). The calculations are greatly simplified by noting that the recurrence relations between the Laguerre polynomials lead to

$$\gamma \left[1 + \Pi \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial \xi} + \Pi \xi \frac{\partial^2}{\partial \xi^2} \right] W_k =$$

$$= \frac{1}{2}\gamma(\Pi - 1)(k+1)W_{k+1} - \frac{1}{2}\gamma \left[\Pi(2k+1) - 1 \right] W_k + \frac{1}{2}\gamma(\Pi + 1)kW_{k-1}. \tag{7.159}$$

Substituting (7.158) in (7.157) and using the above expression one gets

$$\dot{C}_n = \frac{1}{2}\gamma(\Pi - 1)nC_{n-1} - \frac{1}{2}\gamma\left[\Pi(2n+1) - 1\right]C_n + \frac{1}{2}\gamma(\Pi + 1)(n+1)C_{n+1}. \tag{7.160}$$

Accordingly, to first order in γ there are transitions between the q-state n and its first neighbours $n+1$, $n-1$. In terms of the *transition probabilities* $P_{n\to n+1} \equiv P_{n,n+1}$, $P_{n\to n-1} \equiv P_{n,n-1}$, $P_{n\to n} \equiv P_{n,n}$, equation (7.160) takes the form

$$\dot{C}_n = P_{n+1,n}C_{n+1} + P_{n-1,n}C_{n-1} + (P_{n,n} - P_{n,n+1} - P_{n,n-1})C_n, \tag{7.161}$$

which is a master equation for q-state n.[24] The first two (positive) contributions describe the increase of the population of n due to downward transitions from $n+1$ (emissions, both stimulated and 'spontaneous') and upward transitions from $n-1$ (stimulated absorptions); the last two (negative) terms describe the depopulation of n due to transitions towards $n-1$ and $n+1$. A comparison of the last two expressions furnishes the formulas

$$P_{n,n+1} = \frac{1}{2}\gamma(\Pi - 1)(n+1); \quad P_{n,n} = 0; \quad P_{n,n-1} = \frac{1}{2}\gamma(\Pi + 1)n. \tag{7.162}$$

These results show a very interesting and important phenomenon, of which a hint was found in §5.3.1, namely, that whereas diffusion and radiation reaction reinforce each other to induce emissions , the rate of absorptions is determined by the difference of these two contributions, as indicated by the subtraction of terms in the coefficient of $P_{n,n+1}$. For the zeropoint field, $\Pi(\omega) = 1$ and these two contributions are equal, so that no (spontaneous) absorptions are fired by the zeropoint field. Only in the presence of an external field, $\rho_e \neq 0$, can there be absorptions. More explicitly,

$$P_{n,n-1} = \frac{1}{2}\gamma(\Pi + 1)n = \frac{1}{2}\gamma\left(1 + \frac{\rho_e}{\rho_0}\right)n + \frac{1}{2}\gamma n = \frac{\gamma n}{2\rho_0(\omega_0)}\rho_e(\omega_0) + \gamma n, \tag{7.163}$$

so that in terms of the $B_{n,n-1}$ probability coefficient of stimulated emissions and the $A_{n,n-1}$ coefficient of spontaneous emissions as defined in §5.3.1, the above expression reads $P_{n,n-1} = B_{n,n-1}\rho_e(\omega_0) + A_{n,n-1}$, and we get

$$B_{n,n-1} = \frac{\gamma n}{2\rho_0(\omega_0)}; \quad A_{n,n-1} = \gamma n = \frac{\gamma}{\hbar\omega_0}(E_n - E_0). \tag{7.164}$$

The second equality in equation (7.163) shows explicitly that the vacuum field fluctuations and radiation reaction contribute in equal shares to the value γn of the probability of spontaneous emission, in full agreement with the results of QED.[25] An analogous calculation leads to the following result for the absorption probabilities:

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[24] For an introductory discussion of the master equation see, e.g., van Kampen 1981, chapter 5.

$$B_{n,n+1} = \frac{\gamma(n+1)}{2\rho_0(\omega_0)}; \quad A_{n,n+1} = 0. \tag{7.165}$$

These formulas coincide with the results furnished by quantum mechanics (for the B's) and by QED (for the A's and B's); they exhibit clearly the meaning of the Einstein relations $B_{n-1,n} = B_{n,n-1}$, $A_{n,n-1}/B_{n,n-1} = 2\rho_0(\omega_0)$. Most important is the result $A_{n,n+1} = 0$ establishing that no spontaneous absorptions are predicted by SED; this is fundamental for the stability of the ground state and far from trivial. In chapter 11 we will come back to the study of these important coefficients.

From equation (7.161) with $P_{n,n} = 0$ we see that the decay rate of the q-state n, defined by $\dot{C}_n = -\Gamma_n C_n$, or $C_n = C_{0n} e^{-\Gamma_n t}$, is[26]

$$\Gamma_n = P_{n,n+1} + P_{n,n-1} = \frac{\gamma}{2}\left[(2n+1)\Pi(\omega_0) - 1\right] = \frac{\tau\omega_0}{\hbar}\left[\Pi(\omega_0)E_n - E_0\right], \tag{7.166}$$

so that upon separation of the stimulated and the spontaneous part one gets

$$\Gamma_n = \Gamma_n^{\text{stim}} + \Gamma_n^{\text{spont}} = \frac{\tau\omega_0}{\hbar}E_n\frac{\rho_e(\omega_0)}{\rho_0(\omega_0)} + \frac{\tau\omega_0}{\hbar}(E_n - E_0). \tag{7.167}$$

Hence the oscillator in its ground state can be excited only by an external field above the vacuum ($\rho_e \neq 0$), and it is stable when immersed in the zeropoint field. This agrees with Nernst's point of view discussed in chapter 4, in the sense that the stability of the ground state ($\Gamma_0 = 0$ for $\rho_e = 0$) requires the existence of the zeropoint field; but the theory goes further, by allowing *in principle* for spontaneous emissions from excited states. Similar arguments in QED, leading to the explicit recognition of the need of the zeropoint field to explain the stability of the ground state of quantum systems, may sometimes be found in the literature [Fain 1966, 1982, Fain and Khanin 1969, Dalibard et al. 1982, Milonni 1994].

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[25] The QED problem was clarified in Milonni et al. (1973) and Senitzky (1973); for detailed discussions see Cohen-Tannoudji (1986) and Milonni (1994). An introductory quantum discussion on the linewidth and the related Wigner-Weisskopf theory can be found in the last cited book, chapter 4. The first calculation of A_{nm} for the SED harmonic oscillator was given in Marshall (1968).

[26] For more details see de la Peña and Cetto (1979). Note that in this and related papers, D_{xp} differs in sign with respect to the one used here.

7.5.2. LAMB SHIFT OF THE EXCITED STATES

The correction to the energy levels resulting from the diffusion coefficient D_{xp} can be obtained by restoring the corresponding term in the Fokker-Planck equation. A detailed calculation of the correction to the energy of a given q-state can thus be performed [de la Peña and Cetto1979]; however, with the results at hand we can skip it by noting that the formula $\delta E_L = \frac{1}{2} D_{xp}$ that was derived in §7.2.4 is general and applies to any q-state. This can be easily verified by resorting to equation (7.65), since on multiplying by xp and integrating over the whole phase space, after some integrations by parts to eliminate the derivative operators one gets twice the correction to the energy, which results in $\int dxdpD_{xp}W = D_{xp}$ for any W, with D_{xp} given by equation (7.75). Hence equation (7.106) gives the Lamb shift for any state of the harmonic oscillator with the radiation field at zero temperature, $\rho(\omega) = \rho_0(\omega)$.

Now suppose there is an external (stationary) radiation field, characterized by $\rho_e(\omega)$. To calculate D_{xp} under this condition one can use again the procedure of §7.2.4, starting with equation (7.81); one thus arrives at

$$D_{xp} = D_{xp}^{(\rho_0)} + D_{xp}^{(\rho_e)}, \tag{7.168}$$

with the original $D_{xp}^{(\rho_0)}$ given by (7.105) and the additional contribution given by

$$D_{xp}^{(\rho_e)} = \frac{\tau \hbar \omega_0^2}{\pi} \int_0^{\omega_c} d\omega \frac{\omega(\omega^2 - \omega_0^2)}{|\Delta(\omega)|^2} \frac{\rho_e(\omega)}{\rho_0(\omega)} = \frac{\hbar\beta}{\pi\omega_1} \left[T(\omega_1) - T(-\omega_1)\right], \tag{7.169}$$

$$T(\omega_1) \equiv \int_0^{\omega_c} du \left[\frac{\beta^2 + \omega_1(u + \omega_1)}{\beta^2 + (u + \omega_1)^2}\right] \frac{\rho_e(u)}{\rho_0(u)}. \tag{7.170}$$

It is clear from the calculations that led to equation (7.105) that the main contribution to D_{xp} comes from the higher frequencies, $\omega \sim \omega_c$. Therefore, since normally $\rho_e/\rho_0 \ll 1$ at high frequencies, the value obtained for the extra term $D_{xp}^{(\rho_e)}$ will be negligible compared to the zeropoint term $D_{xp}^{(\rho_0)}$. For instance, for blackbody radiation $\rho_e/\rho_o \rightarrow_{\omega\rightarrow\infty} 2\exp(-\beta\hbar\omega)$, which is very small except at extremely high temperatures.

A similar reasoning leads us to conclude that the contribution of ρ_e to the mass correction, calculated from the formula (7.78), is also negligibly small compared with the corresponding zeropoint term. A brief discussion of related questions can be found in §7.5.4.

7.5.3. COHERENT EXCITATIONS

Let us now consider a harmonic oscillator that is subject to an external radiation with a fixed amplitude F and a frequency ω close to ω_0, in addition to the zeropoint field. The equation of motion is then

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{e}{m}E_x(t) - \frac{e}{m}F\sin\omega t \tag{7.171}$$

and its stationary solution can be written in the form $x(t) = x_0(t) + x_F(t)$, where x_0 is the usual solution for the ground state and x_F describes the forced, stationary solution with fixed amplitude

$$x_F = \frac{1}{m}\mathrm{Re}\left[\frac{ieFe^{i\omega t}}{\omega_0^2 - \omega^2 + i\gamma\omega}\right]. \tag{7.172}$$

This complete stationary solution applies for t much greater than the lifetimes of the excited states involved; for $\omega = \omega_0$ it represents a coherent state of the harmonic oscillator, as can be seen as follows [França and Marshall 1988]. The phase-space distribution is given by the joint distribution of the variables $x_0 = x - x_F$, and $p_0 = p - p_F$, so that (see (7.24))

$$W(x,p) = \frac{1}{\pi\hbar}e^{-(x-x_F)^2/2\sigma_x^2-(p-p_F)^2/2\sigma_p^2}. \tag{7.173}$$

This distribution can be analyzed in terms of q-states with the help of equations (7.154), which give (here $p_F = m\dot{x}_F$)

$$C_{mn}(t) = \int_{-\infty}^{\infty} dxdpW_{mn}^*(x,p)W(x,p,t) = \sqrt{\frac{\sigma^{m+n}}{m!n!}}e^{-\sigma+i(m-n)(\omega t+\delta)}, \tag{7.174}$$

$$\sigma(t)e^{i\delta(t)} = \frac{1}{2}\left[\frac{x_F(t)}{\sigma_{0x}} + i\frac{p_F(t)}{\sigma_{0p}}\right]e^{-i\omega t}. \tag{7.175}$$

From the last expression and equation (7.172) it follows that at $\omega = \omega_0$, the functions $\sigma(t)$ and $\delta(t)$ become time-independent and take on the constant values $\sigma = e^2F^2/2m\gamma^2\omega_0\hbar$, $\delta = 0$; the expansion (7.174) corresponds then to the usual quantum mechanical description of a coherent state for the harmonic oscillator [see, e.g., Schiff 1965, p. 74]. From this point of view, the coherent state represents the state in which the effects of the zeropoint field and the driving field are balanced by the oscillator's radiation reaction, and the ground state represents a particular case of coherent state obtained for $F = 0$. Hence, equations (7.172), (7.174) and (7.175) can be considered to generalize the description of the coherent state by including the radiative correction.

The oscillator driven by the external field F will of course be radiating. The dominant emitted radiation at low incident intensities is still dipolar, with a spectrum given by the Fourier transform of the correlation function, which consists now of two separate terms:

$$\langle \ddot{x}(s)\ddot{x}(s+t)\rangle = \langle \ddot{x}_0(s)\ddot{x}_0(s+t)\rangle + \langle \ddot{x}_F(s)\ddot{x}_F(s+t)\rangle\,, \tag{7.176}$$

since the motions of x_0 and x_F are uncorrelated. The first term is the usual one characteristic of the ground-state oscillator; the corresponding radiated power, proportional to $\langle \ddot{x}_0^2(s)\rangle$, is compensated by the power absorbed by the oscillator from the zeropoint field. It follows that only the sharp line of frequency ω is observable.

Let us now consider an atom driven by external radiation of a frequency close to one of the fundamental transition frequencies, say between the ground state and the first excited state. At low intensities the emitted radiation, due to both scattering and resonance fluorescence, is reasonably well accounted for by electric dipole radiation or, in the usual language of QED, by one-photon processes. Under this circumstance the Lorentz model of the atom is useful and one may apply the results obtained above for the harmonic oscillator. Normally the incident radiation is not strictly monochromatic but is distributed around ω, so that the fixed amplitude F should be replaced by a function $F(\omega)$. It is clear from (7.172) that for a broadband $F(\omega)$, the scattered radiation acquires the natural lineshape, determined by the inverse of the parameter γ in the denominator; but when the incident radiation has a sharp spectrum, the dominant contribution to the emitted radiation also has a sharp spectral distribution, (almost) independent of the natural line shape of the atom.

This is of course evident for the Rayleigh component, but it turns out to be true also for the resonance fluorescence [see, e.g., Knight and Milonni 1980]. The phenomenon was predicted on the basis of QED [Weisskopf 1931, Heitler 1966] much before the advent of tunable narrow-band coherent sources of radiation made its observation feasible. However, normally the observed phenomenon is much more complex, there being several factors that contribute to this complexity, such as the finite profile of the incoming radiation, the transient behaviour of the system and higher-order contributions to the radiation. In simple terms the phenomenon can be better described as a kind of (resonant or not) dynamical Stark effect, with elastic and inelastic decay modes.[27]

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[27] Detailed accounts of this classical problem of QED can be seen, e.g., in Knight and Milonni (1980), Loudon (1973), chapter 8, or Walls and Milburn (1994), chapter 11. At higher incident intensities the stimulated emissions can contain nonlinear contributions, such as the Rabi oscillations of the state probabilities and other effects that have been entirely neglected in the simple calculation presented above.

7.5.4. CAVITY SED

The atomic radiative corrections can be modified by altering the radiation field in which the atoms are immersed, which can be achieved in different ways. On one hand it is possible to modify the energy content of the field modes, for instance by adding an intense external radiation or raising the temperature of the whole system; alternatively it is possible to change the structure of the field, for instance with the help of conducting plates that affect the boundary conditions (recall the examples given in chapter 6).

Effects on the atomic properties produced by such modifications of the field have been the subject of much theoretical and experimental work; various environmental phenomena, such as modifications to the mass, the emitted frequencies, the lifetimes, and so on, have been predicted and observed, already for almost half a century. For example, Purcell as early as 1946 predicted the influence of a resonant cavity on the relaxation time of a spin system, and Power (1966a) predicted changes in the self energy and the mass of a charged particle confined within conducting plates; Drexhage and Kuhn (1966) observed a modulation of the fluorescence decay time of a dye molecule close to a mirror, and others have observed changes in the lifetimes of highly excited states of atoms strongly coupled to the electromagnetic field through their large polarizabilities [Goy et al. 1983, de Martini 1986, etc.]. By enclosing an atomic system in a small, high-Q cavity so as to expel the field modes corresponding to the decay modes, the lifetimes of excited states have been considerably prolonged [see, e.g., Hulet et al. 1985 and Goy et al. 1983].

Such phenomena constitute what has come to be called *cavity quantum electrodynamics.*[28] We shall briefly discuss this kind of phenomena from the standpoint of SED, following the expositions in Cetto and de la Peña (1988a, b, c) and using, as before, the Lorentz-Drude model for a simplified description of the atoms. One should bear in mind that the frequency ω_0 of the oscillator depends now on the state and transition of interest, but it is assumed fixed in every instance.

Modifications of the energy

Let us start with a discussion of the modifications of the atomic Lamb shift and the self-energy. As follows from §7.2.3, the Lamb shift is given in the three-dimensional case by $\delta\mathcal{E}_L = \frac{1}{2}\mathrm{Tr}(D_{xp})$, assuming that the nondiagonal terms in $\langle\sum(D_{xp})_{ij}x_i p_j\rangle$ average to zero. The explicit expression for the required diffusion coefficient can be obtained from (7.41), where $\mathrm{x}_\alpha(t)$ is the solution of equation (7.37). In the present case this equation is the Abraham-Lorentz equation for the harmonic oscillator, so that (7.41) gives

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[28] An introduction to cavity quantum electrodynamics is given in Haroche and Kleppner (1989); see also Berman (1994).

$$\delta\mathcal{E}_L = -\frac{e^2\omega_0}{2mc^2}\int_0^\infty ds\,\langle \mathbf{A}(t)\cdot\mathbf{A}(t-s)\rangle \sin\omega_0 s. \tag{7.177}$$

Now we assume that the background radiation field is changed from the original configuration, distinguished with the subindex 0, to a final one, denoted by the subindex e. The change in $\delta\mathcal{E}_L$ is then given by

$$\varepsilon_1^{\mathrm{LS}} = -\frac{e^2\omega_0}{2mc^2}\int_0^\infty ds\,\left[\langle \mathbf{A}(t)\cdot\mathbf{A}(t-s)\rangle_e - \langle \mathbf{A}(t)\cdot\mathbf{A}(t-s)\rangle_0\right] \sin\omega_0 s. \tag{7.178}$$

Analogously one obtains for the modification of the self-energy of the free particlemass corrections!cavity effects, given by the $\langle \mathbf{A}^2\rangle$-term in equation (7.76),

$$\varepsilon_1^{\mathrm{FP}} = \frac{e^2}{2mc^2}\left[\left\langle \mathbf{A}^2\right\rangle_e - \left\langle \mathbf{A}^2\right\rangle_0\right]. \tag{7.179}$$

Consider the simple example of a system which is artificially irradiated, giving rise to a 'Lamp shift', as it was named by Kastler [Dupont-Roc et al. 1967]. Assuming the somewhat artificial situation in which the spectral energy is raised to $\rho_e(\omega) = (2\bar{n}+1)\rho_0$ for $a < \omega/c < b$, but leaving ρ_0 untouched outside this range, the above equations lead to

$$\varepsilon_1^{\mathrm{FP}} = \bar{n}\frac{\alpha\hbar^2}{\pi m}(b^2-a^2), \quad \varepsilon_1^{\mathrm{LS}} = \bar{n}\frac{\alpha\hbar^2\omega_0^2}{\pi mc^2}\ln\left|\frac{b^2-\omega_0^2/c^2}{a^2-\omega_0^2/c^2}\right|. \tag{7.180}$$

These results coincide with those obtained from QED by Knight (1972).

Alternatively, when all modes of frequency $\omega < \omega_m$ are eliminated without greatly affecting the rest of the modes, as is approximately the case when the system is placed in the field of a waveguide with a characteristic length $L \simeq 2\pi c/\omega_m$, the free and the bound energies are shifted by the amounts

$$\varepsilon_1^{\mathrm{FP}} = -\frac{\alpha\hbar^2\omega_m^2}{2\pi mc^2}, \quad \varepsilon_1^{\mathrm{LS}} = -\frac{\alpha\hbar^2\omega_0^2}{2\pi mc^2}\ln\left|1-\omega_m^2/\omega_0^2\right|. \tag{7.181}$$

We see that for $\omega_0 \gg \omega_m$ both contributions tend to cancel out, which means that a tightly bound particle does not 'feel' the presence of the wall. The reason for this behaviour is that the field modes which contribute most to the kinetic energy of the particle, namely those with ω near to ω_0, were not altered by the waveguide. For an almost free particle, on the other hand, $\omega_0 \ll \omega_m$ and $\varepsilon_1^{\mathrm{LS}}$ becomes a small correction to $\varepsilon_1^{\mathrm{FP}}$; in this case the modes of frequency close to ω_0 are absent and their contribution to the bound particle becomes negligible.

Modifications of the lifetimes and mass correction

Let us now consider the effects of the field alterations on the atomic lifetimes and associated phenomena. Since the isotropy of the radiation field is in general broken down by the introduction of macroscopic bodies such as mirrors and conducting surfaces, it is convenient to use the notation for the field introduced in §3.1.2, in terms of the orthonormal functions $\mathbf{G}_\alpha$. From section 3.2 one sees that in the dipole approximation and for long times, the radiation reaction $\mathbf{F}_{\text{self}}$ becomes

$$\mathbf{F}_{\text{self}} = -2\pi e^2 \sum_\alpha \mathbf{G}_\alpha \mathbf{G}_\alpha^* \cdot \int_0^\infty ds\, \dot{\mathbf{x}}(t-s) e^{-i\omega_\alpha s} + \text{cc}. \tag{7.182}$$

Using for $\dot{\mathbf{x}}(t)$ the approximate zero-order solution of the oscillator and integrating by parts, one gets for the components of this self-force

$$F_i^{\text{self}} = -\gamma_{ij}\dot{x}_j - \delta m_{ij}\ddot{x}_j, \tag{7.183}$$

with

$$\gamma_{ij} = 2\pi e^2 \sum_\alpha G_{\alpha i} G_{\alpha j}^* I_\alpha^c + \text{c.c.}; \quad \delta m_{ij} = -\frac{2\pi e^2}{\omega_0} \sum_\alpha G_{\alpha i} G_{\alpha j}^* I_\alpha^s + \text{c.c.}, \tag{7.184}$$

$$I_\alpha^c = \int_0^\infty ds \cos\omega_0 s e^{-i\omega_\alpha s}, \quad I_\alpha^s = \int_0^\infty ds \sin\omega_0 s e^{-i\omega_\alpha s}. \tag{7.185}$$

A change in the geometry of the environment will affect the functions $G_{\alpha i}$ that enter into both the friction coeffficient γ_{ij} and the mass correction δm_{ij}, and the anisotropy of the field will reflect itself in the nondiagonal character of these tensors. The first calculations of the mass shift for a particle within a rectangular cavity can be found in Power (1966a), where it is shown that the leading contribution is inversely proportional to the shortest length of the cavity, L, and that it is possible in principle to obtain a measurable effect. To illustrate this effect with a simple order-of-magnitude calculation, let us here consider the free particle within a waveguide of length L that excludes all modes of frequency $\omega < \omega_m \simeq 2\pi c/L$ (without paying attention to the anisotropy). The mass shift becomes

$$\frac{\delta m}{m} = \frac{2}{\pi}\tau \left(\int_{\omega_m}^\infty d\omega_\alpha - \int_0^\infty d\omega_\alpha \right) = -\frac{4\alpha}{3\pi}\frac{\hbar\omega_m}{mc^2} \simeq -\frac{8\alpha}{3}\frac{\hbar}{mcL}, \tag{7.186}$$

which is of order α^4 for $L \simeq 5{,}000$ Å, corresponding to optical frequencies.

Note, further, that according to equation (7.184), the rate at which the particle dissipates energy depends on the environment through the coefficient γ_{ij}. We have repeately seen that from the standpoint of SED, the stationary atomic states are characterized by a dynamical equilibrium in which the loss of energy through radiation reaction is compensated by a gain of energy from the background field. This balance condition became expressed rather indirectly in the one-dimensional case through the fluctuation-dissipation relation (7.43). A similar relation guarantees that the equilibrium situation is maintained when the oscillator is introduced inside the cavity; a calculation of the components of the diffusion tensor $\mathbf{D}_{xx}(x,t)$ from equation (7.41) for a spherically symmetrical harmonic oscillator gives

$$D_{ij}^{xx} = \frac{\mathcal{E}(\omega_0)}{m\omega_0^2}\gamma_{ij}, \tag{7.187}$$

of which (7.43) is a particular instance. This fluctuation-dissipation relation lies at the root of an astonishing property of atomic systems, namely, the almost absolute insensitivity of their stationary states to perturbations of the surroundings, despite the fact that it is the surrounding vacuum field which sustains these states. To see the mechanism behind this stability, recall that in the anisotropic case the Fokker-Planck equation becomes instead of (7.66)

$$\gamma_{ij}x_j W_0 + D_{ij}^{xx}\frac{\partial W_0}{\partial x_j} = 0, \tag{7.188}$$

which, with the help of the previous relation and $\mathcal{E}(\omega_0) = m\omega_0^2\sigma_x^2$, can be recast in the form $\gamma_{ij}\left(x_j W_0 + \sigma_x^2 \partial W_0/\partial x_j\right) = 0$, whose solutions $W_0(x)$ are independent of the tensor γ. Although the presence of conducting objects may alter the components of the tensors γ and $\mathbf{D}$, it does not influence the equilibrium state of the oscillator in the radiationless approximation, owing to the proportionality between the two tensors; the environmental effects show up only in the radiative corrections. From the point of view of QED this conclusion sounds quite trivial, since the radiative corrections are supposed to be the *only* effect of the vacuum field on the atomic electron; according to this view, switching off the field (including the vacuum) simply takes us from QED to quantum mechanics. For SED, however, the whole of the quantum behaviour of the electron is a result of the action of the random background, and switching off the zeropoint field would transform the electron into a classical particle. In practice, even if the zeropoint field cannot be switched off, its structure can be modified, but such modifications do not substantially affect the quantum behaviour of the particle, as we have just seen.

It is important to observe that equation (7.187) —as well as more general fluctuation-dissipation relations, such as the one referred to in section 8.4— applies to a single frequency, i.e., it establishes a detailed balance taking place separately for every frequency ω_0. This means that in solving equation (7.188) a spectral analysis is implicitly made, and in the general case of a nonlinear force the insensitivity to environmental changes will only take place if the response of the system to each relevant frequency can be isolated from the rest. Such a spectral analysis (by the way, characteristic of Einstein's method for the calculation of the A, B coefficients) of course cannot be applied to classical (nonlinear) systems. This observation anticipates problems to be met whenever the present approach is applied to arbitrary nonlinear bound systems, in particular to atoms, as will be discussed in chapter 9. For the moment we simply stress the fundamental role played by the sharp resonance of the harmonic oscillator in the preceding derivations.

As a further interesting application of the theory, one can calculate the effects of the field configuration on the spontaneous emission rates; this is another instance of a phenomenon that becomes transparent in the light of SED. By combining equation (7.166) and $\Pi = 1 + \rho_e/\rho_0$, the decay rate of the oscillator in the q-state n in the presence of an external field (which we consider isotropic, for simplicity), can be cast in the form

$$\Gamma_n = \frac{\gamma}{2}\left[(2n+1)\Pi(\omega_0) - 1\right] = \frac{\gamma}{2}\left[(2n+1)\frac{\rho_e(\omega_0)}{\rho_0(\omega_0)} + 2n\right]. \tag{7.189}$$

It is easy to realize that the contribution proportional to Π comes from the diffusion, whereas the rest comes from radiation damping; therefore, for a particle oscillating along the axis k we may write

$$\Gamma_n^e = \frac{m\omega_0^2}{2\mathcal{E}_0}(2n+1)D_{kk}^{xx} - \frac{\gamma_{kk}}{2} = \frac{\gamma_{kk}}{2}\left[(2n+1)\frac{\rho_e(\omega_0)}{\rho_0(\omega_0)} + 2n\right]. \tag{7.190}$$

From this result it follows in particular that the radiation rate can be increased or decreased by changing the density of modes at the corresponding frequency. Whereas the decay can be significantly enhanced using a resonant cavity, it is also possible to inhibit it by enclosing the oscillator in a cavity that does not admit stationary field modes of frequency ω_0. Such effects have been produced repeatedly in the laboratory, and the results derived from (7.190) coincide with those obtained from quantum calculations [Haroche and Kleppner 1989]. For instance, for particles oscillating on a plane parallel or perpendicular to the plates, respectively, with $q = \omega_0 L/\pi c$ and $[q]$ representing the integral part of q, one obtains[29]

$$\left.\frac{\Gamma^e}{\Gamma}\right|_{\parallel} = \frac{3}{2q}\sum_{n=0}^{[q]}\left(1+\frac{n^2}{q^2}\right)' \sin^2\frac{\pi z}{L}n, \qquad \left.\frac{\Gamma^e}{\Gamma}\right|_{\perp} = \frac{3}{q}\sum_{n=0}^{[q]}\left(1-\frac{n^2}{q^2}\right)' \cos^2\frac{\pi z}{L}n \tag{7.191}$$

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[29] Cetto and de la Peña (1988c); first results of this kind were derived within SED already by Marshall (1965a).

(the prime indicates that the $n = 0$ term is multiplied by $\frac{1}{2}$). From these expressions it follows that for very close plates, when $q < 1$, only the $n = 0$ term contributes and $\Gamma^e/\Gamma|_{\parallel} = 0$, $\Gamma^e/\Gamma|_{\perp} = 3/2q$, which means that an oscillator vibrating parallel to the plates does not radiate, while the transverse oscillator decays at a rate higher than in vacuum; as $q \longrightarrow 0$, its lifetime goes to zero. A more elaborate problem is studied in França et al. (1992); these authors consider a weak magnetic field applied so that the effects of the anisotropy can be observed, and thus obtain a prediction for the angular dependence of the fraction of atoms that survive after traversing the gap between the mirrors, in very close correspondence with the experimental results of Jhe et al. (1987).

CHAPTER 8

QUANTUM PROPERTIES OF OTHER SIMPLE SYSTEMS

In this chapter we pursue our investigations on the behaviour of simple SED systems and compare the results with the quantum description in similar circumstances. Four problems are treated, namely, the free particle, the properties of atoms immersed in a magnetic field, the nature of the electron spin and the specific heat of solids. The free particle does not present in itself any particular problem, at least in the nonrelativistic, long-wavelength approximation used here; anyway, the fluctuation of its energy offers the opportunity to take a fresh glance at a complex phenomenon which in QED is described in terms of absorptions and emissions of virtual photons. On the other hand, diamagnetism is considered a purely quantum phenomenon because it does not occur in a classical context; similarly, Debye's law for the specific heat of solids is conventionally derived within a quantum framework. So, the issue of whether or not SED can correctly describe these phenomena becomes most fundamental for the assessment of its potentialities. We will find out that in both cases SED discloses the quantum properties of matter as originating in its interaction with the zeropoint field, as opposed to the primitive character conferred on them by conventional quantum theory.

The problem of the spin of the electron has received little attention within SED, but the few known results suggest that this theory may be of value in the (not very popular) effort to unravel its mysteries, by offering a dynamical explanation of it. The results obtained up to now are still partial and unripe, so the exposition of this topic is of a more speculative nature.

8.1. The free particle

In the derivations of the Abraham-Lorentz equation it is normally assumed that the external force prevails everywhere over the radiation reaction, so that this equation is not expected to apply if there is no external force, i.e., for the free particle [see, e.g., Jackson 1975]. Now, although this is true in usual classical physics, in SED it is not the case, since the Lorentz force of the zeropoint field is always present. If desired, one may consider the more general case of the particle subject to a constant external force with equation of motion

$$m\ddot{\mathbf{x}} = \mathbf{F} + m\tau\dddot{\mathbf{x}} + e\mathbf{E}(t), \tag{8.1}$$

and take eventually the limit of a null external force; the procedure shows that there is no fundamental difference in the behaviour of the solution and that one can take $\mathbf{F} = 0$ right from the beginning.[1]

According to the discussion in §3.2.2, the simplest way to avoid the runaway solutions of equation (8.1) is by transforming it into

$$m\,\ddot{\mathbf{x}} = \mathbf{F} + \frac{e}{\tau}\int_t^\infty dt'\mathbf{E}(t')e^{(t-t')/\tau}, \tag{8.2}$$

which contains a preacceleration effect of the field (the constant force $\mathbf{F}$ is not modified by this transformation). The spectral density of the transformed electric field is given by $\rho(\omega)/(1+\tau^2\omega^2)$; however, since a cutoff is necessary anyway (as discussed in §3.2.3), and $\tau\omega << 1$ even for frequencies as high as Compton's value ω_C, one can forget about the small correction term in the denominator, which amounts to neglecting the radiation damping in equation (8.1). Hence in what follows $\mathbf{E}(t)$ represents either the original or the modified field; when performing the required integrations with cutoff, the denominator $1+\tau^2\omega^2$ will normally be ignored.

In writing equation (8.1) we have already neglected the magnetic force and introduced the long-wavelength approximation. The first approximation does not in itself represent a problem of principle, at least in the nonrelativistic treatment; the second one is harder to justify, for lack of a natural dimension of the system that could serve as reference to compare with the wavelengths of the field components. But forgetting (for the moment) about the possible oscillations of very high frequencies induced by the field, it does not seem necessary to resort to more elaborate theories, such as the one given by the causal equation (3.128), for a first investigation of the nonrelativistic motions.

8.1.1. BASIC THEORY OF THE FREE PARTICLE

Let us rewrite equation (8.1) in terms of the canonical variables introduced in section 7.1,

$$\dot{\mathbf{x}} = \frac{1}{m}\mathbf{p} + \tau\ddot{\mathbf{x}} - \frac{e}{mc}\mathbf{A}, \quad \dot{\mathbf{p}} = 0. \tag{8.3}$$

According to the above discussion one may approximate equation (8.3) by neglecting the radiation reaction force, i.e,

$$\dot{\mathbf{x}} = \frac{1}{m}\mathbf{p} - \frac{e}{mc}\mathbf{A}, \quad \dot{\mathbf{p}} = 0. \tag{8.4}$$

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[1] For the present exposition we draw from de la Peña and Jáuregui (1983); further details can be found there. The problem of the free particle in SED is analyzed with the path-integral technique in Pesquera and Santos (1977, 1979); in Cavalleri (1981) the propagator of the free particle is studied. Other papers on the subject are Santos (1974a), de la Peña (1981) and Rueda (1984).

Note that nothing that happens to $\mathbf{x}$ affects $\mathbf{p}$, which maintains its initial value; of course the velocity is a fluctuating quantity, so that the distinction between momentum and velocity (or rather, mechanical momentum) becomes as important as it is in quantum theory. We write the solutions in the form

$$\mathbf{p} = \mathbf{p}_0; \quad \mathbf{x} = \mathbf{x}_0 + \frac{\mathbf{p}_0}{m} t + \mathbf{x}_s, \quad \mathbf{x}_s = \frac{e}{mc} \int_0^t dt' \mathbf{A}(t'), \tag{8.5}$$

where $t = 0$ is the time at which particle and field start to interact. The initial variables $\mathbf{x}_0$ and $\mathbf{p}_0$ can be distributed according to any law, but with total independence of the zeropoint field. If they have fixed values, in particular, an average over the realizations of the field amplitudes gives

$$\bar{\mathbf{x}} \equiv \langle \mathbf{x} \rangle = \mathbf{x}_0 + \frac{\mathbf{p}_0}{m} t, \quad \bar{\mathbf{p}} \equiv \langle \mathbf{p} \rangle = \mathbf{p}_0 = \mathbf{p}, \tag{8.6}$$

i.e., the averaged variables follow the classical laws and the momentum variable remains dispersionless. The covariances of the components of the position variable are given by

$$\begin{aligned} \Gamma_{x_i x_j}(t, t+s) &= \langle x_i(t) x_j(t+s) \rangle - \langle x_i(t) \rangle \langle x_j(t+s) \rangle \\ &= \frac{1}{m^2} \int_0^t dt' \int_0^{t+s} dt'' C_{ij}(t', t''), \end{aligned} \tag{8.7}$$

with

$$C_{ij}(t, t') = \frac{e^2}{c^2} \langle A_i(t) A_j(t') \rangle = e^2 \int_0^t dt_1 \int_0^{t'} dt_2 \langle E_i(t_1) E_j(t_2) \rangle . \tag{8.8}$$

Upon integrating one gets for the x component, at $s = 0$

$$\sigma_x^2(t) = \Gamma_{xx}(t, t) = \frac{2e^2}{m^2} \int_0^{\omega_c} d\omega \frac{S(\omega)}{\omega^4} \left[\frac{1}{2} \omega^2 t^2 - \omega t \sin \omega t + 1 - \cos \omega t \right], \tag{8.9}$$

i.e.,

$$\sigma_x^2(t) = \frac{2\hbar\tau}{\pi m} \left[\frac{1}{4} \omega_c^2 t^2 + \gamma - 1 + \cos \omega_c t + \ln \omega_c t - \operatorname{ci}(\omega_c t) \right], \tag{8.10}$$

where $\gamma = 0.577\ldots$ is the Euler constant. Thus, for $\omega_c t \gg 1$

$$\sigma_x^2(t) = \frac{\hbar\tau}{2\pi m} \omega_c^2 t^2. \tag{8.11}$$

The phase-space distribution that corresponds to this solution is

$$W(x, p, t) = \frac{1}{(2\pi\sigma_x^2)^{3/2}} \delta(\mathbf{p} - \mathbf{p}_0) e^{-(\mathbf{x} - \bar{\mathbf{x}})^2 / 2\sigma_x^2}. \tag{8.12}$$

In the limit $\omega_c t \to \infty$, this density describes an ensemble of free particles with a fixed momentum and a uniform distribution over the whole x-space. This solution corresponds to what is described in quantum mechanics by a plane wave, an eigenstate of the momentum operator. Observe that this result, which corresponds to the quantum mechanical limit ($\tau = 0$), is obtained by taking first the limit $\omega_c t \to \infty$ and only afterwards $\tau = 0$; otherwise one simply recovers the classical limit with no zeropoint field. In other words, the quantum mechanical description is obtained from SED in the radiationless limit *only* after having allowed the random field to exert its basic influence on the dynamics of the particle. When the particle reaches this state, it enters the quantum regime. A first instance of this quantum regime was already encountered when analyzing the stationary part of the general solution of the harmonic oscillator, in §7.1.1, and it will recur in the following chapters.

One can also construct solutions that correspond to the quantum mechanical wave packets, by considering distributed initial values. Thus, assume that $\mathbf{x}_0$ and $\mathbf{p}_0$ are normally distributed and have initial dispersions $\sigma_x^2(0) = \sigma_{0x}^2$ and $\sigma_p^2(0) = \sigma_0^2$ for every component, with $\mathbf{x}_0$ and $\mathbf{p}_0$ uncorrelated, $\langle x_{0i} p_{0j} \rangle = 0$. Then the dispersion of x for long enough times is given by

$$\sigma_x^2 = \sigma_{0x}^2 + \frac{\sigma_0^2}{m^2} t^2 + \frac{\hbar\tau}{2\pi m} \omega_c^2 t^2. \tag{8.13}$$

We can now take the radiationless limit, with ω_c finite, without any problem; assuming $\sigma_0^2 \gg m\hbar\tau\, \omega_c^2/2\pi$ in this approximation the last term disappears and one gets (again for the x-component)

$$\left\langle p^2 \right\rangle = \bar{p}^2 + \sigma_0^2; \quad \langle xp \rangle = \bar{x}\bar{p} + \frac{\sigma_0^2}{m} t; \quad \left\langle x^2 \right\rangle = \bar{x}^2 + \sigma_{0x}^2 + \frac{\sigma_0^2}{m^2} t^2, \tag{8.14}$$

which are just the quantum-mechanical results [see, e.g., French and Taylor 1979, section 8.9]. These results are of course not specific to SED, but apply quite generally to classical stochastic problems.

Let us now compare with the quantum description in terms of the Wigner distribution. The equation of evolution of the Wigner function $P_W(\mathbf{x}, \mathbf{p}, t)$ for particles in a potential $V(\mathbf{x})$ follows from a Weyl transformation of the equation of evolution of the density matrix, as discussed in §7.4.1; the result for the three dimensional case is [see, e.g., Moyal 1949, de Groot and Suttorp 1972]

$$\frac{\partial P_W}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla P_W - \frac{2}{\pi^3 \hbar^4} \int d^3x' d^3p' P_W(\mathbf{x}, \mathbf{p}') V(\mathbf{x}') \sin \frac{2}{\hbar} (\mathbf{x} - \mathbf{x}') \cdot (\mathbf{p} - \mathbf{p}') = 0. \tag{8.15}$$

For a free particle the potential function is zero, and the equation of evolution of the Wigner function becomes simply

$$\frac{\partial P_W}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla P_W = 0. \tag{8.16}$$

It is interesting to observe that this fundamental quantum law for the free particle is indistinguishable from the Liouville equation for a free classical particle —just as the averaged, radiationless SED equations (8.6) are indistinguishable from the correponding classical equations. Thus, judged alone by (8.16) it would seem that the description of the electron is entirely classical. This, of course, would be erroneous, because the solution must be consistent with the quantum laws, and in particular with the Heisenberg inequalities. From what we have learned in SED we know that equation (8.16) is lacking the radiative correction terms that account for the dispersion induced by the field. Anyway, the solution that is of interest for us here has initially distributed variables such that the Heisenberg inequalities are satisfied for all t. It is easy to verify that the general solution to equation (8.16) can be written in the form

$$P_W(\mathbf{x},\mathbf{p},t) = W_1(\mathbf{p})W_2(\mathbf{z}), \quad \mathbf{z} = \mathbf{x} - \frac{\mathbf{p}}{m}t = \mathbf{x} - \bar{\mathbf{x}} - \frac{\mathbf{p}-\bar{\mathbf{p}}}{m}t + \mathbf{x}_0, \tag{8.17}$$

where W_1 and W_2 are arbitrary differentiable functions. The solution discussed above corresponds to the following selection, with $\sigma_{0x}^2\sigma_0^2 \geq \frac{1}{4}\hbar^2$:

$$P_W(\mathbf{x},\mathbf{p},t) = Ne^{-(\mathbf{p}-\bar{\mathbf{p}})^2/2\sigma_0^2}e^{-(\mathbf{x}-\bar{\mathbf{x}}-\frac{\mathbf{p}-\bar{\mathbf{p}}}{m}t)^2/2\sigma_{0x}^2}. \tag{8.18}$$

Two apparently contradictory images have thus been reconciled, one in terms of stochastic motions impressed by the field, the other attributing all dispersion to initial conditions and intrinsic indeterminacies. Further, as we have seen, the small contribution to σ_x^2 neglected in performing the passage from equation (8.13) to (8.14) is not contained in the quantum mechanical description given by the Wigner function, but corresponds to radiative effects, so that in the radiationless approximation $\tau \to 0$, equations (8.13) and (8.18) agree, as is verified by integrating (8.18) over $\mathbf{p}$. One can even take now the limit $\sigma_{0x} \to 0$, in which equation (8.18) still holds with the second factor substituted by a δ function; but then the solution is valid only for times $t > t_H$ with $\sigma_0^4 t_H^2 \geq \hbar^2/4$. In its turn, the SED solution (8.13) holds good for any selection of the parameters, but corresponds to the quantum solution only after the quantum regime is reached, i.e., for $t > t_H$. In other words, according to SED all initial values of the dispersions are acceptable, quite independently of the Heisenberg inequalities; only the time needed to reach the quantum regime changes with the selection.

The average energy associated with the above solution is $\mathcal{E} = \frac{1}{2}m\langle \dot{\mathbf{x}}^2 \rangle$, up to first order in τ (see §8.1.3 below). With $\dot{\mathbf{x}} = (\mathbf{p}_0/m) + \dot{\mathbf{x}}_s$, from equation (8.5), one has

$$\mathcal{E} = \frac{\mathbf{p}_0^2}{2m} + \frac{e^2}{2mc^2}\left\langle \mathbf{A}^2(t) \right\rangle . \tag{8.19}$$

The second term is the free-particle contribution to the Lamb shift, which was identified in §7.3.2 as the radiative correction to the energy of the oscillator in the limit $\omega_0 \rightarrow 0$. It corresponds obviously to the expectation value of the quadratic term in the Schrödinger equation for the free particle with minimal coupling to the vacuum field:

$$\delta\mathcal{E} = \frac{e^2}{2mc^2}\left\langle \mathbf{A}^2(t) \right\rangle = \frac{e^2\hbar}{\pi mc^3}\int_0^{\omega_c} d\omega\, \omega = \frac{\alpha}{2\pi}\frac{(\hbar\omega_c)^2}{mc^2} \tag{8.20}$$

with $\alpha = e^2/\hbar c$. Since it does not depend on the state of motion of the particle, it is customary in quantum theory to absorb this term by a proper mass renormalization. However, as was discussed in §7.5.4, by changing the configuration of the background field it is possible in principle to produce observable modifications of this term, as shown in equation (7.179).

It is possible to generalize the above results to the case of a free particle embedded in the Planckian field at temperature T, or indeed in any stationary radiation field with an arbitrary spectral density. In the case of blackbody radiation, the thermal contribution to the energy calculated from the second term in (8.19) turns out to be negligible up to very high temperatures, compared with the zeropoint contribution. Further, by following the procedure developed in section 7.4 for the harmonic oscillator in the radiationless approximation, it is possible in the Planckian case to separate the temperature from the mechanical variables; the process leads as before to the familiar quantum-mechanical description and to Schrödinger's equation with the Hamiltonian operator $\hat{H} = (-\hbar^2/2m)\nabla^2$. Since the calculations are entirely similar to those performed for the oscillator, we do not include them here and refer the reader to the literature for the details.[2]

8.1.2. ARE FREE PARTICLES ACCELERATED BY THE ZEROPOINT FIELD?

Acccording to the approximate, nonrelativistic description just developed, under the persisting effects of the zeropoint field the free particle acquires a final mean square velocity $\langle \dot{\mathrm{x}}^2 \rangle^{1/2}$ given by equation (8.20), which amounts to an effective translational velocity of the order of

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[2] See de la Peña and Jáuregui (1983). The point of view used in this paper differs somewhat from the present one, since here we are distinguishing the quantum-mechanical description from the one provided by QED, which includes the radiative corrections. In the QED literature the radiative corrections to the free particle are normally considered part of the corrections to the mass, as stated above.

$$\frac{v_{\text{eff}}}{c} \simeq \sqrt{\alpha}\frac{\hbar\omega_c}{mc^2}. \tag{8.21}$$

This is confirmed by equation (8.11): by hypothesis the particles start with a well-defined position and momentum, but they reach in a finite time interval a spatial dispersion $\sigma_x^2(t) \simeq v_{\text{eff}}^2 t^2$. This phenomenon is what was referred to as the Boyer effect in section 5.2. As long as no value is assigned to the cutoff frequency, the idea that the vacuum produces arbitrarily large fluctuations in the velocities for the free particles travelling through intergalactic space cannot be excluded a priori. As said before, Rueda and others have studied in detail this phenomenon as a possible mechanism to explain at least in part the very high velocities observed in cosmic rays [see Rueda 1990c and references therein].

However, according to the estimates made in chapter 4 and several other places (particularly chapter 12), the most reasonable value for the cutoff frequency is $\omega_c \approx \omega_C = mc^2/\hbar$, and under these circumstances the effective translational velocity, $\sim \sqrt{\alpha}c$, is far from being highly relativistic. It seems therefore more reasonable to conclude that this effect does not refer to such an acceleration from the vacuum, but represents a mere (radiative) correction to known results, at least in the present nonrelativistic theory. The same point of view was adopted in section 5.2 when discussing the equilibrium of momentum. In support of this conclusion, recall that quantum electrodynamics in terms of retarded potentials predicts likewise no acceleration from the vacuum, so that the consistency between the two theories seems to be maintained [Rueda 1990c].

8.1.3. ENERGY BALANCE AND ENERGY FLUCTUATIONS

It is interesting to make a closer analysis of the energy balance for the free particle. To this end we start from the energy rate equation that follows from equation (8.3), namely,

$$\frac{d}{dt}\left(\frac{1}{2}m\,\dot{\mathrm{x}}^2 - m\tau\,\dot{\mathrm{x}}\cdot\ddot{\mathrm{x}}\right) = -m\tau\,\ddot{\mathrm{x}}^2 + e\,\dot{\mathrm{x}}\cdot\mathbf{E}. \tag{8.22}$$

The second term within the parentheses is the Schott energy $-\frac{1}{2}m\tau(d\dot{\mathrm{x}}^2/dt)$ that arises from the interference between the velocity and the acceleration fields of the radiating particle. It has been ascribed in the literature both to the internal energy of the particle [e.g., Fulton and Rohrlich 1960] and to the bounded field [e.g., Coleman 1961, Teitelboim et al. 1980]; in any case, it seems to 'belong' to the particle. For periodic motions this term averages to zero. For the free particle, according to the solution studied in the foregoing section, this term gives on the average a correction of second order in τ to the energy (8.19).

The right-hand side of (8.22) contains the radiated and the absorbed power, whose difference accounts for the changes in the energy of the particle. Once the quantum regime has been reached, which implies an average balance between absorbed and emitted powers, both sides must average to zero. Now let us look at this balance in some detail. Using the Parseval-Plancherel identity (3.124) and formula (4.65), one can write the radiated and the absorbed power, neglecting second-order corrections, as

$$\dot{W}_r = m\tau\left\langle\ddot{\mathbf{x}}^2\right\rangle = \frac{\tau e^2}{m}\left\langle\left(\mathbf{E}+\tau\dot{\mathbf{E}}\right)^2\right\rangle \simeq \frac{\tau e^2}{m}\left\langle\mathbf{E}^2\right\rangle = \frac{3\tau e^2}{m}\int_0^{\omega_c} d\omega S(\omega), \tag{8.23}$$

$$\begin{aligned}\dot{W}_a &= -\frac{e^2}{m}\left\langle\left(\frac{1}{c}\mathbf{A}-\tau\mathbf{E}\right)\cdot\mathbf{E}\right\rangle \simeq \dot{W}_r - \frac{e^2}{mc}\left\langle\mathbf{E}\cdot\mathbf{A}\right\rangle \\ &\simeq \dot{W}_r + \frac{3e^2}{m}\int_0^{\omega_c} d\omega\frac{S(\omega)}{\omega}\sin\omega t.\end{aligned} \tag{8.24}$$

The last term gives zero for $t \longrightarrow \infty$, indicating that global energy balance is attained in the long run. However, by performing a spectral analysis of the absorbed and radiated powers by means of the expressions

$$\dot{W}_a = \int_0^{\omega_c} d\omega w_a(\omega), \quad \dot{W}_r = \int_0^{\omega_c} d\omega w_r(\omega), \tag{8.25}$$

one gets for the net energy exchanged in a small frequency interval $d\omega$ at a given time t

$$\left(w_a(\omega) - w_r(\omega)\right)d\omega = \frac{3e^2}{m}\frac{S(\omega)}{\omega}\sin\omega t d\omega. \tag{8.26}$$

We see that even though the total average energy remains constant, the energy of each separate mode oscillates continuously; at each frequency the radiated and the absorbed energies only compensate on a time average. Thus, there is never a true balance of energy at every frequency and the system does not reach a strict equilibrium state. Note that in the radiationless limit ($\tau = 0$) this phenomenon disappears and an apparent state of detailed equilibrium is reached in this description, which corresponds to that of usual quantum mechanics. In the language of QED the phenomenon is described by saying that the free electron continuously exchanges virtual photons with the vacuum, which dresses the particle and contributes to its mass [see, e.g., Milonni 1994, chapter 3]. Although the languages and imageries of SED and QED are entirely different, they agree as to the effects of the field on the particle.

8.2. Diamagnetism

According to classical physics the isotropy of the phase-space distribution of particles is not perturbed by the presence of a constant magnetic field, so that no diamagnetic effects can be expected to occur in classical systems; this result is referred to as the *van Leeuwen-van Vleck theorem* [see, e.g., Mattis 1965, p. 21; Rosenfeld 1965, sec. IV.2; van Vleck 1965, pp. 94-102]. Therefore, diamagnetism, and in particular the diamagnetic properties of a free charge as described by Landau (1930), appear as a characteristically quantum phenomenon, without a classical analog.

In SED the problem of the existence of diamagnetism was tackled for the first time by Marshall (1963), who studied the three-dimensional isotropic harmonic oscillator in a weak magnetic field and found that the theory predicts the correct diamagnetic properties of atomic systems (modelled as harmonic oscillators). On their side, Adirovich and Podgoretskii (1954), Braffort and Taroni (1967) and Surdin (1970a) considered the free SED particle and showed that its average kinetic energy takes on the approximate value $\frac{1}{2}\hbar\omega_B$ in the presence of a magnetic field, where $\omega_B = eB/mc$ is the cyclotron frequency. Later on the problem was studied anew in more general terms by Boyer (1975b, 1978b, 1980a) and de la Peña and Jáuregui (1982, 1983).

In this section the SED theory of diamagnetism will be briefly presented, borrowing mainly from the works of Boyer (1975b, 1980a). We use the harmonic oscillator, even though our main concern is the free particle. There are at least two reasons for doing this: Firstly, for the study of the most basic aspects of atomic magnetism the oscillators represent an appropriate model for atoms in a given state. Further and very important, it happens that the diamagnetic behaviour of a free particle appears as a residual effect of the binding, i.e., it remains present in the limit of a free particle when the natural oscillation frequency becomes very small compared with the cyclotron frequency, but it would be absent for a strictly free particle.

As will become clear below, diamagnetism can be visualized within SED as the nonvanishing magnetic effects that result from the permanent precessional average motion acquired by the bound atomic particle around the direction of the magnetic field, under the combined effect of the zeropoint field and radiation reaction; in the absence of the background field this precessional motion would not be maintained. The specific relevance of the zeropoint field is that it causes these magnetic effects to coincide with the corresponding quantum behaviour. For instance, in the free-particle limit and with the z-axis in the direction of the magnetic field, one finds for the average absolute angular momentum $|\langle L_z\rangle| = \hbar$, independently of the magnitude of the magnetic field and of the mass of the particle.

8.2.1. THE HARMONIC OSCILLATOR IN A MAGNETIC FIELD

The equation of motion for the isotropic oscillator in a magnetic field and in the long-wavelength approximation is

$$\ddot{\mathbf{r}} = -\omega_0^2\mathbf{r} + \frac{e}{mc}\dot{\mathbf{r}} \times \mathbf{B}_0 + \tau\,\dddot{\mathbf{r}} + \frac{e}{m}\mathbf{E}. \tag{8.27}$$

Let us consider a constant magnetic field $\mathbf{B}_0$ in the z direction; it can be obtained from the vector potential $\mathbf{A}_0 = \frac{1}{2}\mathbf{B}_0 \times \mathbf{r}$. The equations of motion for the x and y components become coupled, but the system is still linear:

$$\ddot{x} + \omega_0^2 x - 2\omega_L'\dot{y} - \tau\,\dddot{x} = \frac{e}{m}E_x(t), \tag{8.28}$$

$$\ddot{y} + \omega_0^2 y + 2\omega_L'\dot{x} - \tau\,\dddot{y} = \frac{e}{m}E_y(t), \tag{8.29}$$

$$\ddot{z} + \omega_0^2 z - \tau\,\dddot{z} = \frac{e}{m}E_z(t). \tag{8.30}$$

The sign of ω_L' is that of the electric charge and its magnitude is the Larmor frequency, given by half the cyclotron frequency:

$$\omega_L' = \frac{e}{e_0}\omega_L, \quad e_0 \equiv |e|, \quad \omega_L = \frac{1}{2}\omega_B = \frac{e_0 B_0}{2mc}. \tag{8.31}$$

Approximate expressions for the radiation reaction terms can be found by using iteratively the equations of motion; thus, for example,

$$\dddot{x} \simeq -\omega_0^2\dot{x} + 2\omega_L'\,\ddot{y} \simeq -\omega_0^2\dot{x} - 2\omega_L'\left(\omega_0^2 y + 2\omega_L'\dot{x}\right) = -\left(\omega_0^2 + 4\omega_L^2\right)\dot{x} - 2\omega_0^2\omega_L' y. \tag{8.32}$$

There are three different frequencies of oscillation. One occurs along the z axis and corresponds (approximately) to the natural oscillation frequency ω_0, as follows from equation (8.30); the other two, ω_1 and ω_2, correspond to the independent modes of oscillation of the coupled oscillators along the x and y axes. They are given by

$$\omega_1 = \omega_S + \omega_L', \quad \omega_2 = \omega_S - \omega_L', \quad \omega_3 = \omega_0, \quad \text{with} \quad \omega_S = \sqrt{\omega_0^2 + \omega_L^2}. \tag{8.33}$$

Note that a change in the sign of the charge is equivalent to the interchange of the frequencies ω_1 and ω_2. The system of equations can now be solved directly for the stationary motions by using Fourier transforms of the position variables; the procedure is slightly lengthier than for the simple harmonic oscillator, but offers no special difficulty.

We are interested here in calculating the average magnetic moment $\langle \mathbf{M} \rangle$, with $\mathbf{M} = -\nabla_B H$ where H is the Hamiltonian of the oscillator in the magnetic field, $H = \frac{1}{2m}(\mathbf{p} - \frac{e}{c}\mathbf{A}_0)^2 + \frac{1}{2}m\omega_0^2\mathbf{r}^2$; hence

$$\mathbf{M} = \frac{e}{2c}\mathbf{r} \times \dot{\mathbf{r}} = \frac{e}{2mc}\mathbf{L}, \tag{8.34}$$

where $\mathbf{L}$ is the mechanical part of the angular momentum

$$\mathbf{L} = \mathbf{mr} \times \dot{\mathbf{r}} = \mathbf{r} \times (\mathbf{p} - \frac{e}{2c}\mathbf{B}_0 \times \mathbf{r}). \tag{8.35}$$

The calculation of the average quantities can be simplified using action-angle variables, which are related to the x, p variables by the canonical transformation[3]

$$x = \sqrt{\frac{J_1}{m\omega_S}}\cos\theta_1 + \sqrt{\frac{J_2}{m\omega_S}}\cos\theta_2, \quad y = -\sqrt{\frac{J_1}{m\omega_S}}\sin\theta_1 + \sqrt{\frac{J_2}{m\omega_S}}\sin\theta_2,$$

$$z = \sqrt{\frac{2J_3}{m\omega_0}}\cos\theta_3; \qquad p_z = m\dot{z}; \tag{8.36}$$

$$p_x = m(\dot{x} - \omega_L' y) = -\sqrt{m\omega_S}(\sqrt{J_1}\sin\theta_1 + \sqrt{J_2}\sin\theta_2),$$

$$p_y = m(\dot{y} + \omega_L' x) = \sqrt{m\omega_S}(-\sqrt{J_1}\cos\theta_1 + \sqrt{J_2}\cos\theta_2), \tag{8.37}$$

with $i = 1, 2, 3$; the angles are $\theta_i = \omega_i t + \delta_i$, with arbitrary phases δ_i. Since in terms of the new variables there are three independent one-dimensional stationary oscillators, each one obeys a distribution similar to (7.63) and the complete probability density becomes

$$W(J_i, \theta_i) = \frac{1}{(2\pi)^3 \bar{J}_1\bar{J}_2\bar{J}_3} e^{-J_1/\bar{J}_1 - J_2/\bar{J}_2 - J_3/\bar{J}_3}, \tag{8.38}$$

where $\bar{J}_i = \Pi(\omega_i)\hbar/2$. In particular, if $\Pi(\omega) = \coth\frac{1}{2}\hbar\omega\beta$, corresponding to a Planckian field according to equation (7.115), one gets for the mean energy

$$\langle H \rangle = \sum_1^3 \langle \omega_i J_i \rangle = \sum_1^3 \frac{1}{2}\hbar\omega_i \coth\frac{1}{2}\hbar\omega_i\beta. \tag{8.39}$$

The components L_x and L_y average to zero, as follows from symmetry considerations; for the component along the direction of the field one gets, from equations (8.36) and its time derivatives,

$$\langle L_z \rangle = m\left\langle x\dot{y} - y\dot{x} \right\rangle = \frac{1}{\omega_S}\left\langle -\omega_1 J_1 + \omega_2 J_2 - 2\omega_L'\sqrt{J_1 J_2}\cos(\theta_1 + \theta_2) \right\rangle, \tag{8.40}$$

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[3] A full discussion of the action-angle variables can be seen in Born (1960). For the specific details of the present calculation see Boyer (1980a).

which gives

$$\langle L_z\rangle = -\frac{\omega_1}{2\omega_S}\hbar\coth\frac{1}{2}\hbar\omega_1\beta + \frac{\omega_2}{2\omega_S}\hbar\coth\frac{1}{2}\hbar\omega_2\beta. \tag{8.41}$$

Hence, for the average magnetic moment one has

$$\langle M_z\rangle = \frac{e\hbar}{4mc\omega_S}\left(-\omega_1\coth\frac{1}{2}\hbar\omega_1\beta + \omega_2\coth\frac{1}{2}\hbar\omega_2\beta\right). \tag{8.42}$$

Several limiting cases are of interest, which will be examined cursorily.

Low-temperature limit. In this limit $\coth\frac{1}{2}\hbar\omega_i\beta \to 1$ and one gets

$$\langle H\rangle = \frac{\hbar}{2}(2\omega_S+\omega_0); \quad \langle L_z\rangle = \frac{\omega_2-\omega_1}{2\omega_S}\hbar = -\frac{\hbar\omega_L'}{\omega_S}; \quad \langle M_z\rangle = -\frac{e^2\hbar B_0}{4m^2c^2\omega_S}. \tag{8.43}$$

Observe that whereas $\langle L_z\rangle$ depends on the sign of the charge, $\langle M_z\rangle$ always opposes the magnetic field, since it depends on e^2; this is the characteristic feature of diamagnetism. The (dia)magnetic susceptibility predicted by the last equation, $\chi = -e^2\hbar/4m^2c^2\omega_S$, coincides with the usual quantum result. To write $\langle M_z\rangle$ in more familiar terms, note that

$$\left\langle x^2+y^2\right\rangle = \frac{\hbar}{2m\omega_S}\left(\coth\frac{1}{2}\hbar\omega_1\beta + \coth\frac{1}{2}\hbar\omega_2\beta\right) \tag{8.44}$$

reduces in the limit of low temperatures to $\langle x^2+y^2\rangle = \hbar/m\omega_S$, whence

$$\langle M_z\rangle = -\frac{e^2B_0}{4mc^2}\left\langle x^2+y^2\right\rangle. \tag{8.45}$$

This expression is known in the theory of diamagnetic susceptibility as the *Langevin formula*; it was derived by Langevin in 1905 by considering that the application of a magnetic field to the molecules creates an induced magnetic moment. Observe that even though both the angular momentum and action variables have a continuous range of values in the present theory, the diamagnetic properties given by equation (8.43), which correspond to the observable physical effects, are exactly the same as those predicted by quantum theory.

High-temperature limit. In this case $\frac{1}{2}\hbar\omega_i\beta \ll 1$ and the Planck spectrum goes into the Rayleigh-Jeans spectrum, for which no diamagnetic effect whatsoever should arise, i.e., $\langle \mathrm{M}\rangle = 0$. To see how this happens we take $\coth x = \frac{1}{x} + \frac{1}{3}x - \cdots$ in equation (8.41), to get

$$\langle L_z\rangle = -\frac{\hbar\omega_1}{\omega_S}\left[\frac{1}{\hbar\omega_1\beta} + \frac{\hbar\omega_1\beta}{12}\right] + \frac{\hbar\omega_2}{\omega_S}\left[\frac{1}{\hbar\omega_2\beta} + \frac{\hbar\omega_2\beta}{12}\right]$$

$$= \frac{\hbar^2}{12}\frac{\omega_2^2-\omega_1^2}{\omega_S k_B T} \to 0, \quad T \to \infty. \tag{8.46}$$

This calculation reveals the mechanism leading to the absence of diamagnetic effects in the classical domain, namely, the cancellation of the dominant terms proportional to the temperature [Boyer 1980a]. For the Rayleigh-Jeans spectrum, and only for it, do the contributions to the average angular momentum for the opposite directions of rotation cancel exactly to the leading order of approximation, independently of the strength of the magnetic field relative to the binding potential; the remaining contributions are essentially nil at high temperatures. This result is in agreement with Miss van Leeuwen's proof that when classical Boltzmann statistics applies completely to a dynamical system, the magnetic susceptibility is zero (see, e.g., van Vleck 1965, p. 94).

Weak-magnetic field limit. A weak magnetic field relative to the binding force corresponds to $\omega_L \ll \omega_0$; in this case

$$\omega_S = \sqrt{\omega_0^2+\omega_L^2} \simeq \omega_0; \quad \omega_1 = \omega_S + \omega_L' \simeq \omega_0 + \omega_L'; \quad \omega_2 = \omega_S - \omega_L' \simeq \omega_0 - \omega_L', \tag{8.47}$$

and from equation (8.41) it follows that

$$\langle L_z \rangle = -\frac{\omega_0+\omega_L'}{2\omega_0}\hbar \coth\frac{1}{2}\hbar(\omega_0+\omega_L')\beta + \frac{\omega_0-\omega_L'}{2\omega_0}\hbar\coth\frac{1}{2}\hbar(\omega_0-\omega_L')\beta, \tag{8.48}$$

which can be approximated by

$$\langle L_z \rangle = -\frac{\omega_L'}{\omega_0}\hbar\left[\coth\frac{1}{2}\hbar\omega_0\beta + \omega_0\frac{\partial}{\partial\omega_0}\coth\frac{1}{2}\hbar\omega_0\beta\right]. \tag{8.49}$$

From (8.34) the average magnetic moment is $\langle M_z \rangle = (e/2mc)\langle L_z \rangle$, which corresponds to a mean magnetic energy

$$\mathcal{E}_{\text{mag}} = -\left\langle \int_0^{B_0} \mathbf{M}\cdot d\mathbf{B} \right\rangle = \frac{e^2\hbar B_0^2}{8m^2c^2\omega_0}\left[\coth\frac{1}{2}\hbar\omega_0\beta + \omega_0\frac{\partial}{\partial\omega_0}\coth\frac{1}{2}\hbar\omega_0\beta\right]. \tag{8.50}$$

Again, these results coincide with those predicted by quantum theory in the weak-field limit.[4]

[4] A detailed comparison between the results of SED and quantum theory is made by Boyer (1975b). A model of simple molecules considered as composite particles has recently been studied by Barranco et al. (1989) within SED, using previous work by Boyer (1984c) on the behaviour of spinning magnetic moments; with the adjustment of a single parameter, this model gives results for the magnetic moments which are in excellent agreement with experiment for a large range of values of B_0/T.

8.2.2. MAGNETIC BEHAVIOUR OF FREE PARTICLES

To apply the above results to the free particle, consider a negligible binding force compared with the magnetic field trapping, so that $\omega_0 \ll |\omega_B|$. Under this circumstance one can use the approximations

$$\omega_S = \sqrt{\omega_0^2 + \frac{1}{4}\omega_B^2} \simeq \frac{1}{2}\omega_B, \quad \omega_1 = \omega_S + \frac{\omega_B}{2} \simeq \omega_B, \quad \omega_2 = \omega_S - \frac{\omega_B}{2} \simeq \frac{\omega_0^2}{\omega_B} \tag{8.51}$$

(assuming ω_B positive; otherwise ω_1 and ω_2 must be interchanged), and one thus gets from (8.39)

$$\langle H \rangle = \frac{1}{2}\hbar\omega_B \coth \frac{1}{2}\hbar\omega_B\beta + \frac{1}{2}\hbar\frac{\omega_0^2}{\omega_B} \coth \frac{1}{2}\hbar\frac{\omega_0^2}{\omega_B}\beta + \frac{1}{2}\hbar\omega_0 \coth \frac{1}{2}\hbar\omega_0\beta, \tag{8.52}$$

which simplifies to

$$\langle H \rangle = \frac{1}{2}\hbar\omega_B \coth \frac{\hbar\omega_B}{2k_BT} + 2k_BT. \tag{8.53}$$

Similarly, from equation (8.41) it follows that

$$\langle L_z \rangle = -\frac{\omega_B}{\omega_B}\hbar \coth \frac{1}{2}\hbar\omega_B\beta + \frac{\omega_2}{\omega_B}\hbar \coth \frac{1}{2}\hbar\omega_2\beta, \tag{8.54}$$

or

$$\langle L_z \rangle = -\hbar \coth \frac{\hbar\omega_B}{2k_BT} + \frac{2k_BT}{\omega_B}. \tag{8.55}$$

We see that as a result of the binding, there still appear in the free-particle limit the residual additive contributions $2k_BT$ and $2k_BT/\ \omega_B$ to $\langle H \rangle$ and $\langle L_z \rangle$, respectively, which play an essential role in recovering the diamagnetic behaviour of free particles. For high temperatures the above results give

$$\langle H \rangle = 3k_BT; \quad \langle L_z \rangle = -\frac{\hbar^2\omega_B}{6k_BT} \to 0 \quad \text{as} \quad T \to \infty. \tag{8.56}$$

The residual term has cancelled the contribution to the angular momentum from the coth term proportional to the temperature. This is in exact correspondence with the fact that the vanishing of diamagnetism for a classical free particle depends crucially upon the existence of a finite binding force [e.g., van Vleck 1965, pp. 100-102]. In the opposite temperature extreme, when the radiation field is reduced to its zeropoint component, the above equations give

$$\langle H \rangle = \frac{1}{2}\hbar\omega_B = \hbar\omega_L, \quad \langle L_z \rangle = -\frac{e}{e_0}\hbar, \quad \langle M_z \rangle = -\frac{e_0\hbar}{2mc} \equiv -\mu_0 \quad (T \to 0), \tag{8.57}$$

where μ_0 stands for the Bohr magneton. There are clearly no residual effects from the binding. Note that $\langle L_z \rangle$ has always the same magnitude $\hbar$ at zero temperature, irrespective of the magnitude of the charge and mass of the particle and of the strength of the magnetic field. This result is in agreement with the fact that from equation (8.44) in the limit $T \to 0$ and $(\omega_0/\omega_L) \ll 1$, the mean square radius for the electronic motion on the xy-plane is given by $\langle \rho^2 \rangle = \langle x^2 + y^2 \rangle = \hbar/m\omega_L$, whence the average magnitude of the corresponding angular momentum component is $\langle L \rangle = m\omega_L \langle \rho^2 \rangle = \hbar$. In fact, the same results (8.57) can be obtained more directly by solving the equation of motion for the free particle subject to the magnetic field alone and moving on the xy-plane, i.e., equation (8.27) with $\omega_0 = 0$ [Sachidanandam 1983].

8.3. An attempt to model the spin of the electron

The previous results afford a nice physical explanation of the magnetic behaviour of simple quantum systems in terms that are quite transparent and direct. However, the description is still fundamentally incomplete, since the spin of the electron is missing. In nonrelativistic quantum theory the spin is introduced by hand, and the conviction apparently prevails that there is no need (nor possibility) to give a dynamical explanation for it. But although the possibility in principle to construct a dynamical model for the spin is usually disclaimed, some plausible models have been proposed, such as those by Huang (1952), Barut and Bracken (1981) and Hestenes (1985).

If SED attempts to explain the origin of the quantum behaviour of matter, it should say something about the nature of the spin of the electron, considered as a dynamical variable. Despite its importance, though, the subject has received little attention, and except for some phenomenological discussions by Moore and Ramírez (1982) and Moore (1984), only one model of the electron spin has been explored within SED [Jáuregui and de la Peña 1981, de la Peña and Jáuregui 1982], which still remains in a preliminary stage of development. This is the model that will be sketched in what follows, hoping that its exposition may stimulate further inquiry into this important subject.

8.3.1. ANGULAR MOMENTUM EQUATIONS

Just as the zeropoint field impresses fluctuations on the velocity and the energy of the particle, it also gives rise to fluctuations of the angular momentum. Since this fluctuating angular momentum should be expected to be essentially independent of the orbital motion, at least under circumstances in which the long-wavelength approximation can be reasonably applied, one is led to consider two different angular motions of the electron immersed in the zeropoint field. A more detailed analysis of the behaviour of the system is therefore in order, to clearly identify these components.

The task is made easier by introducing a constant magnetic field to probe the particle. In order to avoid unnecessary complications and obtain the correct free–particle limit, let us once more use a harmonic oscillator as the model, as was done in §8.2.1. For the external magnetic field we write $\mathbf{A}_0 = \frac{1}{2}\mathbf{B}_0 \times \mathbf{r}$ as before, and $\mathbf{B}_0 = B_0\hat{\mathbf{n}}$ with $\hat{\mathbf{n}}$ pointing in an arbitrary, fixed direction; we shall make the usual nonrelativistic, long-wavelength approximations. Since at some point the radiative corrections will be calculated, it is important to use a consistent definition of the canonical variables, so we keep the definitions introduced earlier, equations (8.37), and write

$$\mathbf{p} = m\dot{\mathbf{r}} + \frac{e}{c}\mathbf{A}_0 = m(\dot{\mathbf{r}} + \omega_L'\hat{\mathbf{n}} \times \mathbf{r}) \qquad (8.58)$$

with $\omega_L' = eB_0/2mc$. Neglecting terms of order higher than the first in τ, the equation of motion is

$$m\ddot{\mathbf{r}} = -m\omega_0^2\mathbf{r} + 2m\omega_L'\dot{\mathbf{r}} \times \hat{\mathbf{n}} - m\tau\omega_0^2\dot{\mathbf{r}} + e\mathbf{E}(t). \qquad (8.59)$$

Let us now consider the angular momentum defined by

$$\mathbf{J} \equiv \mathbf{r} \times \mathbf{p} = \mathbf{L} + m\omega_L'\mathbf{r} \times (\hat{\mathbf{n}} \times \mathbf{r}) \qquad (8.60)$$

with $\mathbf{L}$ given by (8.35). As noted above, this $\mathbf{p}$ is not the canonical momentum, but just the 'classical' part of it; the complete canonical momentum is $\mathbf{p}_c = \mathbf{p} + (e/c)(\mathbf{A} + \mathbf{A}_{\mathrm{rr}})$, with $\mathbf{A}_{\mathrm{rr}} = (2e^2/3c^3)\ddot{\mathbf{x}}$, and the corresponding angular momentum is

$$\mathbf{J}_c = \mathbf{r} \times \mathbf{p}_c = \mathbf{J} + \frac{e}{c}\mathbf{r} \times (\mathbf{A} + \mathbf{A}_{\mathrm{rr}}). \qquad (8.61)$$

However, it is clear that in the radiationless approximation the difference between the two definitions is irrelevant. In §8.3.3, where we deal with radiative corrections, this delicate point will be reconsidered, but for the present purposes the use of (8.60) is appropriate. To study the evolution of $\mathbf{J}$ we consider the expression $d(\mathbf{r} \times \mathbf{p})/dt = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}}$ and use for the single time derivatives the values given by equations (8.58), (8.59). We thus arrive at the following equation, written to first order in $\gamma = \tau\omega_0^2$ (or second order in the electric charge),

$$\frac{d\mathbf{J}}{dt} = -\omega_L'\hat{\mathbf{n}}\times \left[\mathbf{J} + m\omega_L'(\hat{\mathbf{n}} \cdot \mathbf{r})\mathbf{r}\right] - \gamma\mathbf{J} + e\mathbf{r} \times \mathbf{E}. \qquad (8.62)$$

This result shows that there are essentially two different sources of angular momentum of the particle: one is the external magnetic field that gives rise to the Larmor precession; the other is the coupling of the particle to the radiation field, both the zeropoint and the self-field. Hence even in the absence of the external magnetic field there are two terms contributing an instantaneous torque and giving rise to a stochastic angular momentum, namely,

$$\frac{d\mathbf{J}}{dt} = -\gamma \mathbf{J} + e\mathbf{r} \times \mathbf{E}. \tag{8.63}$$

However, when the ensemble average of (8.63) is taken, the term involving the zeropoint field in the antisymmetric combination $\mathbf{r} \times \mathbf{E}$ averages to zero, and one is left only with the damping term. Nothing that could recall the spin of the electron seems to have made its way into the description.

8.3.2. UNFOLDING THE SPIN OF THE ELECTRON

There is, nevertheless, another possible reading of the above equations, according to which the spin is there, but becomes normally hidden by the averaging process. To see this, let us divide the set of all realizations of the field into two mutually exclusive and complementary subensembles, σ and $\tilde{\sigma}$, so that $\langle\cdot\rangle = \langle\cdot\rangle_\sigma + \langle\cdot\rangle_{\tilde{\sigma}}$. If the division into subensembles has a meaning, the averaging over the subensembles is also meaningful, and it may happen that such partial averages contain some information that is cancelled out when averaging over the whole ensemble.

Now assume further that the set of particles belonging to any one of the two subensembles can reach equilibrium separately; below we shall find an example of this kind of separation. By averaging the above equations over one of the subensembles, a set of relations among the angular momentum components and other dynamical quantities is obtained, valid for the corresponding subensemble. As an example, let us consider the angular momentum component along the direction of the magnetic field and study the state of equilibrium for which $\langle dJ_n/dt\rangle_\sigma = 0$. From (8.62) it follows that under this condition,

$$\langle J_n\rangle_\sigma = \frac{e}{\gamma}\left\langle(\mathbf{r} \times \mathbf{E})_n\right\rangle_\sigma; \tag{8.64}$$

clearly $\langle J_n\rangle_\sigma$ represents the equilibrium value attained by the (subensemble) average angular momentum component under the combined action of the zeropoint field and radiation damping.

In looking for the division into subensembles that can possibly disclose the presence of the electron spin, the most natural criterion is to consider the angular momentum contained in the components of the zeropoint field itself. Hence, instead of working with field modes of linear polarization, as has been done up to now, it is convenient to develop the field in terms of states of circular polarization with respect to an arbitrary axis. If no externally preferred direction exists, the choice of this axis is entirely free, but assumed to be fixed; in case there exists an externally preferred direction, such as that defined by the magnetic field, it is selected as the reference direction. Components of the zeropoint field that correspond to different circular polarizations are statistically independent, as follows from the discussion in §3.1.3, and have each a nonzero angular momentum as expressed by equations (3.58) and (3.59). Further, in the above expressions only products of at most two field components are involved, so that their average over all realizations of the field decomposes naturally into the sum of the averages over each one of the circular polarizations. Thus, they can represent the two subensembles σ, $\tilde{\sigma}$.

Actually, when working with linear polarizations it is also customary to first average over the subensembles (of linearly polarized modes) and then calculate the sum over polarizations indicated by the symbol $\sum_\lambda$; now we are simply transforming this sum into $\sum_\sigma$, with σ referring to the states of circular polarization. As will become evident below, with this selection of subensembles it is not anymore true that equation (8.64) gives zero, but an average angular motion, characteristic to this subensemble, will manifest itself. We propose to identify this angular motion as the spin of the electron.

It is clear that the mean orbital angular momentum should be essentially the same for the two subensembles, because averaging over one of them should be equivalent to averaging over the full ensemble in what refers to the orbital motions. In other words, any angular-momentum related variable that distinguishes σ-averages from full averages, should be identified as related to the spin angular momentum.

According to this interpretation, the free electron possesses a spin, but it is normally in an unpolarized spin state. Irrespectively of the details of the calculations that will be made in the next section, one property of the spin thus defined is certain: since the zeropoint field has only two independent states of polarization, the spin of the electron has just two independent states. Its formal quantum description should therefore correspond to spin 1/2.[5] The issue of how the spin is polarized remains unexplored, so to avoid too much speculation we refrain from discussing it here. What should be mentioned, however, is that a close link between spin polarization of particles and circular polarization of radiation is known to exist, as expressed for instance through the polarization-dependence of scattering cross sections of spin-polarized beams of particles [Lipps and Tolhoek 1954]. Hence the assumption underlying the proposed separation into two subensembles, namely that the electron interacts preferentially with the zeropoint radiation of a given circular polarization and reaches a state of equilibrium with it, seems to be physically justified.

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[5] Recently Kracklauer (1992) has reached the same conclusion.

Within this framework, the spin is not an intrinsic property of the electron but a dynamical attribute acquired through its interaction with the zeropoint field. Now, since this interaction is unavoidable, it is harmless in practice to consider the spin as intrinsic, as long as one refers to normal circumstances in which the electron has had ample opportunity to interact with the full zeropoint field and reach the quantum regime. The situation is conceptually similar to what occurs, for instance, with the anomalous magnetic moment, which in the applications is considered as a given attribute of the electron, even though for QED it is a derived quantity.

Now we test the proposal with a couple of calculations. We shall first take $B_0 = 0$ for simplicity and calculate the (subensemble) average of the angular momentum, including radiative corrections. A most interesting result of this calculation is precisely the radiative correction, because it fixes the anomalous magnetic moment, whose determination is considered a province entirely reserved to QED. Given the coarseness of the treatment, the predicted value, similar to the results of non relativistic quantum calculations, is quite satisfactory. As a second step, we tackle the more complicated equation (8.62) to determine the gyromagnetic ratio; this is a crucial test, since the theory should be dismissed as unsatisfactory if the predicted gyromagnetic ratio is not close to 2.

8.3.3. THE ELECTRON SPIN AND MAGNETIC MOMENT

Equation (8.64) can be used to calculate $\langle J_n \rangle_\sigma$ for a given circular polarization σ of the random field, to first order in τ, in the absence of an external magnetic field. The calculation is straightforward but long, so that only some indications of the main steps will be given here, referring the reader to the literature for the details. We take $\sigma = 1$ for right polarization and $\sigma = -1$ for left polarization, and decompose the random field into its σ components:

$$\mathbf{E}^{(+)}(t) = \frac{1}{2\pi}\left[\int_{k_z>0} d^3k\mathbf{E}(t;\mathbf{k}+) + \int_{k_z<0} d^3k\mathbf{E}(t;\mathbf{k}-)\right], \tag{8.65}$$

$$\mathbf{E}^{(-)}(t) = \frac{1}{2\pi}\left[\int_{k_z<0} d^3k\mathbf{E}(t;\mathbf{k}+) + \int_{k_z>0} d^3k\mathbf{E}(t;\mathbf{k}-)\right], \tag{8.66}$$

where $(\lambda = \pm)$

$$\mathbf{E}(t;\mathbf{k}\lambda) = i\sqrt{\frac{\hbar\omega}{2}}\hat{\epsilon}^{\lambda}(\mathbf{k})a_{\lambda}(\mathbf{k})e^{-i\omega t} + \text{c.c.} \tag{8.67}$$

The correlations of the fields $E_i{}^{(\sigma)}$ are then

$$\left\langle E_i{}^{(\sigma)}(t)E_j{}^{(\sigma')}(t')\right\rangle = \delta_{\sigma\sigma'}\frac{\hbar}{8\pi^2}\left\{\int d^3k\,\omega\left(\delta_{ij}-\hat{k}_i\hat{k}_j\right)\cos\omega(t-t')\right.$$

$$\left.-\sigma\left(\int_{k_z>0}-\int_{k_z<0}\right)d^3k\,\omega\varepsilon_{ijl}\hat{k}_l\sin\omega(t-t')\right\}. \tag{8.68}$$

To perform the angular integrations we use

$$\int_{k_z>0}d\Omega\hat{k}_l=-\int_{k_z<0}d\Omega\hat{k}_l=\pi\hat{n}_l, \tag{8.69}$$

whence

$$\left\langle E_i{}^{(\sigma)}(t)E_j{}^{(\sigma')}(t')\right\rangle$$

$$=\delta_{\sigma\sigma'}\frac{\hbar}{3\pi c^3}\int_0^\infty d\omega\,\omega^3\left[\delta_{ij}\cos\omega(t-t')-\sigma\eta\varepsilon_{ijl}n_l\sin\omega(t-t')\right]. \tag{8.70}$$

The factor $\eta = 3/4$ (which will be a nuisance) comes from the angular integration. This result explicitly shows that the two subensembles are orthogonal, as required.

Now we decompose equation (8.59) into its σ components, write

$$\ddot{\mathbf{r}}^{(\sigma)}+\omega_0^2\mathbf{r}^{(\sigma)}+\gamma\dot{\mathbf{r}}^{(\sigma)}=(e/m)\,\mathbf{E}^{(\sigma)} \tag{8.71}$$

and solve for $\mathbf{r}^{(\sigma)}(t)$. After calculating the required integrals, with n_l chosen along the direction $\hat{\mathbf{n}}$, one obtains

$$\langle J_n\rangle_\sigma=\sigma\eta\frac{\hbar}{\pi}I_1 \tag{8.72}$$

with

$$I_1=\tau\int_0^{\omega_c}d\omega\frac{\omega^4}{\left(\omega^2-\omega_0^2\right)^2+\tau^2\omega_0^4\omega^2}. \tag{8.73}$$

The divergent value of this integral, when extended to infinity, is an artifact that arises from having used the causal form of the Abraham-Lorentz equation, with $\tau\,\dddot{x}\to-\tau\omega_0^2\dot{x}$, which led to the substitution $\tau^2\omega^6\to\tau^2\omega_0^4\omega^2$ in the denominator of the integrand. However, the required cutoff at ω_c takes due care of it and makes the value of the integral insensitive to this approximation, provided $\tau\omega_c\ll 1$, which is well satisfied as we know; one can readily see that the error introduced by this approximation is of second order in τ. The integration gives $I_1=\frac{\pi}{2}(1+\frac{2}{\pi}\tau\omega_c)$, which introduced in equation (8.72) leads to

$$\mathbf{S}_\sigma\equiv\langle\mathbf{J}\rangle_\sigma=\frac{\hbar}{2}\eta\sigma\hat{\mathbf{n}}\left(1+\frac{2}{\pi}\tau\omega_c\right). \tag{8.74}$$

This result, obtained from the equilibrium condition $\langle dJ_n/dt\rangle_\sigma = 0$, can alternatively be obtained by calculating $\langle \mathbf{J}\rangle_\sigma = \langle \mathbf{r} \times \mathbf{p}\rangle_\sigma$, with $\mathbf{r}(t)$ given by the solution of (8.71) and $\mathbf{p} = m\dot{\mathbf{r}}$ (since $B_0 = 0$).

Hence if $\mathbf{S}$ is identified as the electron spin, the theory predicts that its projection along the axis $\hat{\mathbf{n}}$ has the value $\hbar/2$, multiplied by the unpleasant factor η. The result is very suggestive, but it also indicates that some mistake is being made in the calculation (in §10.2.2 we offer a possible solution to this intriguing problem). The factor σ, on the other hand, is very pleasant, since it gives the required double sign for this vector, normally introduced by means of the Pauli operator $\hat{\sigma}_3$.[6]

Further, we recall from (8.34) that the magnetic moment $\mathbf{M}$ is proportional to the mechanical part of the angular momentum; hence the average magnetic moment $\boldsymbol{\mu}_\sigma$ of the electron is proportional to $\langle \mathbf{J}\rangle_\sigma$ given in (8.74), so that the extra term within the parentheses of (8.74) gives a partial correction to the magnetic moment of the electron. Since $\mu_3 = -g(e_0/2mc)s_3$, the radiative correction to the spin projection combines with the radiative mass correction to give rise to an anomalous magnetic moment. The anomalous contribution due to the change in the spin projection is, with μ_0 the Bohr magneton,

$$\delta g \equiv \frac{\mu_a}{\mu_0} = \frac{2}{\pi}\tau\omega_c \approx \frac{4}{3}\frac{\alpha}{\pi}, \tag{8.75}$$

where the last expression is obtained by assuming $\omega_c = \omega_C$.[7] This result has the correct sign and about 8/3 the value given by relativistic QED (and confirmed by experiment), which is $\alpha/2\pi$; it is also of the same order as the predictions that can be obtained from nonrelativistic treatments in QED, as is shown in Grotch and Kazes (1977) and discussed in detail in Cohen-Tannoudji (1986). Alternatively, by selecting ω_c such that equation (8.75)

[6] Such construction of the spin can be applied to the theory proposed by Sokolov and Tumanov (1956), discussed in section 2.4. The original particle there is classical, but its interaction with the quantized vacuum field transforms its dynamical variables into operators. The spin can be identified following a procedure similar to the present one; of course, the factor 3/4 appears also here (the unpublished calculations are due to A. Jáuregui). As discussed in section 2.4, Schiller and Tesser (1971) show that a magnetic dipole gives rise to operators satisfying the correct angular momentum commutation relations, which applies also to the spin constructed as just described.

Concerning the comment on the high frequencies, we have in mind the possibility that the spin could be more closely linked to the zitterbewegung than here suggested, and then it would be more dependent on the very high frequencies, which are poorly taken into account in a nonrelativistic treatment as the present one. A model of this kind would be close to the one suggested in Hestenes (1985).

[7] Apparently the first suggestion that SED predicts an anomalous magnetic moment for the electron not merely associated with the mass renormalization, is due to Braffort and Taroni (1967).

gives the correct result, one would get approximately $0.4\omega_C$, which is also in agreement with the much more detailed results of Grotch and Kazes.

To the value reported in (8.75) one should still add the mass correction to the gyromagnetic ratio, as follows from the development

$$g = \frac{|\mu_0 + \delta\mu|}{\frac{e_0}{2c(m+\delta m)}} = \frac{|\mu_0 + \delta\mu|}{\frac{e_0}{2cm}}\left(1 + \frac{\delta m}{m}\right) \simeq 2 + \frac{|\delta\mu|}{\frac{e_0}{2cm}} + 2\frac{\delta m}{m}. \tag{8.76}$$

In writing the last expression the uncorrected spin gyromagnetic ratio —to be studied in the next section— was set equal to 2. Still other corrections are actually needed, to account both for the coupling of the electron to the magnetic part of the Lorentz force produced by the zeropoint field and for the space dependence of the random field modes, whose neglect by making use of the long-wavelength approximation is equivalent to the neglect of the electron recoil, as is shown in Grotch and Kazes (1977). However, an estimate of these corrections shows that they are of the same order of magnitude as those coming from the mass correction, but with opposite sign, so that the two effects tend to cancel each other and the end result is basically that given by equation (8.75). Since the result (8.74) depends on the cutoff frequency it does not make much sense to enter into finer detail; a more complete (relativistic) theory is obviously necessary to overcome this limitation.

8.3.4. THE GYROMAGNETIC RATIO OF THE ELECTRON

Now we come back to equation (8.59) with the external magnetic field included, and use its solution to calculate the average angular momentum along the direction of the field, again with the help of the approximate equation (8.64). We disregard all radiative corrections, but consider a more general radiation field represented by $\rho(\omega) = \Pi(\omega)\rho_0(\omega)$, whereby the integrand in (8.70) appears multiplied by $\Pi(\omega)$. The full calculation gives then for the σ contribution to the mechanical part of the angular momentum along the direction $\hat{\mathbf{n}}$, to lowest order in τ,

$$\langle L_n \rangle_\sigma = \frac{1}{2\omega_S} \sum_{i=1,2} \left[S_n^\sigma + (-1)^i \frac{\hbar}{2} \right] \omega_i \Pi(\omega_i), \qquad S_n^\sigma = \frac{\hbar}{2}\eta\sigma\hat{\mathbf{n}}, \tag{8.77}$$

where ω_1, ω_2 and ω_S are as defined in equations (8.31) and (8.33). The corresponding energy in the same approximation is

$$\mathcal{E}_\sigma = \frac{1}{2}\left[\frac{1}{2}\hbar\omega_0 \Pi(\omega_0) + \sum_{i=1,2} \left(\frac{\hbar}{2} - (-1)^i S_3^\sigma \right) \omega_i \Pi(\omega_i) \right]. \tag{8.78}$$

When these expressions are summed over the two polarizations, with $\Pi(\omega)$ corresponding to Planck's spectrum, the results obtained coincide correctly with equations (8.39) and (8.41). But if the summation over σ is not performed, one gets at $T = 0$, i.e., with $\Pi(\omega) = 1$, and with $\omega_L = -\omega_L'$ for the electron,

$$\langle L_n \rangle_\sigma = S_n^\sigma + \frac{\omega_L}{\omega_S}\frac{\hbar}{2}, \tag{8.79}$$

$$\mathcal{E}_\sigma = \frac{1}{2}\left[\frac{1}{2}\hbar\omega_0 + \hbar\omega_S + 2\omega_L S_n^\sigma\right]. \tag{8.80}$$

The coefficients 1 and 2 of the second and third terms in the last equation are the gyromagnetic ratios of the orbital and spin angular momenta predicted by the present theory. The result $g = 2$ for the spin term confirms that this 'internal' angular motion has the properties of a spin angular momentum. This is a significant result, up to now characteristic of the quantum domain, and then in its relativistic version.[8] In the weak-field limit one may use the approximation $\omega_S \simeq \omega_0 + \omega_L^2/2\omega_0$, whence from (8.79) it follows that

$$\left\langle L_n^{\text{orb}} \right\rangle \simeq \frac{\omega_L}{2\omega_0}\hbar \tag{8.81}$$

and equation (8.80) can be recast into the form

$$\mathcal{E}_\sigma = \frac{1}{2}\left[\frac{3}{2}\hbar\omega_0 + \omega_L\left(\left\langle L_n^{\text{orb}}\right\rangle + 2S_n^\sigma\right)\right], \tag{8.82}$$

which shows explicitly, in the customary form used in quantum mechanics, the gyromagnetic ratio of the spin as twice its orbital value. The origin of this difference of gyromagnetic ratios can be traced back to the relation given in equation (8.31), $\omega_B = 2\omega_L$; in other words, the spin motion refers to a cyclotron-type rotation rather than a Larmor precession, so that writing both contributions in terms of Larmor's frequency entails a correction by a factor of 2. Of course, by summing over the polarizations one gets rid of the overall factor $\frac{1}{2}$, but with it also the spin components.

8.4. Specific heat of solids

In 1907, the early quantum ideas applied by Planck and Einstein to the field oscillators were extended by Einstein to material oscillators, giving birth to the quantum theory of the specific heat of solids. The theory was further developed by Debye (1912) and others into a firmly established piece of knowledge. The specific heat of solids at low temperatures is thus known to have a behaviour correctly explained by quantum theory, and contrary to the classical predictions. This subject constitutes therefore a natural ground to test further the potentialities of SED.

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[8] In Sachidanandam (1983) it is asserted that the gyromagnetic ratio predicted for the electron by SED is 2, if the total canonical angular momentum is used. However, he is referring to orbital motions and the calculations (when corrected for an undue factor $\frac{1}{2}$) do not confirm such a conclusion.

The goal is to show that the mean energy $\mathcal{E}(T)$ of the solid is given by Planck's law, from which the specific heat follows using $c_V = \partial \mathcal{E}/\partial T$. Since the solid body is traditionally modelled as a set of coupled oscillators, the problem has the additional interest of providing an opportunity to study some aspects of a system of (linearly) coupled harmonic oscillators within SED.[9] In a recent calculation that will be sketched below, Blanco et al. (1991) have obtained correct results for the specific heat of a solid using the Lorentz-Drude model to lowest order in the fine structure constant.

Assuming that the temperature is neither too high nor too low, the solid can be modelled by a lattice structure consisting of a set of N three-dimensional oscillators, which represent the small vibrations of the N nuclei (or rather ions, since the electronic contribution turns out to be negligible) that constitute the solid body, around the corresponding equilibrium positions $\mathbf{a}_s, s = 1, 2, \cdots, N$. The ions are considered pointlike, with a mass m and an electric charge e_s, so that the equation of motion for the displacement of ion s around its equilibrium position is

$$m \ddot{\xi}_i (t) = -\sum_{j=1}^{3N} K_{ij}\xi_j(t) + F_i^R + e_s E_\alpha(\mathbf{a}_s, t), \tag{8.83}$$

where $E_\alpha(\mathbf{a}_s, t)$ represents the α electric component ($\alpha = 1, 2, 3;\ i = 3s - \alpha$) of the random field (thermal plus zeropoint) with respect to three orthogonal axes at the equilibrium position $\mathbf{a}_s$ of ion s. The long-wavelength approximation is made only with respect to the variations of the electric field inside the ions, the spatial dependence of the field with respect to the equilibrium position $\mathbf{a}_s$ being taken into account. The matrix K describes all the (linear) couplings through the instantaneous Coulomb force and the electronic clouds; F_i^R contains the radiation reaction and the retarded force produced by all other ions on ion s, up to and including terms of order v/c. The total mean energy of the ions in the lattice is taken as

$$\mathcal{E} = \frac{1}{2}\sum_{i=1}^{3N} m \left\langle \dot{\xi}_i^2(t) \right\rangle + \frac{1}{2}\sum_{i,j=1}^{3N} K_{ij} \left\langle \xi_i(t)\xi_j(t) \right\rangle, \tag{8.84}$$

where $\langle \cdot \rangle$ means time and ensemble average, or either of them if ergodicity is assumed.

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[9] A much simpler example of two coupled oscillators was partially discussed in § 6.2.1.

Since the calculation is straightforward and rather lengthy, we refer the reader to the original paper for the details. It can be shown that if the equal-times linear correlation of the random electric force is written in the form

$$e_s e_{s'} \langle E_\alpha(\mathbf{a}_s, t) E_\beta(\mathbf{a}_{s'}, t)\rangle = \int_0^\infty d\omega \rho(\omega) B_{ij}\left(\frac{\omega\,|\mathbf{a}_s - \mathbf{a}_{s'}|}{c}\right), \qquad (8.85)$$

with $i = 3s - \alpha$, $j = 3s' - \beta$, and the Fourier transform of F_i^R, in its turn, is expressed as

$$\tilde{F}_i^R(\omega) = -\sum_{j=1}^{3N} R_{ij}(\omega)\tilde{\xi}_j(\omega), \qquad (8.86)$$

then the following fluctuation-dissipation relation holds,

$$\operatorname{Im}\left[R_{ij}(\omega)\right] = \frac{1}{4\pi}\omega^3 B_{ij}(\omega). \qquad (8.87)$$

With this important result, up to terms of order α the average energy of the oscillators is given by

$$\mathcal{E} = \frac{\pi c^3}{4}\sum_{,j=1}^{3N} \omega_j \rho(\omega_j)\frac{|B_{jj}(\omega_j)|}{|\operatorname{Im}\left[R_{jj}(\omega_j)\right]|} = \sum_{,j=1}^{3N} \hbar\omega_j\left[\frac{1}{2} + \frac{1}{e^{\hbar\omega_j/k_B T} - 1}\right]. \qquad (8.88)$$

Hence, without the need to introduce any discrete property for the oscillators the theory leads to the correct quantum law, when Planck's law for the equilibrium field is used. According to this calculation, what reveals itself as the root of the quantum thermal behaviour of bulk matter is the coupling of its weakly interacting atoms to the equilibrium radiation field, including the zeropoint component.

CHAPTER 9

BREAKDOWN OF DETAILED ENERGY BALANCE

When attempts were made to extend the theory developed in the preceding chapters to the study of systems subject to nonlinear forces, it was soon realized that the predictions obtained were at variance with well established results of quantum mechanics. The first symptoms of this problem were found by Boyer (1976) when studying an anharmonic oscillator. He showed that the Lorentz-invariant spectrum of zeropoint radiation is not an equilibrium spectrum in presence of the nonlinear oscillator; the latter acts to 'push' the zeropoint radiation towards the Rayleigh-Jeans law of thermal radiation, as it were. Such an oscillator coupled to an infinite reservoir of zeropoint radiation could provide an unlimited source of energy to do useful work; this would constitute an example of perpetual-motion machine of the second kind, so to the extent that the Second Law of thermodynamics holds, the result points toward a failure of the theory.

Shortly after, with the development of techniques for the construction of approximate Fokker-Planck-type equations for coloured noises, expected to be applicable to the SED case, it became clear that similar difficulties are encountered by all SED systems with more than one frequency, whether fundamental or overtones. A whole series of negative results soon accumulated, among them the demonstration that the Fokker-Planck approximation leads to a self-ionizing ground state for the hydrogen atom.[1] The problem was further complicated with the observation that in the case of the H atom the (pseudo) Fokker-Planck equation admits two different (non-integrable) stationary solutions [Claverie et al. 1980, Claverie and Soto 1982]. The emergency of such anomalous situation can be appreciated by recalling that any bound state should be described by a unique integrable density, so that the absence of a finite measure is an expression of its unboundedness, i.e., of self-ionization and the characteristic non-recurrence property of the system under such circumstances.

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[1] The problem of the failures of SED when applied to nonlinear systems is well documented in the literature. The interested reader may find detailed accounts in the reviews by Boyer (1980c), Claverie and Diner (1980), Brody (1983) and de la Peña (1983). Similarly, the literature on the treatment of the H atom and other nonlinear problems within SED with Fokker-Planck-equation techniques is relatively abundant. Some pertinent references are: Boyer (1976, 1978a), Claverie (1979), Claverie and Diner (1980), Claverie et al. (1980), Marshall and Claverie (1980b), Pesquera (1980a), Claverie and Soto (1982), Pesquera and Claverie (1982); the last cited review contains an extensive bibliography. A pioneering manuscript of 1978 on the subject by Claverie, de la Peña and Diner, which is cited here and there under various titles, remained as a draft and was never prepared for publication.

The analysis of the origin of these problems is a complicated task, for which it is desirable to use the smallest possible number of assumptions; in practice this means avoiding the use of Fokker-Planck-type equations and paying direct attention to the detailed balance of energy. In several parts of this book we have found that the energy equilibrium condition can be Fourier-analyzed into a relation that holds for a given frequency, what is usually called a *detailed balance* condition.[2] One such instance was encountered in relation with the Einstein A, B coefficients; also, we saw that the Markov description of the harmonic oscillator in §7.1.2 holds for a fixed frequency; in §7.5.3 and elsewhere the fluctuation-dissipation relations determining the equilibrium state and its stability against changes in the zeropoint field were shown to hold separately at each given frequency, and so on.

When detailed energy balance does not hold, the mechanical system may nevertheless maintain a constant average energy, but there is then a periodic energy exchange with certain modes of the field; although a steady state may exist, no equilibrium is really reached. A similar situation was found in the study of the free particle in chapter 8, where it was noticed that only on a time average is there no net exchange of energy with each single mode.

Over 70 years ago van Vleck (1924a, b) showed that any ensemble of classical multiply periodic mechanical systems in equilibrium with an electromagnetic field follows a Maxwell-Boltzmann distribution, and that the equilibrium field in its turn obeys the Rayleigh-Jeans law. In classical physics nonlinear systems inextricably mingle the components of the motion of different frequencies, hence it becomes possible to construct a detailed energy balance condition only when the individual contributions to the mean energy are frequency-independent, namely, when the Rayleigh-Jeans law holds. This result is of course in plain contradiction with some of our previous derivations, as those in chapter 7 for the archetype of linear systems, the harmonic oscillator. Thus the situation is by no means clear, and the problem requires closer attention to evaluate the real possibilities of the present theory. In this chapter we verify that the difficulties predicted by van Vleck-type calculations when applied to SED systems, stem indeed from a violation of detailed balance, a behaviour incompatible with quantum theory. This result suggests that if SED in its present form is indeed a first step in the right direction, further steps are still wanting.

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[2] As happens with many other expressions in physics, the term 'detailed balance' is used with various not necessarily equivalent meanings. A most common use refers to a microscopic time-reversal property of statistical systems [see, e.g., van Kampen 1981, section 5.6] that gives rise to macroscopic properties such as the zero current associated with a stationary distribution [see, e.g., Marshall and Claverie 1980b]. Here, in contrast, we use it as a weak form of the demand that in equilibrium the power radiated and the power absorbed by a body must be equal for any element of area of the body, for any frequency range and for any polarization (see §5.3.1). These two meanings are not entirely independent, since under certain assumptions microscopic time-reversal invariance leads to the second property. An elementary discussion of this relationship is given in Reif (1965), section 9.15.

9.1. Classical dynamics and detailed balance

9.1.1. COMMENTS ON THE PERTURBATIVE APPROACH IN SED

We will study the SED equilibrium condition using a perturbative treatment. The usual procedure to define the unperturbed state consists in taking the limit $e \to 0$; taken verbatim, this operation disconnects also the Coulomb potential and transforms the atomic problem into that of free particles, which obviously is not what we want. Two courses of action are possible. The first, apparently more physical alternative is to consider that e is e in whatever context it may appear; then no unperturbed atom exists and the problem requires a different kind of analysis. Such study is partially made in Cole (1990a) for the relativistic Coulomb problem [see also Boyer 1989], where by means of a dimensional analysis it is concluded that the equation of motion can be expressed in terms of a characteristic (but arbitrary) length and the fine structure constant α, but then the usual Braffort-Marshall equation (9.1) cannot be anymore justified. More generally, there is no obvious reason to treat the zeropoint field and the radiation reaction as perturbations.

The second alternative, which is the one generally found in the literature (and the one we will follow below), distinguishes between the electric charge that couples the particle to the radiation field and the one that couples it to the static Coulomb field. This can be easily done by considering the applied forces (or potentials) as given, and thus leaving them untouched. The advantage of this procedure is that it treats all external potentials on an equal footing, regardless of their origin, which is quite similar to the situation that prevails in quantum mechanics. However, in the developments to follow we will soon perceive that this perturbative approach takes us away from the quantum behaviour (see particularly footnote 4).

Another significant consideration that warns against an uncritical use of the perturbative approach to analyze the problem at hand is the following. Berry and Mount (1972) showed that in general the wave function of quantum mechanics has a singular behaviour as $\hbar \to 0$, so that it does not admit a power development in this parameter. This suggests that a perturbative introduction of the field (which is the source of $\hbar$) could be misleading if one is expecting the theory to approach the quantum description.

9.1.2. EQUILIBRIUM CONDITION AND FORMULAS FOR THE DIFFUSION COEFFICIENTS

Let us consider the Braffort-Marshall equation of motion for an arbitrary force $\mathbf{F}(\mathrm{x})$, in the nonrelativistic, dipolar approximation,[3]

$$m\ddot{\mathrm{x}} = \mathbf{F}(\mathrm{x}) + m\tau\,\dddot{\mathrm{x}} + e\mathbf{E}(t). \tag{9.1}$$

To construct the equation for the energy rate we take the scalar product of equation (9.1) with $\dot{\mathrm{x}}$ and consider the particle Hamiltonian $H = \frac{1}{2}m\dot{\mathrm{x}}^2 + V(\mathrm{x}) - m\tau\dot{\mathrm{x}} \cdot \ddot{\mathrm{x}}$, where $V(\mathrm{x})$ is the potential associated with the external force and the last term is the Schott energy, defined as in §8.1.3 (see equation (8.22)). After some elementary transformations the total average power exchanged by the particle becomes

$$\left\langle \frac{dH}{dt} \right\rangle = -m\tau \left\langle \ddot{\mathrm{x}}^2 \right\rangle + e \left\langle \dot{\mathrm{x}} \cdot \mathbf{E} \right\rangle. \tag{9.2}$$

We now assume that x can be expressed as a power series in the electric charge, so that

$$\mathrm{x} = \mathrm{x}^{(0)} + \mathrm{x}^{(1)} + \mathrm{x}^{(2)} + \cdots, \tag{9.3}$$

with $\mathrm{x}^{(k)}$ of order k in the electric charge, and accordingly $\mathbf{p}^{(k)} = m\dot{\mathrm{x}}^{(k)}$. The Larmor term in equation (9.2) already contains the factor τ, hence to lowest (second) order in the charge it reduces to $-m\tau\langle(\ddot{\mathrm{x}}^{(0)})^2\rangle$. Similarly, the lowest (first) order contribution to the power supplied by the stochastic field is $e\langle\dot{\mathrm{x}}^{(0)} \cdot \mathbf{E}\rangle$; this is the single first-order term in equation (9.2). Defining the state of equilibrium as that for which the left-hand side is zero, it follows that the unperturbed motion must be such that this term vanishes, namely,

$$\left\langle \dot{\mathrm{x}}^{(0)} \cdot \mathbf{E} \right\rangle = 0. \tag{9.4}$$

We interpret this important equation as the condition for the existence of a zero-order stationary state. For $\dot{\mathrm{x}}^{(0)}$ independent of the random field (such as the one of the present analysis, as follows from equation (9.6) below) this condition is trivially satisfied; however, in §11.2.1 we will meet a more complex situation where it becomes far from trivial. Equation (9.2) gives then for the equilibrium condition, to lowest (second) order in the charge,

$$-m\tau \left\langle \left(\ddot{\mathrm{x}}^{(0)}\right)^2 \right\rangle + e \left\langle \dot{\mathrm{x}}^{(1)} \cdot \mathbf{E} \right\rangle = 0. \tag{9.5}$$

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[3] The first analysis of detailed energy balance in SED is given in Santos (1985b). This section is based on Cetto et al. (1984), Cetto and de la Peña (1991b) and de la Peña and Cetto (1991b).

To get rid of possible remaining oscillations (as those discussed in §8.1.3 in connection with the free particle), we define the average in the last expression as including also a time averaging if necessary.

Now $x_i^{(r)}$ can be constructed by substituting (9.3) into (9.1), which leads to the hierarchy

$$m\ddot{x}_i^{(0)} = F_i(\mathrm{x}^{(0)}), \tag{9.6}$$

$$m\ddot{x}_i^{(1)} = \left.\frac{\partial F_i}{\partial x_j}\right|_0 x_j^{(1)} + eE_i, \tag{9.7}$$

$$m\ddot{x}_i^{(2)} = \left.\frac{\partial F_i}{\partial x_j}\right|_0 x_j^{(2)} + \frac{1}{2}\left.\frac{\partial^2 F_i}{\partial x_j \partial x_k}\right|_0 x_j^{(1)} x_k^{(1)} + m\tau\, \dddot{x}_i^{(0)}, \tag{9.8}$$

and so on. Note that all the equations are linear in the corresponding $x_i^{(r)}$ variable for $r \geq 1$ and have the same general form

$$m\ddot{x}_i^{(r)} = \left.\frac{\partial F_i}{\partial x_j}\right|_0 x_j^{(r)} + R_{i,}^{(r)} \tag{9.9}$$

with $R_i^{(r)}$ an appropriate function of the $x_k^{(s)}$, $s \leq r-1$. This means that the form of the solution for $x_k^{(1)}$ to be constructed below can be used to calculate successively all $x_k^{(r)}$, with the single change $eE_i(t) \equiv R_i^{(1)}(t) \to R_i^{(r)}(t)$ to accommodate the different functions $R_i^{(r)}$.[4] The causal solution to equation (9.7) can be found by introducing

$$x_i^{(1)}(t) = e\int_{-\infty}^{t} dt' \mathcal{G}_{ij}(t,t') E_j(t') \tag{9.10}$$

in (9.7); the Green function $\mathcal{G}_{ij}(t,t')$ must be a solution of the equation

$$m\frac{\partial^2}{\partial t^2}\mathcal{G}_{ij}(t,t') = \left.\frac{\partial F_i}{\partial x_k}\right|_0 \mathcal{G}_{kj}(t,t') \tag{9.11}$$

satisfying the conditions

$$\mathcal{G}_{ij}(t,t) = 0, \quad \lim_{t' \to t} \frac{\partial}{\partial t}\mathcal{G}_{ij}(t,t') = \frac{1}{m}\delta_{ij}. \tag{9.12}$$

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[4] Note carefully that in the present perturbative description the zero-order solutions are given by nonrandom numbers; stochasticity enters with the corrections. Thus the former cannot correspond to the variables of quantum mechanics, which are dispersive. An exception is the harmonic oscillator, for which the solution $\mathrm{x}^{(0)} = 0$ transforms the stochastic equation (9.7) into the leading one (in the radiationless approximation).

From equation (9.6) it follows that for a potential force, the general solution has the form

$$\mathcal{G}_{ij}(t,t') = A\left.\frac{\partial x_i(t)}{\partial x_j(t')}\right|_0 + B\left.\frac{\partial x_i(t)}{\partial p_j(t')}\right|_0 . \tag{9.13}$$

Indeed, using equation (9.11) one gets

$$m\frac{\partial^2\mathcal{G}_{ij}(t,t')}{\partial t^2} = A\left.\frac{\partial F_i(t)}{\partial x_k(t)}\frac{\partial x_k(t)}{\partial x_j(t')}\right|_0 + B\left.\frac{\partial F_i(t)}{\partial x_k(t)}\frac{\partial x_k(t)}{\partial p_j(t')}\right|_0$$

$$= \left.\frac{\partial F_i}{\partial x_k}\right|_0 \mathcal{G}_{kj}(t,t'). \tag{9.14}$$

The particular solution that satisfying (9.12) corresponds to $A = 0$ and $B = 1$, so that

$$\mathcal{G}_{ij}(t,t') = \frac{\partial x_i^{(0)}(t)}{\partial p_j^{(0)}(t')} \tag{9.15}$$

and equation (9.10) leads to

$$x_i^{(1)}(t) = e\int_{-\infty}^{t} dt' \frac{\partial x_i^{(0)}(t)}{\partial p_j^{(0)}(t')} E_j(t'), \tag{9.16}$$

$$p_i^{(1)}(t) = e\int_{-\infty}^{t} dt' \frac{\partial p_i^{(0)}(t)}{\partial p_j^{(0)}(t')} E_j(t'). \tag{9.17}$$

These formulas are of much practical value. As a first application we connect them with the formulas for the diffusion coefficients used in chapter 7. Specifically, from (7.41) applied to the SED case it follows that ($g_{\alpha K} = e$)

$$e\left\langle x_i^{(1)} E_j\right\rangle = e^2\int_{-\infty}^{t} dt' \frac{\partial x_i^{(0)}(t)}{\partial p_k^{(0)}(t')} \langle E_k(t')E_j(t)\rangle = D_{ij}^{px}, \tag{9.18}$$

$$e\left\langle p_i^{(1)} E_j\right\rangle = e^2\int_{-\infty}^{t} dt' \frac{\partial p_i^{(0)}(t)}{\partial p_k^{(0)}(t')} \langle E_k(t')E_j(t)\rangle = D_{ij}^{pp}, \tag{9.19}$$

which shows a relationship between the diffusion coefficients and the second-order contribution to the corresponding correlations; indeed, by comparing with the equations (7.50) and (7.51) obtained for the harmonic oscillator one verifies that the latter are particular cases of the more general equations just obtained. Further, note that $(1/m)\mathrm{Tr}D^{pp} = (e/m)\langle \mathbf{p}\cdot\mathbf{E}\rangle$ gives the power delivered by the field to the particle; this suggests that $\mathrm{Tr}D^{px}$

should refer to a contribution of the field to the energy of the particle, which has indeed already been identified with the Lamb shift in §7.3.2.[5]

Observe that this procedure can be reversed to derive first the equations $e < x_i^{(1)} E_j >= D_{ij}^{px}$ and $e < p_i^{(1)} E_j >= D_{ij}^{pp}$, by considering the relations between moments of phase-space variables predicted from the Fokker-Planck equation and comparing with the corresponding results derived from the equations of motion, as was done in chapter 7. Once such relations are obtained, a direct application of equations (9.16), (9.17) leads to the formulas for the diffusion coefficients, which were given without proof in section 7.2.

Before leaving this section let us write a couple of relations that will be needed below. Consider a classical potential motion with $p = m\dot{x}$ and observe that

$$\frac{\partial x_i(t)}{\partial p_j(t')} = \frac{\partial x_i(t)}{\partial p_k(t')} \frac{\partial x_j(t')}{\partial x_k(t')} = - \left[x_i(t), x_j(t')\right], \tag{9.20}$$

where $[\cdot,\cdot]$ is a Poisson bracket, here evaluated with respect to the set of canonical variables $x_k(t'), p_k(t')$. We use the invariance of the Poisson brackets with respect to canonical transformations [see, e.g., Goldstein 1980, section 9.4] to reexpress this result in terms of a different set of canonical variables; for instance, we can write

$$\frac{\partial x_i(t)}{\partial p_j(t')} = -[x_i(t), x_j(t')] = -\frac{\partial x_i(t)}{\partial x_k(t)} \frac{\partial x_j(t')}{\partial p_k(t)} + \frac{\partial x_i(t)}{\partial p_k(t)} \frac{\partial x_j(t')}{\partial x_k(t)} = -\frac{\partial x_j(t')}{\partial p_i(t)} \tag{9.21}$$

and so on. Such (very helpful) relations are examples of the so-called 'direct conditions' in the symplectic approach to canonical transformations [Goldstein 1980, p. 391]. Another such relation that will be used below is

$$\frac{\partial p_i(t)}{\partial p_j(t')} = [x_j(t'), p_i(t)] = \frac{\partial x_j(t')}{\partial x_i(t)}. \tag{9.22}$$

9.2. Detailed balance for multiply periodic systems

We propose now to apply the results of the preceding section to the study of detailed balance in multiply periodic systems. Before entering into the subject, let us recall some basic properties of multiply periodic classical systems.

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[5] We return to this point in §11.4.

9.2.1. DESCRIPTION OF CLASSICAL MULTIPLY-PERIODIC, NONDEGENERATE SYSTEMS

Multiply periodic problems are separable in some system of generalized coordinates and are then conveniently described in terms of a set of action-angle variables. Consider a motion describable by a set of n periodic coordinates; these are the multiply periodic systems.[6] If $\{q_i, p_i\}$ is a set of generalized coordinates and their partner canonical momenta for which the Hamiltonian is separable, the action variables are defined as (without summing over i, of course)[7]

$$J_i = \frac{1}{2\pi}\oint p_i dq_i, \tag{9.23}$$

where the integral is over a complete period in the $q_i p_i$-plane. The factor 2π is conventional and many authors omit it. More generally, on an n-dimensional torus in phase space that corresponds to the integrals of motion of the problem, we can choose n closed paths not deformable into one another nor into a point; then, denoting a set of them by $\{\Gamma_k\}$,

$$J_i = \frac{1}{2\pi}\sum_k \oint_{\Gamma_i} p_k dq_k. \tag{9.24}$$

The quantity J_i does not depend on the details of the path Γ_i. This property is used when possible to select in each case a path such that only one term in (9.24) contributes to the sum; for instance, such that all dq_k except dq_i are zero, thus recovering equation (9.23). Let us denote by $W(J_i, q_i)$ the characteristic function of the canonical transformation $(q_i, p_i) \longrightarrow (\theta_i, J_i)$, where θ_i is the angle associated with the action J_i. Then, from the theory of canonical transformations it follows that

$$\theta_i = \frac{\partial W}{\partial J_i}, \quad p_i = \frac{\partial W}{\partial q_i}; \qquad W = \sum_i W_i \quad \text{with} \quad W_i = \int_0^{q_i} dq_i p_i(q_i, J_k). \tag{9.25}$$

The first of these equations has a simple geometrical meaning. For consider an area in phase space between a torus with action J_i and another one with action $Ji + \delta J_i$; in q, p space this area is (omitting the subindex i)

$$\delta A = \int\int dq dp = \int_0^q dq\,[p(q, J + \delta J) - p(q, J)] \tag{9.26}$$

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[6] Observe that the complete motion need not be periodic. For example, if in a three-dimensional oscillator the three frequencies are not rational fractions of each other, the orbit will be an open Lissajous figure eventually filling the whole accessible phase space, without any global periodicity.

[7] For a classical detailed introduction to the theory of multiply periodic classical systems see Born (1960).

$$= \delta J \int_0^q dq \frac{\partial p}{\partial J} = \delta J \frac{\partial}{\partial J} \int_0^q dq p(q, J) = \delta J \frac{\partial W}{\partial J}, \tag{9.27}$$

and in θ, J space it is

$$\delta A = \int_0^\theta d\theta \delta J = \theta(q, J) \delta J. \tag{9.28}$$

Since the area does not change with the transformation, by equating the two results one recovers the relation under discussion. To extract the derivative with respect to J from the integral sign in the fourth equality, we took into account that on the invariant torus J is constant.

Complete separability means thus that there exists a set of generalized coordinates so that one can write as above $p_i = \partial W_i(q_i, \{\alpha_k\})/\partial q_i$. In the new coordinates the Hamiltonian becomes $H = K(J_i)$, so that the new generalized coordinates are ignorable, and the new momenta are corresponding integrals of the motion, as revealed by the Hamilton equations

$$\dot{J}_i = \frac{\partial K}{\partial \theta_i} = 0, \quad \dot{\theta}_i = \frac{\partial H}{\partial J_i} = \frac{\partial K}{\partial J_i} = \omega_i(J) = \text{const.} \tag{9.29}$$

From the first of these equations one verifies that the action variables are integrals of the motion; this is easily understood from equation (9.23), since the integration gives a result of the form $J_i = J_i(\alpha_1, ...\alpha_n)$ where α_k are constants of motion, and thus J_i is itself a constant of the motion. The numbers ω_i themselves are also constant, since according to (9.29) they are functions of the constants J_k. They represent the fundamental (angular) frequencies of the motions on the $q_i p_i$-planes, since on integrating one gets

$$\theta_i = \omega_i t + \theta_{0i}, \tag{9.30}$$

where θ_{0i} stand for the initial values. The periods of the orbital motion are given by $T_i = 2\pi/\omega_i$, and each θ_i increases by 2π over a period. Thus one can in principle calculate the periods (as functions of the action variables) using equation (9.30), without the need to construct first a complete solution of the problem. This is of course one of the advantages of the action-angle approach.

Each separable generalized coordinate can be expressed as a periodic function of its angle variable by means of a common Fourier development, with Fourier components

$$\tilde{q}_{ni} = \frac{1}{2\pi} \int_0^{2\pi} d\theta_i q_i e^{-in\theta_i}. \tag{9.31}$$

Now consider a function of the coordinates; its Fourier development contains the θ_i of each coordinate that enters into its definition along with its harmonics. Since there may be as many angle variables as there are degrees of freedom, in the general Fourier development there will appear compound angles of the form

$$\mathbf{n}\cdot\boldsymbol{\theta} \equiv n_1\theta_1 + n_2\theta_2 + \cdots = \mathbf{n}\cdot\boldsymbol{\omega}t + \mathbf{n}\cdot\boldsymbol{\theta}_0. \tag{9.32}$$

In particular, the Cartesian coordinates x_i, although normally not separable, can be developed conforming to the above rule; thus

$$x_i = \sum_{n=-\infty}^{\infty} \tilde{x}_{ni} e^{i\mathbf{n}\cdot\boldsymbol{\theta}} = \sum_{n=-\infty}^{\infty} \tilde{x}_{ni} e^{i\mathbf{n}\cdot\boldsymbol{\theta}_0} e^{i\mathbf{n}\cdot\boldsymbol{\omega}_i t}. \tag{9.33}$$

The condition for commensurability of all frequencies is that they satisfy n–1 relations of the form $\sum k_i\omega_i = 0$, with k_i integers; in this case there exists only one independent frequency (of oscillation or rotation), the system is completely degenerate and the orbit is closed. If there are only $0 < m < n-1$ such relations, the system is said to be m-fold degenerate. In an m-fold degenerate system there are $n-m$ different fundamental frequencies, and by means of a canonical transformation to new action-angle variables one can make m of the new frequencies vanish; the new action variables associated with the nonvanishing frequencies are the so-called *proper action variables* [see, e.g., Dittrich and Reuter 1992, chapter 6].

9.2.2. CONSEQUENCES OF THE DETAILED BALANCE CONDITION

Let us come back to our original problem of the detailed balance condition. We use the above theory to make a multiply periodic development of the Cartesian coordinates and canonical momenta expressed to zero order in e, by writing

$$\mathbf{x}^{(0)}(t) = \sum_{\mathbf{k}} \tilde{\mathbf{x}}_{\mathbf{k}} e^{i\omega_{\mathbf{k}}t}, \quad \mathbf{p}^{(0)}(t) = im\sum_{\mathbf{k}} \omega_{\mathbf{k}}\tilde{\mathbf{x}}_{\mathbf{k}} e^{i\omega_{\mathbf{k}}t}, \quad \omega_{\mathbf{k}} = k_i\omega_{0i} = \mathbf{k}\cdot\boldsymbol{\omega}_0, \tag{9.34}$$

where ω_{0i} are the fundamental frequencies of the orbital motion and the initial phases have been absorbed into the Fourier coefficients, since they play no dynamical role at all. Our purpose is to obtain the average powers emitted and absorbed by the system in a state described by equations (9.34); in a stationary situation these can be calculated by taking a time average over a long time interval, $T \to \infty$, of the rates of emission and absorption of energy. From equations (9.19), (9.22) and (4.55), the rate of energy absorption is given to second order in the charge by

$$e\left\langle \dot{\mathbf{x}}^{(1)}\cdot\mathbf{E}\right\rangle = \frac{e^2}{m}\int_{-\infty}^{t} dt' \frac{\partial p_i^{(0)}(t)}{\partial p_j^{(0)}(t')} \langle E_j(t')E_i(t)\rangle$$

$$= \frac{e^2}{m} \int_{-\infty}^{t} dt' \left[x_j^{(0)}(t'), p_i^{(0)}(t)\right] \langle E_j(t') E_i(t) \rangle$$

$$= \frac{e^2}{m} \delta_{ij} \int_{t}^{\infty} ds \int_{0}^{\infty} d\omega S(\omega) \left[x_j^{(0)}(t-s), p_i^{(0)}(t)\right] \cos \omega s. \qquad (9.35)$$

where $s = t - t'$. The Poisson bracket can be calculated in terms of the action and angle variables

$$\left[x_i^{(0)}(t'), p_i^{(0)}(t)\right] = \sum_j \left[\frac{\partial x_i^{(0)}(t')}{\partial \theta_j} \frac{\partial p_i^{(0)}(t)}{\partial J_j} - \frac{\partial p_i^{(0)}(t)}{\partial \theta_j} \frac{\partial x_i^{(0)}(t')}{\partial J_j} \right], \qquad (9.36)$$

so that its time average gives

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \left[x_i^{(0)}(t-s), p_i^{(0)}(t)\right] =$$

$$= m \sum_j \sum_{\mathbf{k},\mathbf{k}'} \left\{ k_j \left(\omega_{\mathbf{k}} \tilde{x}_{\mathbf{k}i}\right) \frac{\partial \tilde{x}_{\mathbf{k}'i}}{\partial J_j} - k'_j \tilde{x}_{\mathbf{k}'i} \cdot \frac{\partial}{\partial J_j} \left(\omega_{\mathbf{k}} \tilde{x}_{\mathbf{k}i}\right) \right\} e^{-i\omega_{\mathbf{k}'} s} \delta_{\mathbf{k},-\mathbf{k}'}$$

$$= m \sum_j \sum_{\mathbf{k}} k_j \frac{\partial}{\partial J_j} \left(\omega_{\mathbf{k}}\right) \left|\tilde{x}_{\mathbf{k}i}\right|^2 e^{i\omega_{\mathbf{k}} s}. \qquad (9.37)$$

In this calculation we assumed that there is no degeneracy, so that the condition $(\mathbf{k} + \mathbf{k}') \cdot \boldsymbol{\omega}_0 = 0$ imposed by the Kronecker delta can be transformed into $\mathbf{k} + \mathbf{k}' = 0$. This cannot be done if the values of two or more frequencies ω_{0i} coincide, which means that the degenerate case requires a separate analysis. Further, we took into consideration that the real character of the Cartesian coordinates implies $\tilde{x}_{\mathbf{k}i} = \tilde{x}^*_{-\mathbf{k}i}$. Introducing this result into equation (9.35) one gets

$$e \left\langle \dot{\mathbf{x}}^{(1)} \cdot \mathbf{E} \right\rangle = \frac{e^2}{2} \mathrm{Re} \sum_{i,j} \sum_{\mathbf{k}} k_j \frac{\partial}{\partial J_j} \int_{-\infty}^{\infty} d\omega S(\omega) \omega_{\mathbf{k}} \left|\tilde{x}_{\mathbf{k}i}\right|^2 \int_0^{\infty} ds e^{i(\omega + \omega'_{\mathbf{k}})s}, \qquad (9.38)$$

or finally,

$$e \left\langle \dot{\mathbf{x}}^{(1)} \cdot \mathbf{E} \right\rangle = \frac{\pi e^2}{2} \sum_{\mathbf{k}} \mathbf{k} \cdot \nabla_J S(\omega_{\mathbf{k}}) \omega_{\mathbf{k}} \left|\tilde{x}_{\mathbf{k}i}\right|^2. \qquad (9.39)$$

The radiated power, on the other hand, can be calculated directly from (9.34); the time average over t gives again a Kronecker delta and the end result is

$$m\tau \left\langle \ddot{\mathbf{x}}^{(0)2} \right\rangle = m\tau \sum_{\mathbf{k}} \omega_{\mathbf{k}}^4 \left|\tilde{x}_{\mathbf{k}i}\right|^2. \qquad (9.40)$$

The energy balance condition for nondegenerate systems thus becomes, after dividing by $m\tau = 2e^2/3c^3$,

$$\sum_{\mathbf{k}} \left[\omega_{\mathbf{k}}^4 \left| \tilde{x}_{\mathbf{k}i} \right|^2 - \frac{3}{4} \pi c^3 \mathbf{k} \cdot \nabla_J \omega_{\mathbf{k}} S(\omega_{\mathbf{k}}) \left| \tilde{x}_{\mathbf{k}i} \right|^2 \right] = 0. \tag{9.41}$$

A slightly different form can be given to this result by considering an ensemble of particles instead of a single one and taking the average, on the assumption that the equilibrium phase-space density of the unperturbed system is a function of the action variables only, $W_0 = W_0(J_i)$. By multiplying the above result by W and integrating over the whole phase space we get, with $d\mu = \prod_i d\theta_i dJ_i$ and after an integration by parts with respect to the action variables in the second term,

$$\int d\mu \sum_{\omega_{\mathbf{k}}=\omega} \left[\omega_{\mathbf{k}}^4 \left| \tilde{x}_{\mathbf{k}i} \right|^2 W + \frac{3}{4} \pi c^3 \mathbf{k} \cdot (\nabla_J W) \, \omega_{\mathbf{k}} S(\omega_{\mathbf{k}}) \left| \tilde{x}_{\mathbf{k}i} \right|^2 \right] = 0. \tag{9.42}$$

This is the result of van Vleck (1924) and others, referred to above [see also van Vleck and Huber 1977].

It is possible to extract various conclusions from these expressions, with regard to detailed balance. In the first place, let us assume that there exists a spectrum $S(\omega)$ such that the global energy balance, equation (9.42), is a consequence of detailed balance for every separate frequency, so that the equilibrium condition is satisfied term by term. Then for both terms in equation (9.42) to have equal powers of **k**, the spectrum must have the classical form $S(\omega) \sim \omega^2$. Further, consider a one-dimensional system, for which the equilibrium zero-order distribution is a function of the energy only; then $dE/dJ = \omega$, so that $\partial W/\partial J = \omega \, (\partial W/\partial E)$. From equation (9.42) the requirement of detailed balance implies therefore

$$S(\omega) \frac{\partial W}{\partial E} = -\text{const} \, \omega^2 W; \tag{9.43}$$

hence, with $S(\omega) \sim \omega^2$ the dependence on ω disappears, and the general solution of the equation becomes

$$W = \frac{1}{\bar{E}} e^{-E(J)/\bar{E}}, \tag{9.44}$$

i.e., the energy has necessarily a Laplacian distribution. In particular, for $\bar{E} = kT$, corresponding to the classical Maxwell-Boltzmann distribution, $S(\omega)$ must be the Rayleigh-Jeans spectrum, $S(\omega) = 4\omega^2 k_B T/3\pi c^3$; thus the theory leads unavoidably and consistently to classical results. This conclusion can be verified by introducing the Maxwell-Boltzmann distribution

$W \sim e^{-\beta H}$ into equation (9.42), from which it follows that $\omega_{\mathbf{k}}\mathbf{k} \cdot \nabla_J W \sim \beta\omega_{\mathbf{k}} k_i \omega_{0i} W = \beta\omega_{\mathbf{k}}^2 W$, so that indeed $S(\omega) \sim \omega^2/\beta$ is a solution.

Still another way of looking at the problem related to the multiperiodic solution is the following. From equation (9.42) one can define an absorption coefficient per unit phase-space volume at frequency ω such that (reinserting the factor $m\tau$)

$$a(\omega)\rho(\omega)d\mu = -\pi^2 c^3 m\tau\omega\rho(\omega) \sum_{\omega_{\mathbf{k}}=\omega} \mathbf{k} \cdot (\nabla_J W) \left|\tilde{x}_{\mathbf{k}i}\right|^2 d\mu,$$

where $S(\omega) = (4\pi/3)\rho(\omega)$ was used, whence it follows that

$$a(\omega) = -\frac{2}{3}\pi^2 e^2 \omega \sum_i \sum_{\omega_{\mathbf{k}}=\omega} \left|\tilde{x}_{\mathbf{k}i}\right|^2 \mathbf{k} \cdot (\nabla_J W) . \tag{9.45}$$

This expression for the absorption coefficient can alternatively be written in terms of the derivatives $\partial W/\partial\omega_{0i}$, using the second of equations (9.25) to write the actions as functions of the fundamental frequencies. Considering that every nonlinear system changes its frequencies (and actions) with the absorption of energy in a particular way, so that the quantities $W^{-1}\partial W/\partial\omega_{0i}$ are specific for each system and depend on the depopulation and repopulation of the different frequencies with the change of energy [Planck 1917], it follows that the ratio of the absorption and emission coefficients depends in general on the system and its energy state (note from equation (9.42) that the emission coefficient is proportional to W). Such behaviour is contrary to Kirchhoff's law, which asserts the universality of this ratio as a function of temperature and frequency;[8] only with W given by a Lorentzian distribution of the energy does this ratio become state-independent.

We conclude that a general multiply periodic function of the form (9.34) leads to the breakdown of detailed balance, unless it refers to a classical system in equilibrium with a Rayleigh-Jeans spectrum that obeys Maxwell-Boltzmann statistics. In particular, the solution of the SED system subject to the zeropoint field cannot be expressed in the form of (9.34) if it is to be consistent with detailed balance. This in a nutshell is the essence of the problems of SED with frequency mixing and the violation of detailed balance.

9.2.3. SYSTEMS WITH A SINGLE FREQUENCY

The situation changes drastically when the mechanical system has only one frequency of motion, since then global and detailed balance coincide.

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[8] A discussion of Kirchhoff's law can be seen in Landau and Lifshitz (1967), section 63.

This happens in the case of totally degenerate and linear multiply periodic systems, in which no harmonics are generated. There are a very few such systems of physical interest, and we study first its most conspicuous representative, the harmonic oscillator.

The harmonic oscillator
In this case we have (writing ω everywhere instead of ω_0)

$$x^{(0)} = \tilde{x}_1 e^{i\omega t} + \tilde{x}_{-1} e^{-i\omega t}, \quad p^{(0)} = im\omega \left(\tilde{x}_1 e^{i\omega t} - \tilde{x}_{-1} e^{-i\omega t}\right), \tag{9.46}$$

$$\tilde{x}_1 = \frac{x_0}{2} - i\frac{p_0}{2m\omega}, \quad \tilde{x}_{-1} = \tilde{x}_1^*, \tag{9.47}$$

and the equilibrium condition reduces to

$$\omega^4 \left|\tilde{x}_1\right|^2 - \frac{3}{4}\pi c^3 \frac{\partial}{\partial J}\omega S(\omega) \left|\tilde{x}_1\right|^2 = 0. \tag{9.48}$$

From (9.46), the action is given by

$$J = \frac{1}{2\pi} \oint p^{(0)} \dot{x}^{(0)} dt = 2m\omega \left|\tilde{x}_1\right|^2, \tag{9.49}$$

and since the frequency is a fixed parameter, we get from (9.48)

$$J = \frac{3\pi c^3}{4\omega^3} S(\omega), \quad E = 2m\omega^2 \left|\tilde{x}_1\right|^2 = \omega J. \tag{9.50}$$

With the power spectrum for the zeropoint field $S(\omega) = 2\hbar\omega^3/3\pi c^3$ these formulas reproduce the old Bohr-Sommerfeld quantization rules for the ground state (with the zeropoint term included),

$$J = \frac{1}{2}\hbar, \quad E = \omega J = \frac{1}{2}\hbar\omega. \tag{9.51}$$

Other simple applications
The example of the harmonic oscillator is not very meaningful in itself because of the nonspecificity of many of its properties; for example, equations (9.50) are equally valid in quantum theory, in SED and in classical mechanics. For this reason it is convenient to consider other systems with a single frequency ω_0, but where ω_0 depends on the state of motion instead of being externally fixed. With some minor changes, the above theory can indeed be applied to the study of this class of problems, which includes the rigid

rotor, the circular ground-state orbit of the hydrogen atom, and so on.[9] Here we briefly report some main results at $T = 0$.

For a dipole rotator moving on the xy-plane, characterized by a moment of inertia I and a dipole moment $\mathbf{d}_0$, the SED equation of motion is

$$\frac{d(I\omega)}{dt} = -\frac{2d_0^2}{3c^3}\omega^3 + [\mathbf{d}_0 \times \mathbf{E}(t)]_z . \tag{9.52}$$

The problem lends itself to a description in the action-angle phase space. After getting rid of the angle variable, one obtains the reduced Fokker-Planck equation for the stationary probability density in ω space

$$\frac{d}{d\omega}\left[\omega^3 W + \frac{\hbar}{2I}\frac{d}{d\omega}|\omega|^3 W\right] = 0, \tag{9.53}$$

the physical solution of which is

$$W(\omega) = \frac{1}{\omega_0}e^{-\omega/\omega_0} \quad \text{with} \quad \omega_0 = \frac{\hbar}{2I}, \quad \text{for } \omega > 0. \tag{9.54}$$

Note that this result is a particular instance of equation (9.44), with $E = \omega J = I\omega^2/2$; as before, equilibrium is obtained at each frequency between the energy absorbed from the vacuum and the energy lost by radiation. In particular, for the absorption coefficient equations (9.45) and (9.54) lead to a formula due to Planck,

$$a(\omega) = -\frac{2\pi^2 d_0^2 \omega}{3I}\frac{dW}{d\omega} = \frac{2\pi^2 d_0^2}{3I\omega_0^2}\omega e^{-\omega/\omega_0}. \tag{9.55}$$

The absorption goes through a maximum at $\omega = \omega_0 = \hbar/2I$, which is just the value obtained in quantum mechanics for the transition frequency between the ground state and the first excited state.

For the three-dimensional rotator, the solution of the corresponding Fokker-Planck equation is

$$W(\omega) = \frac{\omega}{\omega_0^2}e^{-\omega/\omega_0} \quad \text{with} \quad \omega_0 = \frac{\hbar}{2I}, \quad \text{for } \omega > 0, \tag{9.56}$$

and the Planck formula obtained for the absorption coefficient is, instead of (9.55),

$$a(\omega) = -\frac{2\pi^2 d_0^2 \omega^2}{3I}\frac{d}{d\omega}\left(\frac{W}{\omega}\right) = \frac{2\pi^2 d_0^2}{3I\omega_0}\left(\frac{\omega}{\omega_0}\right)^2 e^{-\omega/\omega_0}. \tag{9.57}$$

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[9] An early study of the rigid rotator immersed in the complete Planck spectrum is made in Boyer (1970b), while Claverie and Diner (1976b, 1977a) give a detailed discussion of the rigid rotator in two and three dimensions at $T = 0$. These authors used the Fokker-Planck-equation technique and (in the later works) linear-response theory applied to the stationary solution of this equation.

This function has a maximum for $\omega = 2\omega_0 = \hbar/I$, which again is the quantum mechanical result for the transition frquency to the first excited state. It is interesting to recall that this difference between the plane and the spatial rotator, which is typical of quantum mechanics, could not be obtained in the old quantum theory. However, for the higher-order moments the coincidences with the quantum mechanical predictions for the rigid rotator disappear [see, e.g., Marshall and Claverie 1980b, Pesquera 1980a].

It should come as no surprise anymore that the solution given by the Fokker-Planck equation for the probability density $W(\omega)$ is completely at variance with quantum mechanics. Even at zero temperature the SED expression $W(\omega)$ implies a continuous distribution of rotation frequencies and hence of energies, with an average value $\bar{E} = \hbar^2/4I$ for the planar rotator and $\bar{E} = 3\hbar^2/4I$ for the three-dimensional rotator, in contrast with the zero value found in quantum mechanics. This difference between the two theories is difficult to test experimentally, however, as remarked by Claverie and Diner (1977a).

It is tempting to close this section with an approximate calculation for the hydrogen atom, which despite its naïveté adds something to the heuristic treatment discussed in §4.1.3, especially when contrasted with the calculations to be performed in the next section. On several occasions it has been suggested to treat heuristically the ground-state orbit, assumed to be circular, as the combined motion of two orthogonal oscillatory motions of equal frequency [Boyer 1975a, Surdin 1975b, de la Peña and Cetto 1986, Puthoff 1987]. For the Kepler problem with potential $V = -k/r$, the energy and the action variable are related through $E = -mk^2/2J^2$; hence, with $J = \hbar$ as follows from (9.51) for two oscillators, and $k = Ze^2$ for the hydrogenic atom, one gets

$$E = -\frac{Z^2 me^4}{2\hbar^2}, \tag{9.58}$$

which coincides with the quantum mechanical value for the ground-state energy. Further, from the virial theorem the average potential energy is $\langle V \rangle = 2E$, whence the mean radius corresponding to this orbit is given by $a = \hbar^2/mZe^2$, precisely the Bohr radius for the hydrogenic atom. We will find in the next section that these pretty results are hopelessly spoiled by more elaborate calculations.

9.3. The hydrogen atom

A crucial test for any theory intended to explain the quantum behaviour of matter, is of course provided by the hydrogen atom, the example of non-linear, multiply periodic system par excellence. In view of its importance, and of the severe difficulties that SED confronts when dealing with it, the system should be analyzed in some detail. As mentioned above, most of the work pertaining to the hydrogen atom in SED focussed on the construction and solution of the relevant Fokker-Planck-type equations. Here, however, we shall look at the problem from the much simpler perspective of the detailed-balance condition.

9.3.1. THE MULTIPLY-PERIODIC DEGENERATE CASE. APPLICATION TO THE KEPLER PROBLEM

In view of the algebraic intricacies of the Kepler problem, it is best to recall first some basic results applicable to this case; the details can be seen in Born (1960). For a general central motion in the plane $x = r\cos\theta$, $y = r\sin\theta$, the polar variables have the following general form in terms of action-angle variables,

$$r = g_1(\theta_\theta, J_r, J_\theta); \quad \theta = \theta_\theta + g_2(\theta_\theta, J_r, J_\theta). \tag{9.59}$$

For a uniform circular motion the dependence of the functions g_1, g_2 on θ_θ disappears. Now, from $x = (r/2)(e^{i\theta} + e^{-i\theta})$ and similarly for y, one obtains the Fourier expansions

$$x = \sum_{n=-\infty}^{\infty} \left[\tilde{x}_{n,1} e^{-i(n\theta_r+\theta_\theta)} + \tilde{x}_{n,-1} e^{-i(n\theta_r-\theta_\theta)}\right], \tag{9.60}$$

$$y = \sum_{n=-\infty}^{\infty} \left[\tilde{y}_{n,1} e^{-i(n\theta_r+\theta_\theta)} + \tilde{y}_{n,-1} e^{-i(n\theta_r-\theta_\theta)}\right], \tag{9.61}$$

the frequencies and Fourier coefficients being functions of the actions. An alternative form of the above equations is

$$x + iy = \sum_{n=-\infty}^{\infty} C_n e^{i(n\theta_r+\theta_\theta)}, \tag{9.62}$$

$$\tilde{x}_{n,1} = \frac{1}{2} C_n^*, \quad \tilde{x}_{n,-1} = \frac{1}{2} C_{-n}; \quad \tilde{y}_{n,1} = \frac{i}{2} C_n^*, \quad \tilde{y}_{n,-1} = -\frac{i}{2} C_{-n}. \tag{9.63}$$

In the Kepler problem the equilibrium orbits are closed, so that the two frequencies are equal and one can set $\theta_r = \theta_\theta \equiv \omega_0 t$, where the frequency of the orbit ω_0 depends on the action variables or, better, on the single proper action variable $J = J_r + J_\theta$ [see, e.g., Goldstein 1980]. Substituting this relation in (9.60) one gets

$$x = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[C_{-n} e^{-i(n-1)\omega_0 t} + C_n^* e^{-i(n+1)\omega_0 t}\right]$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[C_{-n-1} + C^*_{n-1} \right] e^{-in\omega_0 t}, \tag{9.64}$$

and similarly for y; it follows that for the degenerate case one has

$$\tilde{x}_n = \frac{1}{2}(C^*_{n-1} + C_{-n-1}), \quad \tilde{y}_n = \frac{i}{2}(C^*_{n-1} - C_{-n-1}), \quad C_n = \tilde{x}_{-n-1} + i\tilde{y}_{-n-1}, \tag{9.65}$$

so that combining with (9.63) one gets

$$\tilde{x}_{n,1} = \frac{1}{2}(\tilde{x}_{n+1} - i\tilde{y}_{n+1}), \quad \tilde{x}_{n,-1} = \frac{1}{2}(\tilde{x}_{n-1} + i\tilde{y}_{n-1}); \tag{9.66}$$

$$\tilde{y}_{n,1} = \frac{1}{2}(\tilde{y}_{n+1} + i\tilde{x}_{n+1}), \quad \tilde{y}_{n,-1} = \frac{1}{2}(\tilde{y}_{n-1} - i\tilde{x}_{n-1}). \tag{9.67}$$

The Fourier development for the unperturbed Kepler problem can now be written as

$$x^{(0)} = \sum_{n=-\infty}^{\infty} \tilde{x}_n e^{-in\omega_0 t}, \quad y^{(0)} = \sum_{n=-\infty}^{\infty} \tilde{y}_n e^{-in\omega_0 t}, \quad \tilde{x}_{-n} = \tilde{x}^*_n, \quad \tilde{y}_{-n} = \tilde{y}^*_n. \tag{9.68}$$

9.3.2. PROBLEMS WITH THE ENERGY-BALANCE CONDITION

Let us now apply the energy-balance condition to the Kepler problem.[10] For the radiated power the calculation is direct and leads to

$$m\tau \left\langle \ddot{\mathbf{r}}^{(0)2} \right\rangle = m\tau\omega_0^4 \sum_n n^4 \left(|\tilde{x}_n|^2 + |\tilde{y}_n|^2 \right). \tag{9.69}$$

The evaluation of the absorbed power is more complicated. We start from the expression derived in §9.3.2,

$$e \left\langle \dot{\mathbf{r}}^{(1)} \cdot \mathbf{E} \right\rangle = \frac{e^2}{m} \lim_{T\to\infty} \frac{1}{2T} \sum_i \int_{-T}^{T} dt \int_{-\infty}^{t} dt'$$

$$\int_0^{\infty} d\omega \left[x_i^{(0)}(t'), p_i^{(0)}(t) \right] S(\omega) \cos\omega(t-t') \tag{9.70}$$

and use the multiply periodic development of equations (9.60) and (9.61). During the evaluation of the Poisson bracket and taking into account (at the end) that both frequencies are equal, one gets terms such as

$$\left[x^{(0)}(t'), p_x^{(0)}(t) \right] = m \sum_j \sum_{n_1,n_2} \sum_{n'_1,n'_2} (n_1 + n_2)\, e^{-i\omega_0\left[(n_1+n_2)t + (n'_1+n'_2)t'\right]} \times$$

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[10] The following discussion draws heavily on Alcubierre and Lozano (1988).

$$\times\left[n_j\omega_0\tilde{x}_{n_1,n_2}\frac{\partial}{\partial J_j}\tilde{x}_{n_1',n_2'}-n_j'\tilde{x}_{n_1',n_2'}\frac{\partial}{\partial J_j}\tilde{x}_{n_1,n_2}\omega_0\right], \tag{9.71}$$

where n_2, n_2' can take only the values ± 1, as follows from equations (9.66), (9.67). With $s = t - t'$ in equation (9.70), only the Poisson bracket depends on t; hence the average over t gives a delta function imposing the condition $n_1 + n_2 + n_1' + n_2' = 0$, which means that three of the $n's$ are independent. (In the nondegenerate case the same process led to the much stronger condition $\mathbf{n} = -\mathbf{n}'$, eliminating two degrees of freedom from the three-dimensional problem.) Further, a series of operations must be performed on the Poisson brackets: the sums over n_2, n_2' are developed and in the terms affected by the factor $e^{-i\omega_0(n_1\pm 1)s}$ the substitution $n = n_1 \pm 1$ is made; equations (9.66) to (9.68) are used, and finally, a similar expression for the Poisson bracket involving the y variables is added. All this leads to the following expression, where $E_t\{\cdot\}$ refers to the time average:

$$E_t\left\{\left[x^{(0)}(t-s),p_x^{(0)}(t)\right]+\left[y^{(0)}(t-s),p_y^{(0)}(t)\right]\right\}=2\pi m\sum_n ne^{-i\omega_0 ns}\times$$

$$\times\left\{n\frac{\partial}{\partial J_r}\omega_0\left(\left|\tilde{x}_n\right|^2+\left|\tilde{y}_n\right|^2\right)+2i\left[\frac{\partial}{\partial J_r}\omega_0\tilde{x}_n\tilde{y}_n-\frac{\partial}{\partial J_\theta}\omega_0\tilde{x}_n\tilde{y}_n\right]\right\}. \tag{9.72}$$

Now we plug this result into the average rate of energy absorption, perform the integration over s, and introduce the power spectrum for the zeropoint field, $S(\omega) = 2\hbar\left|\omega\right|^3/3\pi c^3$; the energy balance condition (9.5) becomes thus

$$\sum_n\left\{\omega_0^4n^4\left(\left|\tilde{x}_n\right|^2+\left|\tilde{y}_n\right|^2\right)-\pi\hbar n^2\frac{\partial}{\partial J_r}\omega_0\left|n\omega_0\right|^3\left(\left|\tilde{x}_n\right|^2+\left|\tilde{y}_n\right|^2\right)\right.$$

$$\left.-i2\pi\hbar n\left[\frac{\partial}{\partial J_r}\omega_0\left|n\omega_0\right|^3\tilde{x}_n\tilde{y}_n-\frac{\partial}{\partial J_\theta}\omega_0\left|n\omega_0\right|^3\tilde{x}_n\tilde{y}_n\right]\right\}=0. \tag{9.73}$$

To calculate the Fourier amplitudes $\tilde{x}_n$, $\tilde{y}_n$ we start from the parametric equations for the orbit, which can be written in the form

$$r = r_0(1-\varepsilon\cos\zeta), \quad t = r_0\sqrt{-\frac{m}{2E}}(\zeta-\varepsilon\sin\zeta), \tag{9.74}$$

where E is the energy of the orbit, ε its eccentricity and $r_0 = -k/2E$, with $k = Ze^2$ for the hydrogenic atom; r_0 is the length of the major semiaxis of the ellipse. From the relations between the parameters of the orbit, with $J = J_r + J_\theta$,

$$E=-\frac{2\pi^2mk^2}{J^2}, \quad \varepsilon=\sqrt{1+\frac{2EL^2}{mk^2}}, \quad \omega_0=\sqrt{-\frac{8E^3}{mk^2}}=\sqrt{\frac{k}{mr_0^3}} \tag{9.75}$$

it follows that

$$x = r\cos\theta = r_0(\cos\zeta - \varepsilon), \quad y = r\sin\theta = r_0\sqrt{1-\varepsilon^2}\sin\zeta. \tag{9.76}$$

Now the velocity components are given by (leaving aside for the moment the superindex $^{(0)}$ that reminds us that all quantities refer to the unperturbed motion)

$$\dot{x} = -i\sum_{n=-\infty}^{\infty} n\omega_0\tilde{x}_n e^{-in\omega_0 t}, \tag{9.77}$$

so that by multiplying by $e^{-in\omega_0 t}$ and averaging over one period of the orbit,

$$\frac{1}{T}\int_0^T dt\dot{x}e^{in\omega_0 t} = -in\omega_0\tilde{x}_n. \tag{9.78}$$

Since $\dot{x}dt = dx = (dx/d\zeta)d\zeta = -r_0\sin\zeta d\zeta$, this gives for the Fourier amplitude $\tilde{x}_n$, using (9.74) for t, the expression

$$\tilde{x}_n = -i\frac{r_0}{2\pi n}\int_0^{2\pi} d\zeta\sin\zeta e^{in(\zeta-\varepsilon\sin\zeta)} = \frac{r_0}{n}J_n'(n\varepsilon). \tag{9.79}$$

In an analogous way one gets

$$\tilde{y}_n = i\frac{r_0\sqrt{1-\varepsilon^2}}{2\pi n\varepsilon}\int_0^{2\pi} d\zeta e^{in(\zeta-\varepsilon\sin\zeta)} = i\frac{r_0\sqrt{1-\varepsilon^2}}{n\varepsilon}J_n(n\varepsilon). \tag{9.80}$$

The above equations hold for $n \neq 0$; in writing them we used the following formula for the cylindrical Bessel function:

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} d\zeta e^{i(n\zeta - z\sin\zeta)}. \tag{9.81}$$

For $n = 0$ a direct calculation gives

$$\begin{aligned}\tilde{x}_0 &= \frac{1}{T}\int_o^T dtx = \frac{1}{T}\int_o^T d\zeta x\frac{dt}{d\zeta}\\ &= \frac{\omega_0}{2\pi}\int_0^{2\pi} d\zeta r_0(\cos\zeta - \varepsilon)r_0\sqrt{-\frac{m}{2E}}(1-\varepsilon\cos\zeta)\\ &= \frac{r_0}{2\pi}\int_0^{2\pi} d\zeta\left(-\varepsilon + (1+\varepsilon^2)\cos\zeta - \varepsilon\cos^2\zeta\right) = -\frac{3}{2}r_0\varepsilon;\end{aligned} \tag{9.82}$$

in a similar way one gets $\tilde{y}_0 = 0$.

It is not possible to apply these complicated results in a closed form; hence, to study the contributions to the balance of energy we expand in powers of the eccentricity. The usual power expansion of the Bessel functions gives

$$\tilde{x}_n = \frac{r_0}{2n} \sum_{k=0}^{\infty} \frac{(-1)^k (n+2k)}{k! \Gamma(n+k+1)} \left(\frac{n\varepsilon}{2}\right)^{n+2k-1}, \quad n \neq 0, \tag{9.83}$$

$$\tilde{y}_n = i\frac{r_0}{2}\sqrt{1-\varepsilon^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(n+k+1)} \left(\frac{n\varepsilon}{2}\right)^{n+2k-1}, \quad n \neq 0. \tag{9.84}$$

For simplicity we restrict ourselves to the circular orbits, so that we need to develop the balance condition to zero order in the eccentricity. For this purpose we write all $\tilde{x}_n, \tilde{y}_n$ that contain ε up to second order, which are the following few:

$$\tilde{x}_0 = -\frac{3}{2} r_0 \varepsilon; \quad \tilde{y}_0 = 0; \quad \tilde{x}_1 = \frac{r_0}{2}\left(1 - \frac{3\varepsilon^2}{8}\right), \quad \tilde{y}_1 = i\frac{r_0}{2}\left(1 - \frac{5\varepsilon^2}{8}\right),$$

$$\tilde{x}_2 = \frac{r_0}{4}\varepsilon; \quad \tilde{y}_2 = i\frac{r_0}{4}\varepsilon; \quad \tilde{x}_3 = \frac{3}{16} r_0 \varepsilon^2, \quad \tilde{y}_3 = i\frac{3}{16} r_0 \varepsilon^2, \tag{9.85}$$

insert them into equation (9.73) and calculate to zero order in ε. The contribution to the radiated power gives

$$\sum_n \omega_0^4 n^4 \left(|\tilde{x}_n|^2 + |\tilde{y}_n|^2\right) = \frac{m^2 k^2}{r_0^4}. \tag{9.86}$$

Further, from the relations between the parameters it follows that

$$\frac{\partial \varepsilon}{\partial J_r} = \frac{1}{2\pi\sqrt{mkr_0}} \frac{1-\varepsilon^2}{\varepsilon}, \tag{9.87}$$

$$\frac{\partial \varepsilon}{\partial J_\theta} = \frac{1}{2\pi\sqrt{mkr_0}} \frac{1-\varepsilon^2-\sqrt{1-\varepsilon^2}}{\varepsilon}, \tag{9.88}$$

$$\frac{\partial r_0}{\partial J_r} = \frac{\partial r_0}{\partial J_\theta} = \frac{1}{\pi}\sqrt{\frac{r_0}{mk}}, \tag{9.89}$$

and that ω_0 is independent of the eccentricity. These expressions for the derivatives allow us to write the contribution coming from the absorbed energy in the form

$$\frac{2\hbar}{\sqrt{mk}} \sum_{n=1}^{\infty} n^4 \left\{ n\sqrt{r_0} \frac{\partial}{\partial r_0} \omega_0^4 \left(|\tilde{x}_n|^2 + |\tilde{y}_n|^2\right) \right.$$

$$\left. + \frac{\omega_0^4}{\sqrt{r_0}} \left[\frac{n}{2} \frac{1-\varepsilon^2}{\varepsilon} \frac{\partial}{\partial \varepsilon} \left(|\tilde{x}_n|^2 + |\tilde{y}_n|^2\right) + i\frac{\sqrt{1-\varepsilon^2}}{\varepsilon} \frac{\partial}{\partial \varepsilon} \tilde{x}_n \tilde{y}_n \right] \right\}$$

$$\equiv \frac{2\hbar}{\sqrt{mk}}\left(A_1 + A_2 + A_3\right). \tag{9.90}$$

We calculate each of the above terms to zero order in ε. For the first one we get

$$\begin{aligned} A_1 &= \sum_{n=1}^{\infty} n^5 \sqrt{r_0}\frac{\partial}{\partial r_0}\omega_0^4\left(|\tilde{x}_n|^2 + |\tilde{y}_n|^2\right) \\ &\simeq \sqrt{r_0}\frac{\partial}{\partial r_0}\omega_0^4\left(|\tilde{x}_1|^2 + |\tilde{y}_1|^2\right) = -\frac{2k^2}{m^2 r_0^{9/2}}. \end{aligned} \tag{9.91}$$

For the next term it is necessary to go up to second order in ε, due to the appearance of ε in the denominators; one thus gets to the desired order of approximation

$$\begin{aligned} \frac{\sqrt{r_0}}{\omega_0^4}A_2 &= \sum_{n=1}^{\infty}\frac{n^5}{2}\frac{1-\varepsilon^2}{\varepsilon}\frac{\partial}{\partial\varepsilon}\left(|\tilde{x}_n|^2 + |\tilde{y}_n|^2\right) \\ &\simeq \frac{1}{2}\left(\frac{1}{\varepsilon} - \varepsilon\right)\frac{\partial}{\partial\varepsilon}\left[\left(|\tilde{x}_1|^2 + |\tilde{y}_1|^2\right)\right] \\ &\simeq \frac{1}{2}\left(\frac{1}{\varepsilon} - \varepsilon\right)\frac{\partial}{\partial\varepsilon}\left[\left(|\tilde{x}_1|^2 + |\tilde{y}_1|^2\right) + 2^5\left(|\tilde{x}_2|^2 + |\tilde{y}_2|^2\right)\right] \simeq \frac{7}{2}r_0^2. \end{aligned} \tag{9.92}$$

It is important to observe that the second harmonics of the motion contribute to this result; their contribution remains different from zero even in the limit $\varepsilon = 0$, due to the factor $1/\varepsilon$ affecting the whole term. We have here an explicit example of frequency mixing, that gives a result different from zero even for the circular orbits. Concerning the third term we have

$$\frac{\sqrt{r_0}}{\omega_0^4}A_3 = i\sum_{n=1}^{\infty} n^4\frac{\sqrt{1-\varepsilon^2}}{\varepsilon}\frac{\partial}{\partial\varepsilon}\tilde{x}_n\tilde{y}_n \simeq i\left(\frac{1}{\varepsilon} - \frac{\varepsilon}{2}\right)\sum_{n=1}^{\infty} n^4\frac{\partial}{\partial\varepsilon}\tilde{x}_n\tilde{y}_n \simeq -\frac{3}{2}r_0^2. \tag{9.93}$$

This result shows another limitation of the heuristic treatment in terms of orthogonal oscillators: it shows that there are contributions coming from the cross products $\tilde{x}_n\tilde{y}_n$, and therefore the oscillators cannot be treated as independent.

The sum gives $A_1+A_2+A_3 = 0$, so that the contribution from the second harmonic and the correlation between the oscillators conspire to cancel the mean absorbed power to zero order in the eccentricity! The requirement of energy balance to zero order in the eccentricity reduces then simply to

$$\frac{m^2k^2}{r_0^4} = 0. \tag{9.94}$$

The only sensible solution of this equation is $r_0 = \infty$, which is equivalent to $E = 0$, implying a selfionized atom of infinite radius. Thus, according to the present treatment, there is no finite-sized stationary orbit for the hydrogenic atom. This is the same result as the one predicted using the Fokker-Planck method.

It has been argued that upon reconsideration, a selfionizing hydrogen atom might be not so unacceptable as it seems at first sight [Brody 1983]. Indeed, for any potential $V(r)$ that remains everywhere finite there is a finite probability, however small, that the interaction of the electron with the zeropoint field allows it to escape and reach a situation of no return. And since astrophysical arguments show that for any lifetime longer than about 20 years the selfionization of neutral cold monoatomic hydrogen would be unobservable, the SED prediction, though apparently grossly mistaken, could well be the physically correct one. This would lead to conclude that the normal stability is not a feature of an isolated atom, but is due to the nearby presence of other atoms. Further, it is also true that for any $T > 0$ quantum mechanics already predicts spontaneous ionization of the hydrogen atom [Brillouin 1930, chapter 10, and Farley and Wing 1981], in agreement with the above qualitative discussion. But with all the value it might have, this argument misses the point, namely that the basic theory should lead to an internally consistent physics, and in particular it should conform to Kirchhoff's law.

Thus one verifies that the heuristic model discussed in the previous section and in chapter 4 simply does not survive a full calculation; the inclusion of the eccentricity, which in a stochastic world is inevitable, destroys the intuitive and pleasant results of the simple heuristic treatment. The detailed calculation confirms that the entanglement of frequencies of the motion in nonlinear systems leads to results that depart from the known quantum behaviour of matter.

9.4. Critical appraisal of the results

What we have learned in this chapter suggests that SED and quantum mechanics are two different theories, and that further attempts to develop the former are bound to fail. However, there is still room for some additional pondering of the situation.

First of all, it is hard to believe that all positive results of previous chapters are due to simple coincidences and mere chance. The answer that such coincidences result from the linearity of the systems is not free of difficulties either. For according to the results of this chapter, an equilibrium ensemble of harmonic oscillators should follow the Rayleigh-Jeans distribution, whereas in chapters 5 and 7 the correct Planck law came out quite naturally as the germane result. This suggests that the derivation just made is omitting aside essential effects of the zeropoint field, as confirmed by the fact that the zero-order solution follows strictly classical laws according to equation (9.6); this solution is therefore not expected to depend on Planck's constant, as it should in order to satisfy the equilibrium condition (9.41) (recall that $S(\omega)$ is proportional to $\hbar$).

This incompatibility between the zero-order solution and quantum mechanics was noted above by recalling that it is expressed in terms of nonrandom numbers, whereas the corresponding quantum variables are dispersive, or random numbers from our stochastic perspective. In other words, the zero-order solution *must* depend on $\hbar$, and this dependence is a signature of the stochasticity impressed by the zeropoint field.

Another point that casts doubts on the pertinence of the analysis carried out in the present chapter concerns the Fourier decomposition of the motions. By expressing the atomic motions in the terms appropriate for multiply periodic systems, the Fourier frequencies became linear combinations with integer (positive or negative) coefficients of a set of basic frequencies, equal in number or less than the degrees of freedom of the problem. Now, with the exception of linear oscillators, the relevant frequencies of quantum problems are not characteristically overtones of a small set of fundamental frequencies. They constitute a quite different, normally infinite, set of frequencies, given by Bohr's transition rule and having little to do with the frequencies associated to the periods of the orbits and their linear combinations. In other words, the language of the Fourier analysis and that demanded by the theory of spectra are entirely different things, the atomic system behaving rather as a dynamical system with an infinite number of degrees of freedom (and of resonant frequencies), than as a conventional multiply periodic system of point particles.

A better interpretation would apparently consist in assuming that the zeropoint field plays a much more fundamental role in establishing the behaviour of the system, than the one implicitly assumed in the preceding analysis.

From the above considerations one concludes that the failure of SED resides centrally either in the equations of motion or in the preceding perturbative method of analysis (if not in both). In relation with the equations of motion it is easy to corroborate that they are incomplete: they are nonrelativistic and were treated in the dipolar approximation; the possibility that the zeropoint field becomes much affected by its interaction with matter was overlooked, as well as several other physical ingredients that might play their part in defining the dynamics, such as the structure of the particle (whose role has been reduced to merely fixing a cutoff frequency for the response to the field), or the existence of other zeropoint fields that along with their electromagnetic partner constitute the real physical vacuum, and which in a finished theory surely should be present. The theory has been reduced to its simplest form and the risk of oversimplification is real. In the literature all these possibilities have been commented from different perspectives, but without leading up to now to any kind of consensus on their significance as a way out of the difficulties. So, for instance, Marshall and Santos (1988a) have insisted on the need to incorporate additional elements, mainly the electron-positron vacuum, whereas Boyer (1976, 1978a) argues that the completed theory should be essentially relativistic in nature so as to leave invariant the Lorentz invariant spectrum. In a more pessimistic vein, other authors, such as Claverie and Diner (1980) or Cavalleri (1985), expressed in different ways their conviction that SED seems to be unable to cope with the atomic problem.[11]

Undoubtedly, the standard form of SED discussed up to now must be modified if the final theory is to lead to a detailed and precise description of atomic systems. However, it is hard to believe that the extra paraphernalia just recalled should become vital for an explanation of the essential features of such a simple system as a hidrogen atom. That simply does not fit with the notion that quantum mechanics lies at the level of a fundamental description of nature. If this is the case and SED has something to do with such fundamental quantum description, then the basic ingredients are already present, and we must conclude that the procedure of analysis, rather than the substance, is to be blamed for the troubles. In plain words, this means that a new (working) principle is missing, that perhaps we have been looking at the SED system with too classical eyes, so to speak. And we know from relativity theory that, despite our intuitions, the world is not classical in the last analysis, not even on the macroscopic level. A minimum of circumspection —and an appeal to Ockham's razor— demands from us to seriously consider the possibility that a different treatment of the problem, as free of the difficulties discussed above as possible, may lead to a better agreement with quantum mechanics. In the following chapters this possibility is explored along two different but complementary routes, in an attempt to assess its viability.

As Claverie and Diner (1977a) put it, "The relationship between quantum theory and SED, if it exists, is of a more subtle nature than (a) mere formal equivalence".

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[11] In what refers to Pierre Claverie —whose contribution was instrumental to the development of the theory— we recall that his untimely death deprived SED of his deep insight just shortly after the main difficulties had been discovered.

Part III

Coda

CHAPTER 10

LINEAR STOCHASTIC ELECTRODYNAMICS

In the foregoing chapter it was observed that the use of a conventional perturbative approach in SED leads to unperturbed solutions obeying classical equations of motion, whereas they should be stochastic and somehow contain $\hbar$, if they are expected to describe the quantum world; in other words, that not even the zero-order motions of the SED system in the stationary regime should be independent of the zeropoint field and given by purely deterministic laws. It was thus concluded that standard SED is incompatible with the quantum description of nature and should consequently be revised or modified. In a few papers published along the years proposing a direct relationship between these two theories, additional assumptions were indeed introduced on top of those of standard SED;[1] unfortunately they all have a *tour-de-force* flavor, and their rather formal approach to the problem prevents them from helping to establish a clear-cut physical connection with quantum mechanics.

In this chapter we address the problem anew, with a specific proposal for an approach that implies a minimum of changes, although the consequences of these turn out to be far from minor. We have listed at the end of chapter 9 a series of elements that may be missing or poorly represented in standard SED; however how to incorporate them into a fuller theory is not at all clear. So our purpose here is to introduce at least the changes necessary to dissolve the contradiction just recalled, and to analyze and develop the ensuing theoretical body.

The new theory, constructed on the basis of a demand of self-consistency, is clearly different from the standard formulation of SED; for this reason, and to avoid confusion, it will be distinguished with the name *linear stochastic electrodynamics*. In linear SED one makes a simplified description in terms of certain subensembles, rather than a detailed description of individual events, so that the statements of the theory have in general a statistical meaning only.

Before entering into the subject of linear SED in detail, and as a prelude to it, let us briefly reconsider the physics behind the failure of standard SED.

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[1] Some of these works are Surdin (1971a, b), de la Peña and Cetto (1977c, 1978, 1985a, 1986).

10.1. In search of firmer grounds

A central lesson learned from the preceding analysis is that any attempt to treat the two parts of the SED system, the material one and the field, as separate entities, even as a first approximation, is prone to run into difficulties. The vacuum field should not be conceived as a given thermostat or reservoir, as was done in chapter 9, but as something that can be modified by the nonnegligible effects of the reaction of the mechanical system upon it. In short, particle and field codetermine each other,[2] so that just as the equilibrium distribution of matter depends on the properties of the equilibrium field driving and supporting it, so also this equilibrium field depends on the mechanical system upon which it is acting. Therefore, and just as happens with the mechanical part, when solving a dynamical problem the field should not be arbitrarily defined a priori (as the free field), but should become determined by the solution of the complete set of equations of motion referring to the system as a whole.[3]

Now, to get the explicit solution of the complete general SED problem is a hopeless task. Thus, as a practical means to advance until better methods are at hand, we propose to look for solutions of the equations of motion that comply with appropriate requirements, and use them to construct the required field in a self-consistent manner. We choose as the fundamental demand that the solutions describing the equilibrium situation eventually attained by the system, be consistent with detailed energy balance. Such solutions differ radically from the classical ones; indeed, so radically that they correspond to the quantum mechanical description, as will be seen below. But the demand of self-consistency affects also the field, which acquires strong correlations among its relevant Fourier components, that are totally nonexistent in the case of the free field.[4]

As a matter of procedure, in this chapter instead of the demand of detailed balance we impose one that is much simpler to handle at the present level of description; only in the next chapter is the equivalence between both conditions established, although it will become manifest much earlier.

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[2] We borrow the expression from Moore (1983a), where a similar point of view is firmly advocated. The same possibility is expressed in Theimer and Peterson (1977) (and afterwards by Theimer alone on repeated occasions), Pesquera and Claverie (1982), and de la Peña (1983, p. 563), while in Santos (1981) a modification of the zeropoint spectrum by matter is considered as a means of avoiding the Bell inequalities.

[3] Field modifications on a macroscopic level were dealt with in chapter 6, where it was shown that the presence of matter can alter the structure, the distribution of modes and the energy content of the field.

[4] As it happens, not all field frequencies have the same importance in a given problem, but there is a (denumerable) set of them to which the system responds resonantly. The term 'relevant' refers to these frequencies, which are the ones of real interest in a first approximation, as will become evident below.

Without entering into details, we may suggest that the solutions constructed here cannot be exact or complete, since as is known already from the first chapters, SED and quantum mechanics are incompatible theories, the former being local and genuinely statistical, which are properties not shared by the latter. Indeed, from the derivations to follow it will become clear that the present solutions are coarse and approximate, and the nature of the approximations and simplifications will be disclosed.

Even if the approximations lead to numerically acceptable results, they destroy many of the original qualities of the theory. In principle, SED affords a local and causal (realistic) description of the individual particle, even though in terms of random variables; as is well known, this stands in sharp contrast with the quantum description. Further, the detailed description of the mechanical system afforded by standard SED contains more information than is normally supplied by quantum methods. For instance, it was shown in §7.3.4 that the equations of motion allow a direct evaluation of two-time correlations such as $\Gamma_{xp}(t, t')$; however, there is no unique recipe in quantum theory to evaluate them. More generally, in SED one expects that a phase-space distribution, and other (conditional, transition, ...) probabilities, should exist as conventional elements of the theory, which is not the case in the quantum description. Further, quantum theory provides only a statistical account, whereas in SED it is possible, at least in principle, to follow individual trajectories for a given sample of the field. But even this latter notion of 'the samples' of the field is unknown to quantum mechanics. Hence to arrive at a quantum description from SED it is necessary to eliminate or in some way reduce (infinitely) many of the degrees of freedom associated with the random variables, in addition to other required approximations.

An astonishing result of the theory presented here is that the response of the mechanical system to the radiation field becomes linear, irrespectively of the nonlinearities of the binding force; it is essential to stress that this property follows from the theory itself, and is *not* an ad hoc postulate, nor does it imply any linear approximation. The result is so central, that it justifies the name of this specific approach, called hereafter linear SED.

10.2. Essentials of linear SED

Most of this chapter will be devoted to the study of the stationary states attained by the mechanical SED system in equilibrium with the field, under the demand of detailed balance, and to a comparison with their quantum mechanical counterpart. By its nature, the discussion is limited to the case of binding, conservative external forces and to motions that can be described in terms of multiply periodic solutions; however, no starting assumption will be made about the relationships between the relevant frequencies or their number, allowing the theory to determine them. The next chapter is devoted to the study of the radiative corrections and other fine properties predicted by the present theory, which are usually the subject of QED. All the work in this and the next chapter is made in the long-wavelength approximation, leading to a *low-frequency theory*; this remark is important in connection with chapter 12, where a complementary approach is developed in which the high frequencies play a central role.

10.2.1. TRANSIENT AND STATIONARY SOLUTIONS

Since the subject of our attention will be the stationary motions, it is convenient to show first of all that the Braffort-Marshall equation is compatible with the assumption of stationary asymptotic motions in the general case of a binding force. This was shown explicitly to be the case for the harmonically bound particle in §7.1.1; now we proceed to a more general analysis (in one dimension, for simplicity, since the results are generalizable without difficulty to more dimensions). The equation of motion is

$$m\ddot{x} = m\tau \dddot{x} + F(x) + eE(t) \tag{10.1}$$

and our aim is to look for stationary asymptotic solutions of this equation, driven by the random field. We separate x into two parts, $x = x_s + x_t$, use the mean-value theorem to write

$$F\,(x_s + x_t) = F(x_s) + F'(x_s + \theta x_t)x_t, \quad 0 \leq \theta \leq 1, \tag{10.2}$$

and introduce this into equation (10.1)

$$m\ddot{x}_s + m\ddot{x}_t = m\tau \dddot{x}_s + m\tau \dddot{x}_t + F(x_s) + F'(x_s + \theta x_t)x_t + eE(t). \tag{10.3}$$

If x_s is chosen such that

$$m\ddot{x}_s = m\tau \dddot{x}_s + F(x_s) + eE(t), \tag{10.4}$$

then x_t satisfies the equation

$$m\ddot{x}_t = m\tau \dddot{x}_t + F'(x_s + \theta x_t)x_t. \tag{10.5}$$

Equation (10.4) for x_s is the same as (10.1), but specialized for the x_s part, which, as shown below, corresponds to the stationary solutions fixed by the field.

For every (stationary) solution x_s of (10.4) one can in principle solve the (in general nonlinear) equation (10.5) for x_t. The feasibility of such solution is a matter of principle; in practice we cannot proceed along this line, since we ignore the value of the coefficient θ, which may even depend on x_s and x_t. What is important to observe is that for a binding force, equation (10.5) describes a nonlinear, radiating oscillator with a time-varying, stochastic frequency of instantaneous value $\omega_\theta = \sqrt{-F'(x_s + \theta x_t)/m}$. Given the absence of an external source to compensate for the dissipation, x_t goes to zero with time with an (instantaneous) decay rate of order $\tau\omega_\theta^2$ and represents therefore the transient part of x. Of course, any spurious runaway solution generated by the $\dddot{x}$-term should be excluded, for instance by approximating the radiation term from the beginning in equation (10.1) by the substitution $m\tau\,\dddot{x} \rightarrow \tau F'\dot{x}$.

Since the solution of equation (10.5) can be selected to satisfy the initial conditions and it goes to zero with time, the asymptotic stationary solution of x, which coincides with x_s, can be taken independent of them. This corresponds well with the observation that the stationary solutions of (10.4) are determined by the zeropoint field, and are obviously independent of x_t. From now on we will study only the x_s part of the solution, leaving aside the transient part; however, we will continue to call it x, since there is no more risk of confusion. This means loosing track of the initial motions and the fate of each and every particle, and paying attention only to those particles that have reached the stationary state, which greatly simplifies matters.

10.2.2. EQUATION OF MOTION IN THE QUANTUM REGIME: ENTRANCE

Since as discussed above our present aim is to move as closely as possible towards the quantum mechanical description, a first convenient step consists in simplifying the problem by eliminating all superfluous information, as seen from the quantum perspective. A most important simplification is the following.[5] We start from the standard Fourier representation of the zeropoint field in terms of plane waves, equation (4.38), in the long-wavelength approximation,

$$\mathbf{E} = i\sum_{\mathbf{k},\sigma}\sqrt{\frac{\pi\hbar\omega_k}{V}}\hat{\epsilon}_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}e^{-i\omega_n t} + \text{c.c.} \tag{10.6}$$

Now in quantum mechanics (not in QED!) there is no reference to the mode variables describing the direction in $\mathbf{k}$ space; only the magnitude of the vector $\mathbf{k}$ enters, through the frequencies $\omega = ck$ (as transition frequencies, for instance). This means that to recover the statistical description of the stationary states of quantum mechanics it suffices to consider together all modes corresponding to the same frequency. The above development contains therefore too many variables compared with the required information. To eliminate the superfluous variables we rewrite it in the form

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[5] Preliminary and partial discussions of the theory under review are given in Cetto and de la Peña (1991a, b) and de la Peña and Cetto (1991-1995).

$$\mathbf{E} = i\sum_{\sigma}\left[\sum_{\mathbf{k}}\sqrt{\frac{\pi\hbar\omega_k}{V}}\hat{\epsilon}_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}\right]e^{-i\omega_n t} + \text{c.c.}$$

$$= i\sum_{\sigma,k}\sqrt{\frac{\pi\hbar\omega_k}{V}}\left[\sum_{\Omega_{\mathbf{k}}}\hat{\epsilon}_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}\right]e^{-i\omega_n t} + \text{c.c.}, \tag{10.7}$$

where $\sum_{\Omega_{\mathbf{k}}}$ indicates the (double) sum over all vectors $\mathbf{k}$ having the same value of k (or integration over the solid angle, in the continuum limit); so finally,

$$\mathbf{E} = \sum_{\sigma,k}\tilde{E}_k^\sigma \mathbf{a}_k^\sigma e^{-i\omega_n t} + \text{c.c.} = \sum_k \tilde{E}_k \mathbf{a}_k e^{-i\omega_n t} + \text{c.c.}, \tag{10.8}$$

$$\tilde{E}_k\mathbf{a}_k = i\sum_{\sigma}\sqrt{\frac{\pi\hbar\omega_k}{V}}\left[\sum_{\Omega_{\mathbf{k}}}\hat{\epsilon}_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}\right]. \tag{10.9}$$

Another way to arrive at this result is by a time Fourier analysis of the original coupled Maxwell equations. Since the coupling makes the problem intractable in the general case, we must resort to an approximate description, which we do by considering that the radiated field is accounted for by the radiation reaction term in the equation of motion, and assuming that the stationary states can be described in terms of an appropriate 'free' field, to be defined below. Under this asumption the time Fourier transform of the field is given by equation (3.21), or in the present notation,

$$\nabla^2\mathbf{A}_k + k^2\mathbf{A}_k = 0, \qquad k^2 = \omega_k^2/c^2. \tag{10.10}$$

This leads directly to equation (10.8), with $\tilde{E}_k\mathbf{a}_k$ the Fourier amplitude of the (coupled) random field of frequency ω_k. Of course, even if a 'free' field is being considered, its stochastic properties must be left open, to adjust them to the requirements of the specific problem, as mentioned previously.[6] It is clear that the present procedure cannot be used in problems related to van der Waals and Casimir forces and the like (such as those studied in chapter 6), in which the k-dependence of the field plays a fundamental role. But since we are for the moment engaged in recovering quantum mechanics in the radiationless approximation, and such problems pertain to the domain of QED, no conflict is created by excluding them.

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[6] The procedure used here is reminiscent of the interaction picture procedure, where the interacting fields are constructed so as to satisfy the appropriate free field equations.

Equation (10.10) shows that the long-wavelength approximation $k \to 0$ amounts to a neglect of the spatial dependence of the field. An alternative formal way to arrive at the same result is by taking the limit $c \to \infty$, an observation that will be important below. Since the Fourier components $\tilde{E}_k \mathbf{a}_k^0$ result from integrating over the angular variables in **k**-space, they contain an infinite number of modes, each with the same average energy $\frac{1}{2}\hbar\omega_k$. The energy per mode of each component has therefore an extremely sharp distribution, and the amplitudes $a_{kj}^0 = (\mathbf{a}_k^0)_j$ are random variables with a (very nearly) Gaussian distribution, sharply peaked around ω_k; in the limit they have fixed amplitudes. For the free field they may be taken as statistically independent, so we write (cf. equations (4.75) and ((4.79))

$$\left\langle a_{ki}^0 \right\rangle = 0, \quad \left\langle a_{ki}^0 a_{k'i'}^{0*} \right\rangle = \delta_{kk'}\delta_{ii'}, \tag{10.11}$$

$$a_{ki}^0 = e^{i\varphi_k}, \tag{10.12}$$

with the random phases φ_k uniformly distributed over $(0, 2\pi)$. All the stochasticity of the field is now expressed in these random phases. The proposition that the components of the field are statistically independent holds only for the free field and requires careful revision in specific applications. It is of course difficult to ascertain the changes pertinent to a given problem before the whole system of coupled equations has been solved, so we cannot give at this point the general rule that should replace the second of equations (10.11). Therefore for the moment we leave the problem open, to allow the required codetermination of field and particle variables perform its task.[7]

In the above expressions a superindex '0' has been added to the amplitudes of the field, with the purpose of introducing the following useful notation (we omit from now on the Cartesian indices from the random amplitudes wherever they are unnecessary):

$$a_k \equiv a_k(t) = a_k^0 e^{-i\omega_k t}. \tag{10.13}$$

Note that $a_k(t)$ and a_k^0 have the same statistical properties.

Further, owing to the finite dispersion of its dynamical variables, one should consider that the particle does not respond to amplitudes a_k^0 of a perfectly defined frequency $\omega = ck$, but to a narrow band centered around ω. A clear illustration of this is given by the solution of the harmonic oscillator in chapter 7, where the response frequency is seen to be defined up to a precision of the order of $\Delta\omega/\omega \sim \tau\omega_0$. Hence in the description of the particle dynamics an average of the field amplitudes around a central frequency should be considered, within a bandwidth $\Delta\omega << \tau\omega^2$,

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[7] The solution to this problem is discussed in §10.3.3.

$$\bar{a}_k^0(\omega) = \frac{1}{\Delta\omega} \int_{\omega_0-\frac{1}{2}\Delta\omega}^{\omega_0+\frac{1}{2}\Delta\omega} d\omega'\, a_k^0(\omega'). \tag{10.14}$$

This averaging is of little relevance for all practical purposes, but it reinforces the assignment of a fixed value to the random-phase amplitude; also, it reaffirms the meaning of the field amplitudes appearing in the present description as being not the original mode amplitudes, but mathematical constructs derived from them by an averaging process.

We have avoided everywhere the question of the polarization; however, there is an interesting physics underlying it, which the present highly simplified treatment misses. In a more detailed approach, in expressions such as (10.8) one may introduce an appropriate fixed direction and decompose the field in terms of states of circular polarization with respect to it, as was done in section 8.3 for the study of the spin of the electron. One would then end up with two sets of amplitudes $a_{ki}^{0\sigma}$ instead of only one, each with stochastic properties similar to those of the unpolarized amplitudes. The additional degree of freedom $\sigma = \pm 1$ so introduced should be identified with the spin orientation of the electron, as follows from the discussion in section 8.3. An important consequence that ensues from the present consideration concerns the wrong factor 3/4 obtained in section 8.3 for the magnitude of the spin, which should be corrected by the present approach. The reason for this is that the factor comes from the angular integration in k-space (see e.g. equation (8.70)); since such integration is not needed within the present approach, the problem should not arise.[8]

We are now ready to write the modified Braffort-Marshall equation that will be used for the analysis of the quantum regime. The redefinition is the combined result of using averaged modes of a given frequency instead of individual field modes, and —most importantly— of considering that the equilibrium zeropoint field is physically *different* from the free field. In terms of this *effective field* we write (in one-dimensional notation, used in this chapter for simplicity)

$$m\ddot{x} = m\tau \dddot{x} + F(x) + e \sum_k \tilde{E}_k a_k^0 e^{-i\omega_n t} + \text{c.c.}, \tag{10.15}$$

with $\tilde{E}_k$ given by (10.9) and random amplitudes of the form (10.12). The variable x in this equation is different from the variable x in equation (10.1), because it describes the response of the particle to the effective field.

This procedure to deal with the field, which will be called *k-averaging*, is a central feature of the description. The k-averaging may alternatively be seen as a coarse-grain partial average over all components of the field with frequencies close to a relevant frequency of the problem, leading to an effective random amplitude for that frequency. The k-averaging introduces several fundamental limitations into the description of the SED system, which we briefly comment upon now.

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[8] A. Jáuregui (unfinished manuscript of 1991-92) has initiated a study along these lines for the harmonic oscillator in a magnetic field.

a) Since a partial averaging is being performed over all possible realizations in Ω_k space and polarization for each given a_k^0, the description refers not to an individual particle, but to an ensemble (or, alternatively, to the partly averaged behaviour of a given particle). Thus, the ensuing theory will be *essentially statistical*, just as is the case with quantum mechanics.

b) The emerging mechanical variables are to be seen as partly averaged random variables, appropriate at most for a coarse description. In particular, the correlations between variables may be poorly described.

c) The theory is approximate, because it is not based on a systematic solution of the coupled equations and because it refers only to the k-averaged field amplitudes in the long-wavelength approximation.[9]

d) The theory is essentially nonrelativistic and corresponds formally to the limit $c \to \infty$. This might open a door through which nonlocal actions can enter into the description, even in relation with the radiation field.[10]

e) The detailed k-components (field modes) have not only become hidden, but are irretrievably lost for the description to follow (hence also for the quantum description). We cannot any more recover a fully deterministic picture by 'adding the hidden variables' or any similar simple procedure; it would not be a matter of embedding the ensuing description into a larger theory, but of constructing a new one from scratch. As a matter of principle, SED offers the key to that by looking for solutions that are asymptotically consistent with detailed balance; but this observation does not seem to be of much help in present-day practice.

10.2.3. ESSENTIALS OF LINEAR SED THROUGH A SIMPLE EXAMPLE

With the aim of solving equation (10.15) it is convenient to write both the stationary $x(t)$ and the force as Fourier series of the form

$$x = \sum_k \tilde{z}_k(\omega) e^{-i\omega_k t} + \mathrm{cc}, \tag{10.16}$$

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[9] The long-wavelength approximation has reduced the external field to a set of oscillators, which means that the actual nature of the field is of little importance at this level of description. This fact strongly supports the possibility of constructing a more general 'stochastic theory', essentially independent of the specific nature of the stochastic zeropoint, as was discussed in section 2.5; see also §10.5.2.

[10] In connection with this remark, examples such as the one discussed in Hegerfeldt (1994) may acquire a new meaning.

$$F(x) = \sum_k \tilde{\Phi}_k(\omega)e^{-i\omega_k t} + \mathrm{cc}, \tag{10.17}$$

so that from (10.15) one gets for the Fourier coefficients the set of equations

$$m(-\omega_k^2 + i\tau\omega_k^3)\tilde{z}_k = \tilde{\Phi}_k + e\tilde{E}_k a_k^0. \tag{10.18}$$

Both $\tilde{z}(\omega)$ and $\tilde{\Phi}(\omega)$ may depend on arbitrary combinations of the random amplitudes a_k^0 of different relevant frequencies ω_k, and are therefore stochastic; further, $\tilde{\Phi}(\omega)$ depends in general on $\tilde{z}(\omega)$ in a complicated way, with the obvious exception of the linear-force problem. Observe that no specific relation among the frequencies is being considered, in contrast with the treatment of classical multiply periodic systems, where very definite relations are assumed from the start, as discussed in §9.3.1.

Since the calculations that follow are somewhat unconventional, it is better to treat first a simple example that contains already the typical complexities and features of the nonlinear problem, as a prelude to the study of the general case of an arbitrary binding force. A handy example is the anharmonic oscillator consisting of a linear oscillator plus a cubic force term and governed by the equation

$$\ddot{x} + \omega_0^2 x + \kappa x^3 = \tau \dddot{x} + \frac{e}{m}E(t). \tag{10.19}$$

The core of the problem resides in the nonlinear part of the force; its Fourier coefficient is, from equations (10.16) and (10.17),

$$\tilde{\Phi}_k^{(3)} = \lim_{T\to\infty} \frac{\kappa}{2T} \int_{-T}^{T} \left[x^3(t)\right] e^{i\omega_k t} dt = \kappa \sum_{n'n''n'''} \tilde{z}_{n'}\tilde{z}_{n''}\tilde{z}_{n'''}, \tag{10.20}$$

where the sum extends over the set of indices for which the frequencies satisfy the condition

$$\omega_k = \omega_{n'} + \omega_{n''} + \omega_{n'''} \tag{10.21}$$

coming from the delta function generated by the time integration. Note that the limitation reduces the number of independent Fourier indices to two for a given frequency ω_k.

As we have learned from the preceding chapters, if the dissipative effect of the radiation reaction term $m\tau \dddot{x}$ is fully taken into account, and if only the zeropoint radiation field is present, then only one stationary solution of the form (10.16) should exist, and it (hopefully) corresponds to the ground state. However, in the radiationless approximation —which corresponds to the quantum mechanical description, as will be confirmed below— the system may admit more than one stationary solution, and usually an infinite number of them. We therefore introduce an index (α) to distinguish between such different solutions,

$$x_\alpha(t) = \sum_n \tilde{z}_{\alpha n} e^{-i\omega_{n\alpha} t} + \text{c.c.} \tag{10.22}$$

Note the reverse order of the subindices of ω in the exponent; we have adopted this convention to adjust the final results to the normal conventions of quantum mechanics. It will be seen below that the two indices (here denoted by n and α) play actually a symmetric role; therefore Greek letters will be used for both. Further, it will turn out that $\omega_{\beta\alpha} = -\omega_{\alpha\beta}$ (see equation (10.62)), so that one may change the sign of the time exponents by inverting the order of the indices, and write equation (10.22) in the form

$$x_\alpha(t) = \sum_\beta \tilde{z}_{\alpha\beta} e^{i\omega_{\alpha\beta} t} + \text{c.c.} \tag{10.23}$$

The Fourier coefficient of the nonlinear term of frequency $\omega_{\alpha\beta}$ is then

$$\tilde{\Phi}^{(3)}(\omega)\Big|_{\omega_{\alpha\beta}} = \kappa \sum \left(\tilde{z}_{\mu'\nu'} \tilde{z}_{\mu''\nu''} \tilde{z}_{\mu'''\nu'''}\right)_{\omega_{\alpha\beta}}, \tag{10.24}$$

where the summation is performed over the sets of indices such that

$$\omega_{\alpha\beta} = \omega_{\mu'\nu'} + \omega_{\mu''\nu''} + \omega_{\mu'''\nu'''}, \tag{10.25}$$

according to (10.21). Although the number of subindices has increased, it should be kept in mind that in equation (10.24) there is only a double sum, which means that the additional indices must be related to the summation indices; see §10.3.3 below.

With (10.24), equation (10.18) becomes for this problem (in a somewhat more concise notation)

$$\left(-\omega_{\alpha\beta}^2 + i\tau\omega_{\alpha\beta}^3 + \omega_0^2\right) \tilde{z}_{\alpha\beta} + \kappa \sum \left(\tilde{z}' \tilde{z}'' \tilde{z}'''\right)_{\alpha\beta} = \frac{e}{m} \tilde{E}(\omega_{\alpha\beta}) a_{\alpha\beta}^0. \tag{10.26}$$

Now given the structure of the right-hand side term, it is convenient to rewrite the Fourier amplitudes of the position coordinate and the force in the form

$$\tilde{z}_{\alpha\beta} = \tilde{x}_{\alpha\beta} a_{\alpha\beta}^0. \tag{10.27}$$

$$\tilde{\Phi}_{\alpha\beta} = \tilde{F}_{\alpha\beta} a_{\alpha\beta}^0. \tag{10.28}$$

In the particular case of the linear oscillator ($\kappa = 0$), the $\tilde{x}_{\alpha\beta}$ and $\tilde{F}_{\alpha\beta}$ are independent of the $a_{\alpha\beta}^0$, so they are nonrandom numbers; but with $\kappa \neq 0$ the situation is different. In fact, in this case it follows from equation

(10.26) that the coefficient $\tilde{x}_{\alpha\beta}$ depends on the amplitudes $\{a^0_{\alpha\beta}\}$ in a quite complicated way, for on substituting (10.27) there appear products of three a^0's containing all those frequencies which combine to give just $\omega_{\alpha\beta}$, as specified by equation (10.25). Explicitly, and on dividing by $a^0_{\alpha\beta}$, one gets the set of coupled equations

$$\left(-\omega^2_{\alpha\beta} + i\tau\omega^3_{\alpha\beta} + \omega_0^2\right)\tilde{x}_{\alpha\beta} + \kappa\sum\left(\tilde{x}'\tilde{x}''\tilde{x}'''\right)_{\alpha\beta}\frac{\left(a^{0\prime}a^{0\prime\prime}a^{0\prime\prime\prime}\right)_{\alpha\beta}}{a^0_{\alpha\beta}} = \frac{e}{m}\tilde{E}(\omega_{\alpha\beta}). \tag{10.29}$$

These equations determine the *response amplitudes* $\tilde{x}_{\alpha\beta}$ and *characteristic frequencies* $\omega_{\alpha\beta}$ for the problem. The solutions are in general functions of the random amplitudes $a^0_{\alpha\beta}$, and thus are stochastic numbers by themselves; they represent a different stationary solution for every realization of the field.

It is just here where we deviate from the usual calculations. We observe that for certain random fields there are solutions to the set of equations (10.29) that are simultaneously nonrandom numbers and independent of the specific realization of the field. These particularly simple solutions can occur only when the set of equations that determine them contains no random coefficients, i.e., when all explicit dependence on the a^0's vanishes from the nonlinear term, which occurs only when the following equality holds for each relevant frequency,

$$\left(a^{0\prime}a^{0\prime\prime}a^{0\prime\prime\prime}\right)_{\alpha\beta} = a^0_{\alpha\beta}, \tag{10.30}$$

except for a possible (constant) factor of proportionality (which will be seen to be 1, as was here presupposed). This is certainly a very strong condition on the a^0's; it is of course not satisfied by the free zeropoint field, because the random amplitudes corresponding to different field modes are uncorrelated. However, for those fields that conform to these conditions, the characteristic (Fourier) frequencies $\omega_{\alpha\beta}$ and the response amplitudes $\tilde{x}_{\alpha\beta}$ of the stationary states of motion are basically insensitive to the fluctuations of the random variables a^0, and hence practically independent of the specific realization of the field. They are thus remarkably stable solutions. On this basis we propose to consider seriously the solutions determined by asking that the frequencies $\omega_{\alpha\beta}$ become *not stochastic.* Of course, it is not clear at this moment that such solutions have anything to do with detailed energy balance; it will turn out however, that at the present level of approximation both demands are equivalent and that the solutions thus generated indeed describe the quantum world.

Let us then investigate what happens when the demand of nonrandom characteristic frequencies is added to the above equations. Since the right-hand side term of (10.29) is nonrandom, it is clear that the solutions $\tilde{x}_{\alpha\beta}$ of the inhomogeneous equation cannot be stochastic, i.e., they also must be field-realization-independent. Of course, a nonrandom solution exists *only* if (10.30) is satisfied, which fixes the class of fields for which such solutions do exist. In the next section the properties which the field amplitudes must possess for equations (10.30) and their generalizations to hold will be analyzed, and it will be shown that there is a general solution to this problem, compatible with detailed balance. Now with (10.30) introduced in (10.29), the a^0's vanish from the equation and one is left with the system of algebraic equations

$$\left(-\omega_{\alpha\beta}^2 + i\tau\omega_{\alpha\beta}^3 + \omega_0^2\right)\tilde{x}_{\alpha\beta} + \kappa \sum \left(\tilde{x}'\tilde{x}''\tilde{x}'''\right)_{\alpha\beta} = \frac{e}{m}\tilde{E}(\omega_{\alpha\beta}), \qquad (10.31)$$

whose solutions $(\tilde{x}_{\alpha\beta},\ \omega_{\alpha\beta})$ are deterministic. But yet another important consequence follows, which we now consider.

Introducing (10.27) into (10.23) one can write with the help of equations (10.25), (10.28) and (10.30), and in a synthetic notation,

$$x^3 = \sum \tilde{x}'\tilde{x}''\tilde{x}''' e^{i(\omega'+\omega''+\omega''')t} =$$

$$= \sum \left(\sum \tilde{x}'\tilde{x}''\tilde{x}'''\right)_{\alpha\beta} a_{\alpha\beta}^0 e^{i\omega_{\alpha\beta}t} = \sum \left(\tilde{x}^3\right)_{\alpha\beta} a_{\alpha\beta}^0 e^{i\omega_{\alpha\beta}t}, \qquad (10.32)$$

where the element $\left(\tilde{x}^3\right)_{\alpha\beta}$ is calculated as a double sum over indices such that (10.25) and (10.30) are satisfied; in our shorthand notation,

$$\left(\tilde{x}^3\right)_{\alpha\beta} = \left(\sum \tilde{x}'\tilde{x}''\tilde{x}'''\right)_{\alpha\beta}. \qquad (10.33)$$

The detailed meaning of this and other similar relations involving constrained sums will be discussed below. What is important to remark here is that a force which is nonlinear in x has become a linear function of the field amplitudes $a_{\alpha\beta}^0$, as shown in equation (10.32). Thus, despite the presence of nonlinearities, the system responds linearly to the field and behaves as a set of linear oscillators of frequency $\omega_{\alpha\beta}$ and amplitude $\tilde{x}_{\alpha\beta}$. We stress that no linear approximation is being made, but it is the system's response to the field that is linear, with sharply defined frequencies and response coefficients determined by the nonlinear equations (10.31).

10.3. Stationary solutions for a general binding force

10.3.1. EQUATION OF MOTION IN THE QUANTUM REGIME: EXIT

In the next section we study the general problem of a nonlinear binding force and deal with many of the details that were omitted in the preceding section; here we advance some suggestive results, still using the example of the anharmonic oscillator. In terms of the quantities defined through equations (10.23), (10.27) and (10.28), equation (10.18) can be recast in the algebraic form

$$-m\omega_{\alpha\beta}^2\tilde{x}_{\alpha\beta} = \tilde{F}_{\alpha\beta} - im\tau\omega_{\alpha\beta}^3\tilde{x}_{\alpha\beta} + e\tilde{E}(\omega_{\alpha\beta}) \tag{10.34}$$

with $\tilde{F}_{\alpha\beta} = \left(\tilde{F}(\tilde{x})\right)_{\alpha\beta}$. For the specific case of the anharmonic oscillator the force coefficients are given by

$$\tilde{F}_{\alpha\beta} = -\omega_0^2\tilde{x}_{\alpha\beta} - \kappa\sum\left(\tilde{x}'\tilde{x}''\tilde{x}'''\right)_{\alpha\beta}, \tag{10.35}$$

where the (double) sum in the triple product of $\tilde{x}$ must be such that equation (10.25) is satisfied.

Observe from (10.23) and (10.27) that one can use alternatively time-dependent coefficients defined by

$$\tilde{x}_{\alpha\beta}(t) = \tilde{x}_{\alpha\beta}e^{i\omega_{\alpha\beta}t}, \tag{10.36}$$

so that

$$\frac{d^2}{dt^2}\tilde{x}_{\alpha\beta}(t) = -\omega_{\alpha\beta}^2\tilde{x}_{\alpha\beta}(t) \tag{10.37}$$

and equation (10.34) multiplied by $\exp(i\omega_{\alpha\beta}t)$ becomes

$$m\frac{d^2\tilde{x}_{\alpha\beta}(t)}{dt^2} = \tilde{F}_{\alpha\beta}(t) + m\tau\frac{d^3\tilde{x}_{\alpha\beta}(t)}{dt^3} + e\tilde{E}_{\alpha\beta}(t), \tag{10.38}$$

with $\tilde{F}_{\alpha\beta}(t) = \tilde{F}_{\alpha\beta}\exp(i\omega_{\alpha\beta}t)$ and $\tilde{E}_{\alpha\beta}(t) = \tilde{E}_{\alpha\beta}\exp(i\omega_{\alpha\beta}t)$. This is a set of (nonlinear) deterministic equations of motion for the $\tilde{x}_{\alpha\beta}(t)$; all random quantities have disappeared completely from the description. Now one can take the radiationless approximation by writing the above equation to zero order in e (or taking the limit $e \to 0$); it reduces to

$$m\frac{d^2\tilde{x}_{\alpha\beta}(t)}{dt^2} = \tilde{F}_{\alpha\beta}(t). \tag{10.39}$$

Despite its form this (unperturbed) equation is *not* a classical equation of motion, owing to the specific meaning and algebraic properties of the terms $\tilde{x}_{\alpha\beta}$ and $\tilde{F}_{\alpha\beta}$. In fact, it is a Heisenberg equation of motion (as will be shown), which means that its solutions correspond to those of the quantum mechanical description of stationary states. Recall that these are the stationary (time-asymptotic), approximate (coarse grained, k-averaged, long-wavelength approximation) solutions for a vacuum field that is assumed to satisfy equations as (10.30) (and, more generally, (10.54)). Recall also the astonishing fact that equation (10.39), which can be and is used as the fundamental law of the problem, does not contain any element whatsoever reminding us of its stochastic origin. But hidden as they may be, the random amplitudes are doing their job by maintaining the system in the quantum regime.

10.3.2. GENERAL BOUND PROBLEM

We now turn our attention to the solution of the general equation

$$m\ddot{x} = m\tau \dddot{x} + F(x) + e\sum_k \tilde{E}_k a_k^0 e^{-i\omega_k t} \tag{10.40}$$

in the quantum regime. For simplicity, the 'c.c.' terms of the zeropoint field have been embodied in the sum, which now runs over positive and negative frequencies. Following equations (10.23) and (10.27) we write

$$x_\alpha(t) = \sum_\beta \tilde{z}_{\alpha\beta} e^{i\omega_{\alpha\beta} t} = \sum_\beta \tilde{x}_{\alpha\beta} a_{\alpha\beta}^0 e^{i\omega_{\alpha\beta} t} \tag{10.41}$$

and introduce this expansion into equation (10.40), to obtain

$$x_\alpha = -\frac{e}{m}\sum_\beta \frac{\tilde{E}_{\alpha\beta}}{\Delta(\omega_{\alpha\beta})} a_{\alpha\beta}^0 e^{i\omega_{\alpha\beta} t}, \tag{10.42}$$

where

$$\Delta(\omega) = \omega^2 + \frac{1}{m}\frac{\tilde{\Phi}(\omega)}{\tilde{z}(\omega)} - i\tau\omega^3 \tag{10.43}$$

still depends on the random amplitudes. As before, $\tilde{\Phi}(\omega)$ represents the Fourier transform of the external force. In free space the sum in equation (10.42) should be replaced by an integral over the frequency; however it will be assumed that all relevant singularities in the integrand are simple isolated poles, and that they give the dominant contributions to $x_\alpha(t)$.[11] For bounded motions, the equation for the poles $\Delta(\omega) = 0$, or

$$\frac{1}{m}\frac{\tilde{\Phi}(\omega)}{\tilde{z}(\omega)} = -\omega^2 + i\tau\omega^3, \tag{10.44}$$

is satisfied only for certain (discrete, and in general stochastic) frequencies. For instance, for the harmonic oscillator of natural frequency ω_0 there are poles at $+\omega_0$ and $-\omega_0$, but in the case of particles bound by nonlinear forces, the values of ω at the poles will in general depend on the state of motion. Further, anticipating that the frequencies ω of interest are such that $|\tau\omega| \ll 1$, we take the radiationless approximation in which equation (10.44) becomes

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[11] We are assuming that the system behaves resonantly at the frequencies corresponding to the poles. The smallness of the parameter τ entering in the radiation reaction force guarantees that the resonance is extremely sharp, so that the contributions from the poles are clearly dominant.

$$-\frac{1}{m}\frac{\tilde{\Phi}_{\alpha\beta}(\omega)}{\tilde{z}_{\alpha\beta}(\omega)} = \omega_{\alpha\beta}^2. \tag{10.45}$$

Now we introduce the requirement of the nonrandom values for the characteristic frequencies $\omega_{\alpha\beta}$, as before. The ratio $\tilde{\Phi}_{\alpha\beta}/\tilde{z}_{\alpha\beta}$ must then be independent of the field amplitudes $a_{\alpha\beta}^0$, and equations (10.41), (10.42) and (10.45) lead to $\tilde{z}_{\alpha\beta}$ and $\tilde{\Phi}_{\alpha\beta}$ linear in $a_{\alpha\beta}^0$, so that one can write

$$\tilde{z}_{\alpha\beta} = \tilde{x}_{\alpha\beta} a_{\alpha\beta}^0, \quad \tilde{\Phi}_{\alpha\beta} = \tilde{F}_{\alpha\beta} a_{\alpha\beta}^0 \tag{10.46}$$

with $\tilde{x}_{\alpha\beta}$ and $\tilde{F}_{\alpha\beta}$ nonrandom coefficients, related through the system of algebraic equations contained in (10.45), namely,

$$\tilde{F}_{\alpha\beta} = -m\omega_{\alpha\beta}^2 \tilde{x}_{\alpha\beta}. \tag{10.47}$$

From this expression and the second one in (10.46) it follows that the Fourier transform of the external force is a linear function of the stochastic amplitudes; the linear response to the field is thus extended to any binding external force. It is remarkable that this general property follows as a consequence of the demand of nonrandom values for the characteristic frequencies $\omega_{\alpha\beta}$ of the stationary solutions.

10.3.3. MATHEMATICAL STRUCTURE OF THE SOLUTION

Let us now investigate the properties that the field amplitudes $a_{\alpha\beta}^0$ should have for equations (10.46) to hold in general. For this purpose we assume that the external force can be expressed as a power series in x; this leads to a sum of terms containing any number of factors of $\tilde{z}_{\mu\nu}$, of the type

$$\tilde{z}_{\lambda_1\mu_1}\tilde{z}_{\lambda_2\mu_2}\cdots\tilde{z}_{\lambda_n\mu_n} = \tilde{x}_{\lambda_1\mu_1}\tilde{x}_{\lambda_2\mu_2}\cdots\tilde{x}_{\lambda_n\mu_n}\, a_{\lambda_1\mu_1}^0 a_{\lambda_2\mu_2}^0\cdots a_{\lambda_n\mu_n}^0, \tag{10.48}$$

each of which should correspond to a fixed frequency, say $\omega_{\alpha\beta}$, so that

$$(\tilde{z}^n)_{\alpha\beta} = (\tilde{x}^n)_{\alpha\beta}\, a_{\alpha\beta}^0. \tag{10.49}$$

Here $(\tilde{z}^n)_{\alpha\beta}$ or $(\tilde{x}^n)_{\alpha\beta}$ represents any term in the Fourier development of the force for the frequency $\omega_{\alpha\beta}$ that contains n corresponding factors. It follows that each product of amplitudes a^0 should reduce to a single a^0, so that the product terms remain linear in a^0,

$$a_{\lambda_1\mu_1}^0 a_{\lambda_2\mu_2}^0\cdots a_{\lambda_n\mu_n}^0 = a_{\alpha\beta}^0. \tag{10.50}$$

Now of the $2n$ indices appearing on the left hand side, two are fixed (α and β) and $n-1$ are summation indices (due to the implicit δ-functions, as in equation (10.20)), so that $n-1$ indices remain; but since this product of a^0's should be just the required $a^0_{\alpha\beta}$ and not another independent random amplitude, the indices must repeat themselves (otherwise independent random phases would appear). In particular, for $n=2$ one can write either

$$a^0_{\alpha\beta}a^0_{\mu\mu} = a^0_{\alpha\beta} \tag{10.51}$$

or

$$a^0_{\alpha\mu}a^0_{\mu\beta} = a^0_{\alpha\beta} \tag{10.52}$$

for arbitrary α, β and μ. From the first of these equations we get

$$a^0_{\mu\mu} = 1, \tag{10.53}$$

whereas the iteration of (10.52) to arbitrary n gives:

$$a^0_{\alpha\mu_1}a^0_{\mu_1\mu_2}\cdots a^0_{\mu_{n-1}\beta} = a^0_{\alpha\beta}. \tag{10.54}$$

Now recall from equation (10.12) that the (k-averaged) a^0's have constant magnitude equal to 1 and can be written as

$$a^0_{\alpha\beta} = e^{i\varphi_{\alpha\beta}}, \tag{10.55}$$

so equation (10.54) means that the phases are not independent, but should satisfy

$$\varphi_{\alpha\beta} = \varphi_{\alpha\mu_1} + \varphi_{\mu_1\mu_2} + \cdots + \varphi_{\mu_{n-1}\beta}, \quad \text{modulo } 2\pi \tag{10.56}$$

and, as follows from equations (10.52) and (10.53),

$$\varphi_{\mu\mu} = 0, \quad \varphi_{\alpha\beta} = -\varphi_{\beta\alpha}. \tag{10.57}$$

In terms of the amplitudes, the last constraint implies the relations

$$a^0_{\alpha\beta} = \left(a^0_{\beta\alpha}\right)^* = \left(a^0_{\beta\alpha}\right)^{-1}. \tag{10.58}$$

As follows from equation (10.25), the frequencies of the time coefficients corresponding to the a^0's will in their turn satisfy the equality

$$\omega_{\alpha\beta} = \omega_{\alpha\mu_1} + \omega_{\mu_1\mu_2} + \ldots + \omega_{\mu_{n-1}\beta} \tag{10.59}$$

and, in particular,

$$\omega_{\alpha\alpha} = \omega_{\alpha\beta} + \omega_{\beta\alpha}. \tag{10.60}$$

Hence if $(\alpha\alpha)$ is taken to denote the field mode of zero frequency (a mode which is actually absent from the pure radiation field),

$$\omega_{\alpha\alpha} = 0, \quad \text{any } \alpha, \tag{10.61}$$

the above equation implies the important symmetry property:

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha}. \tag{10.62}$$

which was used in advance in connection with equation (10.23).

The general solution to equation (10.56), taking into account (10.57), is

$$\varphi_{\alpha\beta} = \varphi_\alpha - \varphi_\beta \mod 2\pi, \tag{10.63}$$

with φ_α and φ_β independent random phases uniformly distributed over the interval $(0, 2\pi)$.[12] Thus the amplitudes have the general form

$$a^0_{\alpha\beta} = e^{i(\varphi_\alpha - \varphi_\beta)}, \tag{10.64}$$

which shows that the statistically independent random quantities are not the amplitudes $a^0_{\alpha\beta}$ with the combined index $(\alpha\beta)$, as might be originally assumed, but the single-index phases φ_α. The characteristic frequencies share this important property of separability, i.e.,

$$\omega_{\alpha\beta} = \Omega_\alpha - \Omega_\beta, \tag{10.65}$$

where Ω_α are (nonrandom) numbers, as can be verified using equations (10.59) and (10.60). One thus gets for the Fourier coefficient of a typical term $(\tilde{x}^n)$ corresponding to frequency $\omega_{\alpha\beta}$

$$(\tilde{x}^n)_{\alpha\beta}\, a^0_{\alpha\beta} = \sum_{\mu_i} \tilde{z}_{\alpha\mu_1} \tilde{z}_{\mu_1\mu_2} \dots \tilde{z}_{\mu_{n-1}\beta} = \left(\sum_{\mu_i} \tilde{x}_{\alpha\mu_1} \tilde{x}_{\mu_1\mu_2} \dots \tilde{x}_{\mu_{n-1}\beta} \right) a^0_{\alpha\beta}. \tag{10.66}$$

Since μ_1, $\mu_2, \dots$ can take any value, a sum over all allowed values of the (repeated) intermediate indices should be performed, as is explicitly indicated in equation (10.66). One recognizes here the multiplication rule for matrices:

$$(\tilde{x}^n)_{\alpha\beta} = \sum_{\mu_i} \tilde{x}_{\alpha\mu_1} \tilde{x}_{\mu_1\mu_2} \dots \tilde{x}_{\mu_{n-1}\beta}, \tag{10.67}$$

so that the solution arrived at is naturally expressed in terms of matrices. For example, $\tilde{x}_{\lambda\mu}$ is the $\lambda\mu$-element of a matrix $\hat{x}$, and so on.

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[12] Note that if φ_α and φ_β are independent random phases uniformly distributed over $(0, 2\pi)$ then $\varphi_{\alpha\beta} = \varphi_\alpha - \varphi_\beta \mod 2\pi$ is also uniformly distributed over the same interval.

10.3.4. THE HEISENBERG EQUATIONS OF MOTION

As before, one may associate the time factor $\exp(i\omega_{\alpha\beta}t)$ of $\tilde{z}_{\alpha\beta}$ either with $\tilde{x}_{\alpha\beta}$, as was done in (10.36), or with $a_{\alpha\beta}$ (see (10.13)),

$$\tilde{x}_{\alpha\beta}a^0_{\alpha\beta}e^{i\omega_{\alpha\beta}t} = \tilde{x}_{\alpha\beta}(t)a^0_{\alpha\beta} = \tilde{x}_{\alpha\beta}a_{\alpha\beta}(t),$$

$$\tilde{x}_{\alpha\beta}(t) = \tilde{x}_{\alpha\beta}e^{i\omega_{\alpha\beta}t}, \quad a_{\alpha\beta}(t) = a^0_{\alpha\beta}e^{i\omega_{\alpha\beta}t} \tag{10.68}$$

and use whatever combination is more convenient in each case. In particular, from equations (10.37), (10.47) and (10.67) one has in the radiationless approximation

$$m\frac{d^2\hat{x}(t)}{dt^2} = \hat{F}(\hat{x}), \tag{10.69}$$

where $\hat{x}(t)$ and $\hat{F}(\hat{x})$ are now time-dependent matrices with elements $\tilde{x}_{\alpha\beta}(t)$ and $\tilde{F}_{\alpha\beta}(\hat{x})$, respectively. To complete the description we define a matrix $\hat{p}(t)$ with elements that follow from the time derivative of equation (10.36),

$$\tilde{p}_{\alpha\beta}(t) = im\omega_{\alpha\beta}\tilde{x}_{\alpha\beta}(t), \tag{10.70}$$

or in matrix notation,

$$\hat{p} = m\frac{d\hat{x}}{dt}. \tag{10.71}$$

This definition is suitable for making contact with quantum mechanics, which is a zero-order theory as regards the radiative terms and hence unable to distinguish between the mechanical moment and the canonical moment with respect to the zeropoint field (recall equations (10.38) and (10.39)). Combining with equation (10.69) it follows then that

$$\frac{d\hat{p}}{dt} = \hat{F}(\hat{x}). \tag{10.72}$$

Equations (10.71) and (10.72) are evidently the Heisenberg equations of motion, and $\tilde{x}_{\alpha\beta}(t)$ are the elementary oscillators of matrix mechanics [see, e.g., Born et al. 1926]. The matrix algebra of quantum mechanics follows therefore as the algebra that guarantees nonrandom values for the characteristic frequencies of the stationary, k-averaged SED system in the radiationless and long-wavelength approximation; it is for this reason that the corresponding SED solutions are said to describe the quantum regime [de la Peña and Cetto 1986].

The above results can be summarized by stating that in the quantum regime, the solutions of equation (10.40) can be expressed as

$$x_\alpha(t) = \sum_\beta \tilde{x}_{\alpha\beta}a^0_{\alpha\beta}, \quad p_\alpha(t) = \sum_\beta \tilde{p}_{\alpha\beta}a^0_{\alpha\beta}, \quad F_\alpha(x) = \sum_\beta \tilde{F}_{\alpha\beta}a^0_{\alpha\beta}, \tag{10.73}$$

where $\tilde{x}_{\alpha\beta}(t)$, $\tilde{p}_{\alpha\beta}(t)$, etc. are elements of the matrices involved in the Heisenberg equations of motion and $a^0_{\alpha\beta}$ are random quantities of the form (10.64) describing a correlated zeropoint field, specific to the stationary problem at hand. Equation (10.58) means that

$$\tilde{x}^*_{\alpha\beta} = \tilde{x}_{\beta\alpha} \tag{10.74}$$

and so on, so that the matrices corresponding to real dynamical variables are Hermitian.

10.3.5. THE SCALE OF QUANTUM PHENOMENA

The above equations of motion (10.71), (10.72) do not yet fully determine $\tilde{x}_{\alpha\beta}$ and $\omega_{\alpha\beta}$. In writing the stationary solutions (10.45) in the form of (10.41), only the positions of the poles were taken into account, without ever really solving the equation (10.40) which contains the full information, and, in particular, fixes $x(t)$ in terms of Planck's constant. This means that we have to come back to the equations that describe the complete SED system before the radiationless approximation is made and the zeropoint field is dropped altogether, if we want to fix the scale of the solutions. Of course what is still lacking in the above formalism to complete the full system of Heisenberg equations of motion is the value of some fundamental commutator, such as, e.g., $[\hat{x}, \hat{p}]$. But instead of dealing with the complicated original equation of motion to search for such relation, in the next section we introduce a useful alternative procedure that greatly simplifies the problem.

10.4. The Poissonian approach

Let us consider a typical SED problem for a bound particle; the total Hamiltonian H_T of the system is of the form given by equation (3.85), with the appropriate potential function and electromagnetic field inserted. For a Hamiltonian treatment one can take $\{x_i, p_i; q_\alpha, p_\alpha\}$ as a complete set of canonical coordinates; the variables x_i, p_i describe the motion of the particle, whereas the infinite set $\{q_\alpha, p_\alpha\}$ refers to the elementary field oscillators, with the simplified notation $\alpha = (\mathbf{k}, \sigma)$. The Hamiltonian is then

$$H_T = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + V(\mathbf{x}) + \frac{1}{2}\sum_\alpha (p_\alpha^2 + \omega_\alpha^2 q_\alpha^2). \tag{10.75}$$

The Poisson bracket of any two dynamical variables f, g is

$$[f, g] = \sum_i \left[\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial p_i}\right] + \sum_\alpha \left[\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_a} - \frac{\partial g}{\partial q_a}\frac{\partial f}{\partial p_a}\right]$$

$$\equiv [f,g]_{x,p} + [f,g]_{q_\alpha,p_\alpha}. \tag{10.76}$$

The second Poisson bracket can be calculated with respect to the variables a_α, a^*_α related to q_α, p_α by equations (4.40) and its complex conjugate; this transformation is canonical except for a scale factor, so that one gets

$$\sum_\alpha \left[\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial g}{\partial q_\alpha}\frac{\partial f}{\partial p_\alpha}\right] = -\frac{i}{\hbar}\sum_\alpha \left[\frac{\partial f}{\partial a_\alpha}\frac{\partial g}{\partial a^*_\alpha} - \frac{\partial g}{\partial a_\alpha}\frac{\partial f}{\partial a^*_\alpha}\right]. \tag{10.77}$$

Since the last expression will be used extensively, we introduce the shorthand notation

$$\langle f; g\rangle = \sum_\alpha \left[\frac{\partial f}{\partial a_\alpha}\frac{\partial g}{\partial a^*_\alpha} - \frac{\partial g}{\partial a_\alpha}\frac{\partial f}{\partial a^*_\alpha}\right]; \tag{10.78}$$

this is the *Poissonian* of f and g.[13] A simple and illustrative example of a Poissonian is that of two electric components of the zeropoint field; from (4.38) one gets after carrying out the sum over the polarizations

$$\langle E_i(t); E_j(t')\rangle = \frac{2\pi\hbar}{V}\sum_n \omega_n(\delta_{ij} - \hat{k}_i\hat{k}_j)\left(e^{-i\omega_n(t-t')} - e^{i\omega_n(t-t')}\right) \tag{10.79}$$

and in the continuum limit one obtains on integration over the solid angle

$$\langle E_i(t); E_j(t')\rangle = -2i\delta_{ij}\int_0^\infty d\omega S(\omega)\sin\omega(t-t'). \tag{10.80}$$

A comparison with equation (4.65) shows that the Poissonian and (1/2 of) the selfcorrelation function of the field amplitudes are related to each other by a Hilbert transformation,

$$\int_0^\infty d\omega S(\omega)e^{-i\omega s} = \langle E_i(t+s)E_i(t)\rangle + \tfrac{1}{2}\langle E_i(t+s); E_i(t)\rangle \tag{10.81}$$

for any Cartesian component i of the field; the selfcorrelation is the symmetric part, while the Poissonian is the antisymmetric part of the Fourier transform of the power spectrum (up to a numerical factor). Note further that $\langle E_i(t); E_j(t')\rangle$ has a nonrandom value, although no averaging has been performed in its calculation. In other words, the Poissonian of a linear field is the same, whether or not the field is random.

Coming back to the main argument, since we are interested in the stationary states driven by the zeropoint field, we concentrate on the dependence of f and g on the field coordinates, in the quantum regime. For this purpose we select an initial time t_0 to calculate the Poisson brackets with respect to the values $\{x_i^0, p_i^0; a_\alpha^0, a_\alpha^0\}$, so that

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[13] De la Peña and Cetto (1986). A similar concept, although in a different context and with the name of 'commutator', was studied in Santos (1983b).

$$[f,g] = [f,g]_{x_0,p_0} - \frac{i}{\hbar}\langle f;g\rangle_0 , \tag{10.82}$$

and take into account that for the stationary radiation field with $a_\alpha = a_\alpha^0 \exp(i\omega_\alpha t)$, the derivatives in the Poissonian can be calculated with respect to a_α^0 or a_α.[14] Equation (10.82) becomes thus

$$[f,g] \equiv [f,g]_{x_0,p_0} - \frac{i}{\hbar}\langle f;g\rangle . \tag{10.83}$$

Now we consider the asymptotic time limit, assuming that the system is in a stationary state. Under this circumstance the dependence of the mechanical variables on the initial values of x and p can be expected to be essentially lost according to the discussion in §10.2.1, which means that equation (10.83) simplifies into

$$[f,g] = -\frac{i}{\hbar}\langle f;g\rangle . \tag{10.84}$$

Let us work out several examples that will be required in what follows. Denoting by H the purely mechanical part of the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}), \tag{10.85}$$

we find in the asymptotic limit

$$\dot{x}_i = [x_i, H_T] \to -\frac{i}{\hbar}\langle x_i; H\rangle - \frac{e}{mc}A_i, \tag{10.86}$$

$$\dot{p}_i = [p_i, H_T] \to -\frac{i}{\hbar}\langle p_i; H\rangle + \frac{e}{c}\dot{x}_j\frac{\partial A_j}{\partial x_i}, \tag{10.87}$$

and the corresponding Poissonian equations are

$$\dot{x}_i = -\frac{i}{\hbar}\langle x_i; H\rangle - \frac{e}{mc}A_i, \tag{10.88}$$

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[14] To use the Poissonians in conjunction with the former formalism, we need to consider that they are calculated in terms of the partly averaged field amplitudes introduced in §10.2.2 instead of the individual mode amplitudes. Assuming that both sets of variables describe the same field, the scale factor remains the same. Further, of the two Greek subindices appearing in $\{a_{\alpha\beta}, a^*_{\alpha\beta}\}$, one (say, α) serves to specify the stationary state being studied, whereas the other one (β) will be the summation index used in the calculation of the Poissonian; see, e.g., equation (10.93). For simplicity, these indices will not be written down except where necessary.

$$\dot{p}_i = -\frac{i}{\hbar}\langle p_i; H\rangle + \frac{e}{c}\dot{x}_j\frac{\partial A_j}{\partial x_i}. \tag{10.89}$$

Further, for $f = x_i$, $g = p_j$ one has $[x_i, p_j] = \delta_{ij}$, so that equation (10.84) gives

$$\langle x_i; p_j\rangle = i\hbar\delta_{ij}, \tag{10.90}$$

and similarly,

$$\langle x_i; x_j\rangle = 0, \quad \langle p_i; p_j\rangle = 0. \tag{10.91}$$

Note that there is a qualitative difference between (10.90), which expresses a *physical law*, and equation $[x_i, p_j] = \delta_{ij}$, which is a mathematical identity. Indeed, $\langle x_i; p_j\rangle = i\hbar\delta_{ij}$ is an approximate, asymptotic relation, which expresses in a very succinct form that the (bound) stationary states in the quantum regime are sustained by the zeropoint field and that the description of such states is irreducible to classical terms ($\hbar$ is playing here a central role). It is perhaps in this symplectic relation where the core of the difference between the classical and the quantum worlds is most clearly shown.

In the radiationless limit we get for the dynamical equations in the quantum regime, from (10.88) and (10.89),

$$\dot{x}_i = -\frac{i}{\hbar}\langle x_i; H\rangle, \quad \dot{p}_i = -\frac{i}{\hbar}\langle p_i; H\rangle. \tag{10.92}$$

The set of equations (10.90)-(10.92) will be referred to as the *Poissonian equations.* In the next section (10.92) will be shown to correspond to the Heisenberg equations of motion discussed above, under the appropriate conditions and approximations.

Equation (10.90) is just the inhomogeneous equation we were looking for to fix the scale of the stationary solutions. From (10.70) and (10.73) one obtains for the Poissonian of x and p in the stationary state α

$$\langle x; p\rangle_\alpha = -2im\sum_\beta \omega_{\alpha\beta}\left|\tilde{x}_{\alpha\beta}\right|^2, \tag{10.93}$$

which inserted in (10.90) gives

$$-\sum_\beta \omega_{\alpha\beta}\left|\tilde{x}_{\alpha\beta}\right|^2 = \frac{\hbar}{2m}. \tag{10.94}$$

This result can be identified as the Thomas-Reiche-Kuhn sum rule of quantum mechanics, which is just the quantization rule $[\hat{x}, \hat{p}] = i\hbar$ expressed in terms of matrix elements. Equation (10.90) is therefore the Poissonian form of the quantum condition, and $\langle x; p\rangle$ stands for the fundamental Poissonian.

Notice that it is through this result that $\hbar$ enters into the description, so that we can say that the strength of the vacuum field fluctuations is what determines the scale of quantum phenomena.

10.4.1. POISSONIAN AND HEISENBERG EQUATIONS OF MOTION

As we have already derived the Poissonian dynamical equations, it makes sense to establish their relationship with linear SED and, of course, to the usual quantum formalism. For this purpose it is first necessary to generalize the Poissonian introduced in equation (10.78) to include a second free index, required for the derivation of nondiagonal matrix elements. The appropriate generalization can be constructed by taking into account the multiplication properties of the $a_{\alpha\beta}$ established in section 10.3, and it is

$$\langle f; g\rangle_{\alpha\beta} = \sum_{\lambda}\left(\frac{\partial f}{\partial a_{\alpha\lambda}}\frac{\partial g}{\partial a_{\lambda\beta}} - \frac{\partial g}{\partial a_{\alpha\lambda}}\frac{\partial f}{\partial a_{\lambda\beta}}\right) a_{\alpha\beta}. \tag{10.95}$$

The Poissonian formula of equation (10.78) is the particular case for which $\alpha = \beta$. Note that this definition gives stochastic quantities for $\alpha \neq \beta$; note also that again the substitution of $a_{\mu\lambda}$ by $a^{0}_{\mu\lambda}$ everywhere in equation (10.95) leaves the equation invariant, because of (10.59). With f and g written in the linear form (10.73) characteristic of the quantum regime, in terms of coefficients $\tilde{f}_{\alpha\lambda}$ and $\tilde{g}_{\alpha\lambda}$, respectively, one gets

$$\langle f; g\rangle_{\alpha\beta} = \sum_{\lambda}\left(\tilde{f}_{\alpha\lambda}\tilde{g}_{\lambda\beta} - \tilde{g}_{\alpha\lambda}\tilde{f}_{\lambda\beta}\right) a_{\alpha\beta} = \left(\tilde{f}\tilde{g} - \tilde{g}\tilde{f}\right)_{\alpha\beta} a_{\alpha\beta} = \left[\tilde{f}, \tilde{g}\right]_{\alpha\beta} a_{\alpha\beta}. \tag{10.96}$$

In the last equality we have introduced the commutator of $\tilde{f}$ and $\tilde{g}$, with matrix element $(\alpha\beta)$ given by

$$\left[\tilde{f}, \tilde{g}\right]_{\alpha\beta} = \sum_{\lambda}\left(\tilde{f}_{\alpha\lambda}\tilde{g}_{\lambda\beta} - \tilde{g}_{\alpha\lambda}\tilde{f}_{\lambda\beta}\right). \tag{10.97}$$

The time-dependent generalization of this equation is

$$\langle f(t); g(t')\rangle_{\alpha\beta} = \sum_{\lambda}\left(\tilde{f}_{\alpha\lambda}\tilde{g}_{\lambda\beta}e^{i\omega_{\beta\lambda}(t-t')} - \tilde{g}_{\alpha\lambda}\tilde{f}_{\lambda\beta}e^{i\omega_{\alpha\lambda}(t-t')}\right) a_{\alpha\beta}. \tag{10.98}$$

For the Poissonian of x and p one therefore obtains, using equation (10.90) generalized to two indices,

$$\langle x; p\rangle_{\alpha\beta} = [\tilde{x}, \tilde{p}]_{\alpha\beta}\, a_{\alpha\beta} = i\hbar a_{\alpha\beta}\delta_{\alpha\beta},$$

whence the commutator of $\tilde{x}$ and $\tilde{p}$ is given in matrix form by

$$[\tilde{x}, \tilde{p}]_{\alpha\beta} = i\hbar\delta_{\alpha\beta}. \tag{10.99}$$

From (10.92) one obtains in an analogous way, with x, $\dot{x}$, p, $\dot{p}$ and H written in the linear form (10.73), a pair of equations for every term $(\alpha\beta)$, namely,

$$i\hbar\,\tilde{\dot{x}}_{\alpha\beta}\,a_{\alpha\beta} = \langle x; H\rangle_{\alpha\beta} = \left[\tilde{x},\tilde{H}\right]_{\alpha\beta} a_{\alpha\beta} = \frac{i\hbar}{m}\,\tilde{p}_{\alpha\beta}\,a_{\alpha\beta}, \tag{10.100}$$

$$i\hbar\,\tilde{\dot{p}}_{\alpha\beta}\,a_{\alpha\beta} = \left[\tilde{p},\tilde{H}\right]_{\alpha\beta} a_{\alpha\beta} = i\hbar\tilde{F}_{\alpha\beta}a_{\alpha\beta}, \tag{10.101}$$

whence it follows that

$$i\hbar\,\tilde{\dot{x}}_{\alpha\beta} = \left[\tilde{x},\tilde{H}\right]_{\alpha\beta}, \qquad i\hbar\,\tilde{\dot{p}}_{\alpha\beta} = \left[\tilde{p},\tilde{H}\right]_{\alpha\beta}. \tag{10.102}$$

These are just the matrix equations of quantum mechanics; alternatively they can be recast in the form

$$m\,\tilde{\dot{x}}_{\alpha\beta} = \tilde{p}_{\alpha\beta}, \qquad \tilde{\dot{p}}_{\alpha\beta} = \tilde{F}_{\alpha\beta}, \tag{10.103}$$

equivalent to equations (10.71) and (10.72).

Now one can take advantage of the separability of $\omega_{\alpha\beta}$ expressed in equation (10.65), to write

$$\tilde{\dot{x}}_{\alpha\beta} = i\omega_{\alpha\beta}\tilde{x}_{\alpha\beta} = i(\Omega_\alpha - \Omega_\beta)\tilde{x}_{\alpha\beta} = -i\sum_\mu(\tilde{x}_{\alpha\mu}\Omega_\beta\delta_{\mu\beta} - \Omega_\alpha\delta_{\alpha\mu}\tilde{x}_{\mu\beta}); \tag{10.104}$$

on the other hand, from equation (10.102) one has

$$i\hbar\,\tilde{\dot{x}}_{\alpha\beta} = \Sigma_\mu(\tilde{x}_{\alpha\mu}\tilde{H}_{\mu\beta} - \tilde{H}_{\alpha\mu}\tilde{x}_{\mu\beta}), \tag{10.105}$$

whence a comparison gives

$$\tilde{H}_{\alpha\beta} = \hbar\Omega_\alpha\delta_{\alpha\beta}. \tag{10.106}$$

This result shows that the matrix representing H in the present formalism is diagonal, which means that H_α is *not* random,

$$H_\alpha = \sum_\mu H_{\alpha\mu}a_{\alpha\mu} = \hbar\Omega_\alpha a_{\alpha\alpha} = \hbar\Omega_\alpha, \tag{10.107}$$

and further, that the Ω_α introduced via equation (10.65) is proportional to H_α, identified in quantum mechanics as the energy of the particle in state α,

$$\mathcal{E}_\alpha \equiv \langle H_\alpha\rangle = H_\alpha = \hbar\Omega_\alpha. \tag{10.108}$$

From this it follows that equation (10.65) is Bohr's formula for the transition frequencies,

$$\hbar\omega_{\alpha\beta} = \mathcal{E}_\alpha - \mathcal{E}_\beta. \tag{10.109}$$

Hence the characteristic frequencies of the SED states coincide with the transition frequencies of quantum mechanics. Analogously, from the above relations it follows that the response amplitudes $\tilde{x}_{\alpha\beta}$ are the transition amplitudes of quantum mechanics. The characteristic frequencies are the natural elements of the present theory, while quantum mechanics puts the accent on the energy eigenvalues, because they are better suited to its formalism. We can now interpret the usual quantum result that for stationary states the Hamiltonian acquires dispersionless values, represented by a diagonal matrix, as referring to a partly averaged energy over the subensemble determined by the coarse-graining process discussed in section 10.1. It was seen above that the stationary states with nonrandom characteristic frequencies are particularly stable against fluctuations, with properties independent of the realization of the field, up to the approximations of the description. We have just verified that this applies also to their energy.

To conclude these remarks, we note that for any couple of variables (f, g) and with the help of the Schwartz inequality, one can derive an inequality for the product of the corresponding dispersions, which is just the Poissonian version of the Heisenberg inequality,

$$\left\langle (\Delta f)^2 \right\rangle_{\alpha\alpha} \left\langle (\Delta g)^2 \right\rangle_{\alpha\alpha} \geq \frac{1}{4} \left| \langle f ; g \rangle_{\alpha\alpha} \right|^2 . \tag{10.110}$$

10.4.2. THE HILBERT-SPACE DESCRIPTION

A correspondence has been established between the description of linear SED and quantum mechanics, via the Heisenberg equations. Now it is a relatively easy matter to further develop the new description, until a direct contact with the usual Hilbert-space formalism is reached. As a practical means to achieve this we introduce the *a-representation*, as follows.

Consider a set of square matrices $\hat{a}^{\alpha\beta}$, each of which has only the element $\alpha\beta$ different from zero,

$$\left(\hat{a}^{\alpha\beta}\right)_{\mu\nu} = a_{\alpha\beta}\delta_{\alpha\mu}\delta_{\beta\nu}. \tag{10.111}$$

The coefficients $a_{\alpha\beta} \equiv a_{\alpha\beta}(t)$ are given by equation (10.64) multiplied by the corresponding time factor; thus,

$$a_{\alpha\beta} = e^{i\varphi_{\alpha\beta}+i\omega_{\alpha\beta}t} = e^{i(\varphi_\alpha-\varphi_\beta)+i(\Omega_\alpha-\Omega_\beta)t} = e^{i(\varphi_\alpha+\Omega_\alpha)t}e^{-i(\varphi_\beta+\Omega_\beta)t}. \tag{10.112}$$

Note that the off-diagonal elements have random phases, whereas for $\alpha = \beta$ the phase is zero. A product of two of such matrices gives, as follows from equations (10.111) and (10.112),

$$\left(\hat{a}^{\alpha\beta}\hat{a}^{\gamma\delta}\right)_{\mu\nu} = a_{\alpha\delta}\delta_{\alpha\mu}\delta_{\beta\gamma}\delta_{\delta\nu} = \delta_{\beta\gamma}\left(\hat{a}^{\alpha\delta}\right)_{\mu\nu}. \tag{10.113}$$

The fact that this product differs from zero only for $\beta = \gamma$ makes these matrices especially suited for the present purposes, for they can be used as a basis to write the matrix representing an arbitrary dynamical variable. For instance for the variable x we write

$$\hat{x} = \sum_{\alpha,\lambda} \tilde{x}_{\alpha\lambda} \hat{a}^{\alpha\lambda} = \sum_{\alpha} \hat{x}_{\alpha}, \tag{10.114}$$

where

$$\hat{x}_{\alpha} = \sum_{\lambda} \tilde{x}_{\alpha\lambda} \hat{a}^{\alpha\lambda} \tag{10.115}$$

for each state α, as seen from equations (10.73); this is the a-representation of x. The matrix elements of $\hat{x}$ are then just $\hat{x}_{\alpha\lambda} = \tilde{x}_{\alpha\lambda} a_{\alpha\lambda}$. Then from (10.113) and (10.114) one gets for instance for the square of $\hat{x}$

$$\left(\hat{x}^2\right)_{\mu\nu} = \sum_{\lambda} \tilde{x}_{\mu\lambda} \tilde{x}_{\lambda\nu} a_{\mu\lambda} a_{\lambda\nu} = \left(\tilde{x}^2\right)_{\mu\nu} a_{\mu\nu}. \tag{10.116}$$

The matrix x^2 is thus again a linear combination of the $\hat{a}$'s, with coefficients $\left(\tilde{x}^2\right)_{\mu\nu} = \sum_{\lambda} \tilde{x}_{\mu\lambda} \tilde{x}_{\lambda\nu}$. Consequently the operator $\hat{x}$ reproduces the matrix properties which the variable x must possess according to the discussion of §10.3.3. The same applies of course to any other observable, which means that it applies to any variable in the quantum regime that can be expressed in the linear form (10.73).

Now we observe from equation (10.112) that the matrix $\hat{a}^{\alpha\beta}$ can be written as the product of two vectors, namely, a column vector $|\alpha\rangle$ of the form

$$|\alpha\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a_{\alpha} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix} a_{\alpha} \tag{10.117}$$

and a row vector $\langle\beta|$ which is the adjoint of $|\beta\rangle$, $\langle\beta| = (|\beta\rangle)^{\dagger}$, and is given by

$$\langle\beta| = \begin{pmatrix} 0 & 0 & \cdots & a_{\beta}^{*} & \cdots \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 1 & \cdots \end{pmatrix} a_{\beta}^{*}, \tag{10.118}$$

with

$$a_{\alpha} = e^{i(\varphi_{\alpha} + \Omega_{\alpha} t)}. \tag{10.119}$$

These vectors have of course as many components as there are different indices α, which normally means a (denumerable) infinity of them. The only element of $|\alpha\rangle$ which is different from zero is in row α, i.e., $(|\alpha\rangle)_\lambda = a_\alpha \delta_{\alpha\lambda}$, and therefore

$$a_\alpha a_\beta^* = a_{\alpha\beta}, \tag{10.120}$$

in agreement with equation (10.112). Finally, from equations (10.111) and (10.117) follows the factorization rule

$$\hat{a}^{\alpha\beta} = |\alpha\rangle \langle\beta| \, . \tag{10.121}$$

The vectors $|\alpha\rangle$ form a complete orthonormal basis, as follows from the equations

$$\langle\alpha \mid \beta\rangle = \delta_{\alpha\beta}, \quad \sum_\alpha |\alpha\rangle \langle\alpha| = \sum_\alpha \hat{a}^{\alpha\alpha} = \hat{1}. \tag{10.122}$$

They thus span the Hilbert space of the states of the system, and an observable f can be represented by any one of the following expressions:

$$\hat{f} = \sum_\alpha \hat{f}_\alpha = \sum_{\alpha,\beta} \tilde{f}_{\alpha\beta} \hat{a}^{\alpha\beta} = \sum_{\alpha,\beta} \tilde{f}_{\alpha\beta} |\alpha\rangle \langle\beta| \, , \tag{10.123}$$

with

$$\tilde{f}_{\alpha\beta} = \langle\alpha| \hat{f} |\beta\rangle \, . \tag{10.124}$$

In particular for the Hamiltonian we have, as follows from (10.107),

$$\hat{H} = \sum_\alpha \tilde{H}_{\alpha\alpha} \hat{a}^{\alpha\alpha} = \sum_\alpha \mathcal{E}_\alpha |\alpha\rangle \langle\alpha| \, . \tag{10.125}$$

Note finally that

$$\hat{a}^{\alpha\alpha} \hat{f} = \sum_{\mu,\nu} \tilde{f}_{\mu\nu} |\alpha\rangle \langle\alpha \mid \mu\rangle \langle\nu| = \sum_\nu \tilde{f}_{\alpha\nu} |\alpha\rangle \langle\nu| = \hat{f}_\alpha, \tag{10.126}$$

which identifies $\hat{a}^{\alpha\alpha}$ as the projector onto subspace α.

It is interesting to note that the vectors $|\alpha\rangle$ do not involve the $\omega_{\alpha\beta}$, but the quantities $\mathcal{E}_\alpha/\hbar = \Omega_\alpha$; the transition to a Hilbert-space formulation in terms of bras and kets has had the effect of shifting the accent from the characteristic frequencies to the energy eigenvalues for the stationary states, and from the field amplitudes $a_{\alpha\beta}$ to the vector elements a_α. The proposed mathematical transformation has thus changed the conceptual framework into an entirely different one, in which the main objects are vectors in a Hilbert space and eigenvalue equations. Note further that in expressions such as (10.117) and (10.118) the factors a_λ contain random phases that remain hidden in the usual Hilbert-space formulation.

In terms of the matrices $\hat{a}$ the dynamical equations take the form (see equations (10.71, 10.72, 10.90))

$$m\frac{d\hat{x}}{dt} = \hat{p}, \qquad \frac{d\hat{p}}{dt} = \hat{F} \tag{10.127}$$

and

$$[\hat{x}, \hat{p}] = i\hbar\hat{1}. \tag{10.128}$$

The general relationship between these expressions in the Hilbert-space representation and the corresponding Poissonian expressions can be readily established by considering a matrix element of the commutator

$$\left[\hat{f}, \hat{g}\right]_{\alpha\beta} = \sum_{\mu} \left(\tilde{f}_{\alpha\mu}\tilde{g}_{\mu\beta} - \tilde{g}_{\alpha\mu}\tilde{f}_{\mu\beta}\right) a_{\alpha\beta} \tag{10.129}$$

and comparing with equation (10.95); it follows that

$$\widehat{\langle f; g\rangle} = \left[\hat{f}, \hat{g}\right], \tag{10.130}$$

which is equivalent to equation (10.96). Once more, in equation (10.115) the time dependence can be attributed to $\tilde{x}$ by writing $\sum \tilde{x}_{\alpha\beta}(t)\hat{a}^{\alpha\beta}$, or else to $\hat{a}$ by writing $\sum \tilde{x}_{\alpha\beta}\hat{a}^{\alpha\beta}(t)$ and using equation (10.68). The passage from the first expression to the latter is equivalent to a transition from the Heisenberg to the Schrödinger picture.

Finally, we observe that from the most general expression for an operator $\hat{R}$, i.e.,

$$\hat{R} = \sum_{\alpha,\beta} \tilde{r}_{\alpha\beta}\hat{a}^{\alpha\beta} \tag{10.131}$$

it follows that

$$\operatorname{tr}\left(\hat{f}\hat{R}\right) = \sum_{\lambda,\mu} \tilde{r}_{\lambda\mu}\tilde{f}_{\mu\lambda}. \tag{10.132}$$

Two instances of this expression are particularly interesting, according to the structure of the coefficients $\tilde{r}_{\alpha\beta}$:

a) when $\tilde{r}_{\lambda\mu} = c_\lambda c_\mu^*$, with $\sum_\lambda |c_\lambda|^2 = 1$. Then

$$\hat{R} = |\psi\rangle\langle\psi| \quad \text{and} \quad \operatorname{tr}\left(\hat{f}\hat{R}\right) = \left\langle\psi\left|\hat{f}\right|\psi\right\rangle, \tag{10.133}$$

which corresponds to the density-matrix description of a pure state $|\psi\rangle = \sum_\lambda c_\lambda |\lambda\rangle$;

b) when $\tilde{r}_{\lambda\mu} = w_\lambda \delta_{\lambda\mu}$, with $\sum_\lambda w_\lambda = 1$. Then

$$\hat{R} = \sum_\lambda w_\lambda |\lambda\rangle\langle\lambda| \quad \text{and} \quad \operatorname{tr}\left(\hat{f}\hat{R}\right) = \sum_\lambda w_\lambda \tilde{f}_{\lambda\lambda} = \sum_\lambda w_\lambda \langle\lambda|\hat{f}|\lambda\rangle, \tag{10.134}$$

which corresponds to the density-matrix description of a mixture with weights w_λ. The usual statistical formalism of quantum mechanics is thus compatible with the theory.

10.4.3. THE SCHRÖDINGER DESCRIPTION

The above results can be transcribed without further complication into the language of the Schrödinger description; for completeness we shall just add here a few simple considerations. According to the above discussion, it is clear that a complete set of orthonormal functions spanning the appropriate Hilbert space is needed; take for instance $\{\psi_\alpha\}$ in the x-representation, with

$$\int \psi_\alpha^*(x)\psi_\beta(x)dx = \delta_{\alpha\beta}. \tag{10.135}$$

In particular, if the eigenfunctions of the Schrödinger equation are used,

$$\hat{H}\psi_\alpha = E_\alpha \psi_\alpha, \tag{10.136}$$

equation (10.125) is automatically satisfied. From equation (10.128) it follows that in this x-representation,

$$\hat{x} = x, \qquad \hat{p} = -i\hbar\nabla, \tag{10.137}$$

and the Hamiltonian operator acquires its usual form, in agreement with (10.136).

This is an appropriate place to reconsider the statistical derivation carried out in §2.2.3. Recall that very general and simple arguments were shown to lead to equation (2.79),

$$ib\frac{\partial\psi}{\partial t} = -\frac{b^2}{2m}\nabla^2\psi + V\psi, \tag{10.138}$$

for an ensemble of particles represented by a density $\rho(\mathrm{x}, t) = \psi^*\psi$ satisfying the continuity equation; also Ehrenfest's theorem $m\langle\ddot{x}\rangle = \langle F(x)\rangle$ (equation (2.77)) was assumed to be satisfied,[15] as a result of considering that the detailed motion of each particle is described by the Abraham-Lorentz equation (2.75) (which here becomes the Braffort-Marshall equation), and taking the radiationless approximation after equilibrium is attained. The solution $\psi(\mathrm{x}, t)$ of equation (10.138) is a complex quantity, such that

$$\psi^*\psi = \rho \quad \text{and} \quad \frac{ib}{2m}\left(\frac{\nabla\psi^*}{\psi^*} - \frac{\nabla\psi}{\psi}\right) = \mathrm{v}, \tag{10.139}$$

according to equations (2.72) and (2.73). Multiplying the last equation by $m\rho$ and integrating, one gets $m\langle\mathrm{v}\rangle = -ib\langle\nabla\rangle$. Using now equation (10.137) and identifying the ψ of this equation with the one just introduced, one gets $m\langle\mathbf{v}\rangle = (b/\hbar)\langle\mathbf{p}\rangle$, whence it follows that $b = \hbar$.

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[15] As follows from the discussion in chapter 2, the electromagnetic case with minimal coupling can be considered by writing an expression of the form of equation (2.43) instead of (2.70); however, for the present discussion such details can be left aside.

In chapter 2 it was argued that for arbitrary stochastic systems one can expect b to be problem-dependent; now we see that linear SED predicts it to be a universal constant, as in quantum mechanics. Equation (10.138) becomes thus Schrödinger's equation,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi. \tag{10.140}$$

This extremely important property of the parameter b explains the singular role played by the Schrödinger equation in quantum theory. The universality of b is the element that was missing in the analysis of §2.2.3 to complete the derivation of the Schrödinger equation, probably the simplest possible one. As regards the mechanics of the particle, we conclude, therefore, that the Schrödinger equation implies the following minimal set of requirements: i) the continuity equation (with the local flow velocity given by equation (2.70)), ii) an average form of Newton's Second Law (or rather, of the appropriate Braffort-Marshall equation with radiative corrections neglected), and iii) a dynamics such that the parameter b attains a constant universal value. Of these three requirements only the last one represents a true departure from classical physics, and thus we conclude that the quantum postulate is equivalent to the highly nontrivial passage from equation (10.138) to equation (10.140).

10.5. A brief discussion

10.5.1. THE LINEAR RESPONSE

It seems appropriate at this point to comment briefly on some of the features and implications of the theory just described. The approximate solutions studied here are certainly much less noisy than one would intuitively expect at first from the solutions of the Braffort-Marshall equation. Firstly, because the dynamical behaviour of the system becomes determined essentially by a denumerable set of field oscillators of nonrandom frequencies $\omega_{\alpha\beta}$ and correlated amplitudes smoothed out by the k-averaging process. Secondly, because the effects of the infinitely many field components not corresponding to any such frequencies become comparatively so weak that they have been entirely neglected in the description given above; in a more detailed account they can be expected to give rise to some extra background noise and to small (radiative) corrections. That these modifications correspond indeed to the radiative corrections of QED will be verified in the next chapter.

The characteristics of the self-consistent solution show that the coupling between the atomic processes and the field due to radiation, generates in the long run phase correlations between the components of the neighboring field, and between this field and the atomic motions.[16] This leads simultaneously to conditions on the equilibrium field, epitomized by equation (10.64), and on the atomic system, as summarized in the matrix law of multiplication for the response amplitudes. On its own, the demand that the particle responds linearly to the field modes (besides being arbitrary) would not be sufficient to solve the problem posed by equation (10.45), because the quotient $\tilde{\Phi}/\tilde{z}$ becomes independent of the a's only under the matrix rule (10.67), which is essential to guarantee the nonmixing of frequencies and detailed energy balance.[17]

The fact that also the response coefficients have nonrandom values, independent of the specific sample of the field, confers to the description an extra insensitivity to the fluctuations, not shared by other stochastic theories, and by standard SED in particular. This observation suggests the possible existence of a kind of least-stochasticity principle (or maximal stability against field variations, or minimal sensitivity to them) as an alternative to the hypothesis of nonrandom characteristic frequencies, used above as a practical means to identify the appropriate solutions.

The method followed, though obviously successful, has the important shortcoming that the self-consistent solution must be accepted as a matter of fact. It is quite clear that without the use of an auxiliary principle or hypothesis it would be practically impossible to identify the solution, due to the high complexity of the mathematical problem. The central question of deriving the solution from first principles remains hopelessly open, although with a good dose of optimism one may perhaps find here the gate to a promising field of research.

As has been already stated, a detailed solution of the full problem would *not* lead to the present description, but this would be reached only after performing a series of approximations and simplifications. It is in the transition to such an approximate theory that some attributes of the starting description, as that of being genuinely statistical and local realistic, become weakened or adopt their quantum guise. Since we cannot be aware of all of these simplifications due to the lack of knowledge of the full solution, it becomes impossible to make a detailed transition from the initial theory to the final one, which affects particularly the interpretative issues.

It is here that we find an explanation to questions as why the quantum formalism gives an incomplete and seemingly noncausal account of the behaviour of mechanical systems: quantum mechanics appears from the present point of view not as a fundamental theory of matter, but as a derived, approximate theory. And approximate physical theories are not bound to satisfy the same rigorous requirements that fundamental theories are supposed to fulfil; this is particularly true in what refers to consistency with first principles. It is the approximations that are to be blamed for the faults, not the foundations. We may illustrate the observation with the example of the Abraham-Lorentz equation of motion. No fundamental problem at all, and even less any crisis within physics, is generated by its noncausal properties, even if it is derived from the conjunction of two of the most basic and general theories of classical physics, Newtonian mechanics and Maxwellian electromagnetism.

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[16] This was anticipated in Theimer and Peterson (1977); our wording is almost the one used there.

[17] Of course, the mathematics for the trick are just those occurring in the matrix formulation of quantum mechanics, as discussed in Born et al. (1926).

Even though it is unfinished, the theory presented here allows already for a certain reinterpretation of some quantum issues. Of special interest is the finding that operators such as $\hat{x}, \hat{p}$ and so on, which belong to the quantum formalism, should not be interpreted as referring to a single particle, but to the subensemble constructed through the coarse-graining process, and then in a highly abstract form, distant from any direct empirical meaning. Also important is the recognition that the quantum-regime description implies approximations that have deep consequences, among them the k-averaging and the transformation of the zeropoint field into merely a set of (correlated) harmonic oscillators, without any spatial dependence. Thus the theory is as intrinsically nonrelativistic as it is intrinsically statistical.

An important consequence of these observations, that the belief that the quantum variables can be readily identified with those describing the individuals, is not supported by the present theory. However, the fact that the theory is not constructed around the notion of trajectory, due to different individuals being considered in the k-averaging process, does not mean that individual trajectories do not exist, nor that the possibility of constructing a theory in terms of them is cancelled forever. Simply, neither the present formulation nor quantum mechanics is such a theory.

Although the basic equations for the linear SED system in the quantum regime are stochastic by nature, they were recast in nonrandom terms, which happen to be just those of matrix mechanics. In such a most cryptic description (in terms of the Heisenberg equations of motion) the elements responsible for the stochasticity —the field amplitudes $a_{\alpha\beta}$— vanish completely, resulting in a seemingly fully deterministic picture. This simple observation explains by itself much of the enduring interpretative problems of the usual quantum mechanics. It is not without interest to note that the basis itself used to construct the Hilbert space of states can be expressed in terms of the hidden random amplitudes, as is done in equations (10.117) and (10.118).

Recall that the transition to the quantum regime is an irreversible process. The complete SED system composed of particle and field is Hamiltonian; however, the dynamical equations governing the purely mechanical part contains a dissipative (radiation reaction) force in addition to the random driving force, and it is the joint action of these forces that allows a bound system to approach asymptotically what seems to behave as an attractor (or rather, one out of a set of attractors), probably of the limit cycle kind, in a process that is quite independent of the initial conditions.[18] The remarkable stability of the corresponding stationary states against the field fluctuations perhaps explains the success of the (partial averaging) procedure used to construct them.

In the present approximation the noise effects produced by the 'inactive' field modes not corresponding to any of the frequencies $\omega_{\alpha\beta}$ were neglected. Their introduction would make the trajectories appear diffuse and would prevent one from telling when an orbit lies on the attractor's basin or just close to it. In other words, one can say that a system has reached an attractor only within an uncertainty determined by the strength of the background noise, and this noise may be able to induce transitions between the stationary states, just as will be confirmed in chapter 11.

A question that invites us to indulge in further speculation refers to the possibility that under certain circumstances the system responds with random frequencies (as would be the case if the demand of detailed balance were removed), in which case the situation would of course be more chaotic than the one represented by usual quantum states. It is clear that for this to happen the system must leave the quantum regime, but it is unclear whether such a process means merely a return to a classical (stochastic) behaviour, or whether some new behaviour arises.

10.5.2. UNIVERSALITY OF QUANTUM MECHANICS

A remarkable related property of quantum mechanics is its *universality*, in the following sense. In the radiationless approximation, the equations describing the quantum regime can be written in terms of matrix elements in the form

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[18] See de la Peña and Cetto (1995a). The proposal that the quantum transitions may be assimilated to abrupt passages between limit cycles was made apparently for the first time within quantum theory by Cap (1956) and later by other authors, such as de Broglie (1968); within SED, as early as 1970 Surdin (1970b) visualized the ground state of the oscillator as a limit cycle. In his turn, and also within a classical scheme, Fer (1977) suggests the possibility that all quantum states could be associated with limit cycles.

$$\begin{gathered} m\,\ddot{\tilde{x}}_{\alpha\beta} = \tilde{F}_{\alpha\beta}, \\ -2m\sum_\beta \omega_{\alpha\beta}\left|\tilde{x}_{\alpha\beta}\right|^2 = \hbar. \end{gathered} \tag{10.141}$$

These equations do not contain the coupling constant e at all, nor do they make explicit reference to the electromagnetic nature of this field; the (now hidden) amplitudes $a_{\alpha\beta}$ could equally well characterize any other random field with energy $\frac{1}{2}\hbar\omega_{\alpha\beta}$ per normal mode, as was discussed in section 4.1. Thus, equations (10.141) describe a mechanics free of any reference to electromagnetism, even though they refer to SED systems. Also remarkable is the fact that the left-hand side of the second equation involves just the universal constant $\hbar$ and does not depend on the specific problem nor on the state of the quantum system. We are referring to these properties under the concept of universality; let us consider them in more detail.

Assume a mechanical system coupled with coupling constant g to a stationary random external field with power spectrum $S(\omega) = \lambda \mid \omega \mid^s$, and subject to a dissipative force of the type $\beta x^{(r)}$ (the index r denotes time derivative of order r). Assume further that for the purposes at hand the system can be modelled as an oscillator of frequency ω_0 (which may depend on the state of the system); it is thus described by the equation

$$m\ddot{x} + m\omega_0^2 x + m\beta x^{(r)} = gF(t); \quad \langle F^2(t)\rangle = \lambda \int_{-\infty}^{\infty} \mid \omega \mid^s d\omega. \tag{10.142}$$

Following the procedure used in section 10.4, we now consider the bilinear form $\langle x^{(k)} * x^{(l)}\rangle$, where $x^{(k)}$ and $x^{(l)}$ are two dynamical variables that contain the factors ω^k and ω^l, respectively. For example, for $x^{(k)} = x$ and $mx^{(l)} = p$, $k = 0$ and $l = 1$, respectively. The symbol $*$ represents a composition of its two arguments (perhaps involving linear operators), such that the result is even in ω. The precise meaning of this 'product' is to be chosen appropriately in each instance, as shown in the examples below. For $k + l$ even, it may be taken as the usual product; for $k + l$ odd, it should be an antisymmetric form (Poisson bracket or Poissonian, for instance). The calculation is straightforward and gives

$$M_{kl} \equiv \left\langle x^{(k)} * x^{(l)} \right\rangle = (-1)^l i^{k+l} \frac{\pi\lambda g^2}{2m^2\beta}\omega_0^{k+l+s-r-1}. \tag{10.143}$$

From this result it follows that:

a) The coefficient β must be proportional to g^2 for M_{kl} to be independent of the coupling constant g (of the nature of the random field). This is in fact the usual situation in classical stochastic systems and holds in SED as well (because $\tau \sim e^2$).

b) Only if

$$k + l + s - r = 1 \tag{10.144}$$

is M_{kl} state-independent (independent of the frequency ω_0); this sets a restriction to the values of k and l, given s and r (i.e., given the theory).

Let us apply these results to a few simple cases of interest.[19] In the classical Brownian motion problem the relationship $\beta \sim g^2$ holds and the friction is proportional to the velocity ($r = 1$); the noise is white ($s = 0$), with intensity $\lambda = (2m\beta/\pi g^2)kT$ [see, e.g., Papoulis 1965]. From equation (10.144) it follows that $k + l = 2$, so that the relevant bilinear form can be taken as the average kinetic energy ($k = l = 1$), for which equation (10.143) gives

$$\left\langle \frac{1}{2} m\dot{x}^2 \right\rangle = \frac{1}{2} kT. \tag{10.145}$$

The state-independence is thus expressed in the equipartition of energy: the molecules of an ideal gas, or oscillators of any frequency, have the same mean kinetic energy.

A second interesting case is the charged particle subject to the classical radiation field with a Rayleigh-Jeans spectrum; in this case $\lambda = 4kT/3\pi c^3$, $g = e$ and $s = 2$. The dissipative term is the radiation reaction force, so that $\beta = \tau = 2e^2/3mc^3$ and $r = 3$. From equation (10.144) one gets $k + l = 2$, leading once more to equipartition of energy.

In the case of SED we meet another situation. Once again, $g = e$ and the dissipative force is radiation reaction, so that $\beta = \tau$, $r = 3$. However, for the zeropoint field $\lambda = 2\hbar/3\pi c^3$ and $s = 3$, so that equation (10.144) gives $k + l = 1$. Hence the relevant universal bilinear form is the Poissonian $\langle x; p \rangle$, which has been shown to be the stochastic equivalent of the commutator $[\hat{x}, \hat{p}]$. This result reveals an interesting conceptual link between the energy equipartition of classical physics and the fundamental commutator of quantum mechanics; in the latter theory the universal relation that holds in place of the classical equipartition of energy is the quantization rule.

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[19] In Santos (1990a) a related qualitative analysis is performed, with similar conclusions.

CHAPTER 11

RADIATIVE CORRECTIONS IN LINEAR SED

In this chapter we continue the study initiated in the preceding one and pay specific attention to higher-order effects on matter arising from its interaction with the radiation field, in the quantum regime of SED. More explicitly, we reinsert into the basic dynamical equations the zeropoint field and radiation reaction terms that were dropped in the passage to the radiationless limit, and use peturbation theory to study detailed energy balance, atomic stability, stimulated and spontaneous transitions, the Lamb shift, etc. In a single phrase, we proceed to the analysis of phenomena that belong to the realm of quantum electrodynamics but can be explained by resorting to the random (non-quantized) vacuum field. The chapter opens with the essentials of SED perturbation theory, which can be formulated in close correspondence with conventional perturbation theory in quantum mechanics.[1]

11.1. Perturbation theory in linear SED

The complete equations of motion for the mechanical system, as given i.e., by (10.88) and (10.89), contain terms that depend explicitly on the radiation field; these terms, which were omitted in the passage to the radiationless equations (10.92) describing the system in the quantum regime, have their origin in the interaction Hamiltonian

$$H_\text{I} = -\frac{e}{mc}\mathbf{p}\cdot\mathbf{A} + \frac{e^2}{2mc^2}\mathbf{A}^2. \tag{11.1}$$

Since the fundamental (zero-order) effects of the field on the mechanical system have already been taken into account in the radiationless approximation, the effects coming from H_I can be considered indeed as perturbations. An important point to observe is that in principle *all* the modes of the radiation field, and not only what were previously called the relevant frequencies, contribute to these corrections. Since the latter frequencies constitute a set of measure zero within the set of all frequencies, the field $\mathbf{A}$ appearing in H_I is essentially uncorrelated from the atomic motions, which means that its zeropoint component still corresponds to the free field (in open space). In chapter 14 we elaborate this point.

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[1] Most of the material used for this chapter can be found in Cetto and de la Peña (1991b) and de la Peña and Cetto (1991a, 1992, 1995a).

11.1.1. POISSONIAN PERTURBATIVE APPROACH

The purpose of this section is to get explicit expressions up to second order for the effects of a perturbation on a generic dynamical variable $B(x,p)$, assuming the unperturbed system to be in a stationary state α, so that B has the form given by equations (10.68) and (10.73), namely,

$$B_\alpha(t) = \sum_\lambda \tilde{B}_{\alpha\lambda} a_{\alpha\lambda}(t), \tag{11.2}$$

with $\tilde{B}^*_{\alpha\lambda} = \tilde{B}_{\lambda\alpha}$. Let us first consider the unperturbed system and use equations (10.92) to write the dynamical equation for the corresponding variable $B_0(x,p)$ (we will omit the Greek subindices as long as they are not necessary),

$$i\hbar\dot{B}_0 = \langle B_0; H\rangle \equiv \mathcal{L}_0 B_0. \tag{11.3}$$

Here $H = \mathbf{p}^2/2m + \Phi(\mathbf{x})$; the second equality defines the (time-independent) Liouvillian operator $\mathcal{L}_0$, for which no explicit expression will be necessary. With the same notation, the perturbed equation can be written as

$$i\hbar\dot{B} = \langle B; H\rangle \equiv \mathcal{L}B = (\mathcal{L}_0 + \mathcal{L}_1)B, \tag{11.4}$$

with

$$H = H_0 + H_1, \qquad \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \tag{11.5}$$

where H_1 and $\mathcal{L}_1$ stand for the perturbing Hamiltonian and Liouvillian operator, respectively. The solution of the unperturbed equation that goes over to $B_0(0)$ at some time $t = 0$ is

$$B_0(t) = e^{\mathcal{L}_0 t/i\hbar} B_0(0). \tag{11.6}$$

Now consider a transformation to the interaction picture. Denoting by a bar the variables in this representation, we have

$$B(t) = e^{\mathcal{L}_0 t/i\hbar}\bar{B}(t), \tag{11.7}$$

so that combining the above expressions one gets

$$\frac{\partial \bar{B}(t)}{\partial t} = \frac{\partial}{\partial t} e^{-\mathcal{L}_0 t/i\hbar} B(t) = e^{-\mathcal{L}_0 t/i\hbar}\left(\frac{\partial B}{\partial t} - \mathcal{L}_0 B\right) = \frac{1}{i\hbar} e^{-\mathcal{L}_0 t/i\hbar}\mathcal{L}_1 B, \tag{11.8}$$

whence

$$i\hbar\frac{\partial \bar{B}(t)}{\partial t} = \overline{\mathcal{L}}_1(t)\bar{B}(t). \tag{11.9}$$

The operator

$$\overline{\mathcal{L}}_1(t) = e^{-\mathcal{L}_0 t/i\hbar}\mathcal{L}_1 e^{\mathcal{L}_0 t/i\hbar} \tag{11.10}$$

is the perturbing Liouvillian in the interaction representation. The solution to equation (11.9) may be written in terms of the unperturbed solution B_0 as

$$B(t) = e^{\mathcal{L}_0 t/i\hbar}\left[\bar{B}_0(0) + \frac{1}{i\hbar}\int_0^t dt'\overline{\mathcal{L}}_1(t')\bar{B}(t')\right],$$

or

$$B(t) = B_0(t) + \frac{1}{i\hbar}\int_0^t dt'\mathcal{L}_1(\bar{q}', t')e^{\mathcal{L}_0(t-t')/i\hbar}B(t'), \tag{11.11}$$

where we have set

$$\mathcal{L}_1(\bar{q}, t') = e^{\mathcal{L}_0(t-t')/i\hbar}\mathcal{L}_1(t')e^{-\mathcal{L}_0(t-t')/i\hbar}. \tag{11.12}$$

The variables $\bar{q}' = \bar{q}'(t'; t)$ are those that evolve from t' towards $q(t)$ $(t > t')$ following an unperturbed motion controlled by the Hamiltonian H_0.

A repeated iteration of equation (11.11) gives $B(t)$ in terms of $B_0(t)$ to the desired order of approximation. With $B = B_0 + \delta B^{(1)} + \delta B^{(2)} + \cdots$ and using equation (11.6), the first- and second-order corrections to $B_0(t)$ come out to be

$$\delta B^{(1)}(t) = \frac{1}{i\hbar}\int_0^t dt'\mathcal{L}_1(\bar{q}', t')B_0(t), \tag{11.13}$$

$$\delta B^{(2)}(t) = \left(\frac{1}{i\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt''\mathcal{L}_1(\bar{q}', t')\mathcal{L}_1(\bar{q}'', t'')B_0(t). \tag{11.14}$$

A closed formal solution to equation (11.9) is

$$\bar{B}(t) = \hat{\mathsf{T}}e^{(1/i\hbar)\int_o^t dt'\overline{\mathcal{L}}_1(t')}\bar{B}(0), \tag{11.15}$$

as is easily verified by a time derivation. $\hat{\mathsf{T}}$ is the time-ordering operator that arranges the factors of a product in chronological order from right to left; it is needed to move totally to the left the factor $\overline{\mathcal{L}}_1(t)$ produced by the time derivation.[2]

Now the Poissonian expression for $\mathcal{L}_1$ can be introduced in the above results, using (11.3-11.5); for convenience, the perturbation Hamiltonian will be denoted by V from now on. One gets

$$\delta B^{(1)}(t) = \frac{1}{i\hbar}\int_0^t dt'\langle B_0(t); V(t')\rangle, \tag{11.16}$$

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[2] This closed expression for the solution is probably the one most frequently used within the interaction representation. The interested reader is referred to the literature for the details [see, e.g., Roman 1965, section 4-3; van Kampen 1981, chapters XIII and XIV; Sterman 1993, Appendix A].

$$\delta B^{(2)}(t) = \left(\frac{1}{i\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt'' \langle\langle B_0(t); V(t'')\rangle; V(t')\rangle. \tag{11.17}$$

Further one obtains

$$i\hbar\frac{d}{dt}\delta H_0^{(1)} = \langle H_0; V\rangle\,, \tag{11.18}$$

$$i\hbar\frac{d}{dt}\delta H_0^{(2)} = \left\langle \delta H_0^{(1)}; V\right\rangle + \left\langle H_0; V^{(1)}\right\rangle. \tag{11.19}$$

These equations follow from (11.13) and (11.14) applied to H_0; alternatively, they can be derived as particular cases of equations (11.3) and (11.4):

$$i\hbar\frac{d\delta B^{(1)}}{dt} = \left\langle \delta B^{(1)}; H_0\right\rangle + \left\langle B_0; \delta H_0^{(1)} + V\right\rangle, \tag{11.20}$$

$$i\hbar\frac{d\delta B^{(2)}}{dt} = \left\langle \delta B^{(2)}; H_0\right\rangle + \left\langle \delta B^{(1)}; \delta H_0^{(1)} + V\right\rangle + \left\langle B_0; \delta H_0^{(2)} + V^{(1)}\right\rangle. \tag{11.21}$$

11.1.2. FIRST- AND SECOND-ORDER ENERGY SHIFTS

According to the above equations there are two kinds of corrections to the value of the Hamiltonian, namely, the modification of H_0 due to the action of $\mathcal{L}_1$ given by (11.18) and (11.19), and the contribution of the perturbation term V,

$$\delta H^{(1)} = \delta H_0^{(1)} + V, \tag{11.22}$$

$$\delta H^{(2)} = \delta H_0^{(2)} + \delta V^{(1)}. \tag{11.23}$$

Assume that the original state is one of the stationary eigenstates of H_0, so that (see equation (10.107))

$$H_{0\alpha}(t) = \sum_\mu \tilde{H}_{0\alpha\mu} a_{\alpha\mu}(t) = \sum_\mu \mathcal{E}_{0\alpha}\delta_{\alpha\mu} a_{\alpha\mu}(t) = \mathcal{E}_\alpha, \tag{11.24}$$

and also the perturbed states are stationary (i.e., eigenstates of H) if H does not depend explicitly on time. This can be easily seen to first order by applying equation (11.3) to V and using (11.18) to get

$$\frac{d\delta H^{(1)}}{dt} = \frac{d}{dt}\left(\delta H_0^{(1)} + V\right) = 0; \tag{11.25}$$

analogously, to second order one gets from equations (11.19) and (11.20) applied to $\delta V^{(1)}$,

$$\frac{d\delta H^{(2)}}{dt} = \frac{d}{dt}\left(\delta H_0^{(2)} + \delta V^{(1)}\right) = 0. \tag{11.26}$$

Hence, if the Hamiltonian for the perturbed α-state is written in the form $H_\alpha(t) = \sum_\lambda \tilde{H}_{\alpha\lambda} a_{\alpha\lambda}(t)$, from the above results it follows that $\tilde{H}_0$, $\delta\tilde{H}^{(1)}$, $\delta\tilde{H}^{(2)}$, ... are all diagonal,

$$\delta\tilde{H}^{(1)}_{\alpha\lambda} = \delta\mathcal{E}^{(1)}_\alpha \delta_{\alpha\lambda}, \quad \delta\tilde{H}^{(2)}_{\alpha\lambda} = \delta\mathcal{E}^{(2)}_\alpha \delta_{\alpha\lambda}, \tag{11.27}$$

and the perturbed eigenenergy is $\mathcal{E}_{0\alpha} + \delta\mathcal{E}^{(1)}_\alpha + \delta\mathcal{E}^{(2)}_\alpha + \cdots$. From equations (11.25-11.26) it follows that the non-diagonal elements of $\delta\tilde{H}_0$ are given by

$$\delta\tilde{H}^{(1)}_{0\alpha\lambda} = -\tilde{V}_{\alpha\lambda}, \quad \delta\tilde{H}^{(2)}_{0\alpha\lambda} = -\delta\tilde{V}^{(1)}_{\alpha\lambda} \quad (\alpha \neq \lambda). \tag{11.28}$$

According to equations (11.22) and (11.23), the energy shift for state α is given to first and second order by

$$\delta\mathcal{E}^{(1)}_\alpha = \delta\tilde{H}^{(1)}_{0\alpha\alpha} + \tilde{V}_{\alpha\alpha}, \quad \delta\mathcal{E}^{(2)}_\alpha = \delta\tilde{H}^{(2)}_{0\alpha\lambda} + \delta\tilde{V}^{(1)}. \tag{11.29}$$

For the evaluation of $\delta\tilde{H}^{(1)}_{0\alpha\alpha}$we apply (11.16) to $B = H_0$ and use the general formula (10.95) to calculate the Poissonian of H_0 and $V(t')$; the diagonal element $(\alpha\alpha)$ is

$$\langle H_0; V(t')\rangle_{\alpha\alpha} = \sum_\mu \left(\tilde{H}_{0\alpha\mu} V_{\mu\alpha} \mathrm{e}^{i\omega_{\alpha\mu}(t-t')} - \text{c.c.}\right) = E^{(0)}_\alpha \tilde{V}_{\alpha\alpha} - \text{c.c.}, \tag{11.30}$$

whence it follows that

$$\delta\tilde{H}^{(1)}_{0\alpha\alpha} = \frac{1}{i\hbar} \sum_\mu \int_0^t dt' \left\langle H_0; V(t')\right\rangle_{\alpha\alpha} = 0 \tag{11.31}$$

and one arrives thus at the well-known result

$$\delta\tilde{H}^{(1)}_{\alpha\alpha} = \delta\mathcal{E}^{(1)}_\alpha = \tilde{V}_{\alpha\alpha}, \tag{11.32}$$

whereas all nondiagonal elements of $\delta\tilde{H}^{(1)}$ are zero, as seen from equation (11.28).

Now we proceed to calculate $\delta\tilde{H}^{(2)}_{\alpha\alpha}$, which is given by equation (11.19) integrated over time,

$$\delta\tilde{H}^{(2)}_{0\alpha\alpha} = \frac{1}{i\hbar} \int_0^t dt' \left(\left\langle \delta\tilde{H}^{(1)}_0; V \right\rangle (t') + \left\langle H_0; \delta V^{(1)} \right\rangle (t') \right)_{\alpha\alpha}. \tag{11.33}$$

The evaluation of this expression requires some intermediate steps. First we use equations (10.95) and (11.16), to get

$$\delta\tilde{V}^{(1)}_{\alpha\beta} = \frac{1}{i\hbar} \int_0^t dt' \langle V; V(t')\rangle_{\alpha\beta}$$

$$= \frac{1}{i\hbar} \sum_{\mu} \int_0^t dt' \tilde{V}_{\alpha\mu} \tilde{V}_{\mu\beta} \left[\mathrm{e}^{i\omega_{\beta\mu}(t-t')} - \mathrm{e}^{i\omega_{\mu\alpha}(t-t')} \right]$$

$$= -\frac{1}{\hbar} \sum_{\mu} \tilde{V}_{\alpha\mu} \tilde{V}_{\mu\beta} \left[\frac{\mathrm{e}^{i\omega_{\beta\mu}t} - 1}{\omega_{\beta\mu}} - \frac{\mathrm{e}^{i\omega_{\mu\alpha}t} - 1}{\omega_{\mu\alpha}} \right] + \frac{1}{i\hbar} \tilde{V}_{\alpha\beta} \left(\tilde{V}_{\beta\beta} - \tilde{V}_{\alpha\alpha} \right) \int_0^t e^{-\epsilon s} ds. \tag{11.34}$$

Note that in the first term of the last expression, $\mu \neq \beta$, and in the second one, $\mu \neq \alpha$; the last term comes from $\mu = \alpha$ or $\mu = \beta$, which corresponds to $\omega_{\alpha\alpha} = \omega_{\beta\beta} = 0$. The latter is a secular term that arises because the developments are made around the unperturbed frequencies (and energy eigenvalues); an (adiabatic) convergence factor $\mathrm{e}^{-\epsilon t}$, $\epsilon \longrightarrow 0^+$ has been added to assign to this term a well defined value for $t \longrightarrow \infty$. A similar convergence factor is required also when $\omega \neq 0$. Upon integration this term gives

$$\frac{1}{i\hbar} \tilde{V}_{\alpha\beta} \left(\tilde{V}_{\beta\beta} - \tilde{V}_{\alpha\alpha} \right) \frac{1 - e^{-\epsilon t}}{\epsilon} \underset{\epsilon \to 0^+}{\longrightarrow} -\frac{it}{\hbar} \tilde{V}_{\alpha\beta} \left(\tilde{V}_{\beta\beta} - \tilde{V}_{\alpha\alpha} \right), \tag{11.35}$$

which can be treated as the first order approximation to

$$\tilde{V}_{\alpha\beta} \left(\mathrm{e}^{-it\tilde{V}_{\beta\beta}/\hbar} - \mathrm{e}^{-it\tilde{V}_{\alpha\alpha}/\hbar} \right) \underset{t \to \infty}{\longrightarrow} 0 \tag{11.36}$$

with the convergence factor duly inserted. With this procedure, also the oscillatory terms disappear for $t \longrightarrow \infty$ and we are left with

$$\delta \tilde{V}^{(1)}_{\alpha\beta} = \sum_{\mu \neq \beta} \tilde{V}_{\alpha\mu} C_{\beta\mu} - \sum_{\mu \neq \alpha} \tilde{V}_{\mu\beta} C_{\mu\alpha}, \tag{11.37}$$

where

$$C_{\beta\mu} = \frac{\tilde{V}_{\mu\beta}}{\mathcal{E}_\beta - \mathcal{E}_\mu} \left(1 - e^{i\omega_{\beta\mu}t} \right) \underset{t \to \infty}{\longrightarrow} \frac{\tilde{V}_{\mu\beta}}{\mathcal{E}_\beta - \mathcal{E}_\mu}, \quad \mu \neq \beta \tag{11.38}$$

and $\mathcal{E}_\alpha \equiv \mathcal{E}^{(0)}_\alpha$.

Not much work is required to extend this result to any observable B; starting again from (11.16), one obtains

$$\delta \tilde{B}^{(1)}_{\alpha\beta} = \frac{1}{i\hbar} \int_0^t dt' \langle B; V(t') \rangle_{\alpha\beta} = \sum_{\mu} \left(\tilde{B}_{\alpha\mu} C_{\beta\mu} - \tilde{B}_{\mu\beta} C_{\mu\alpha} \right), \tag{11.39}$$

where the sum runs over the appropriate values of μ in each case. For H_0, in particular, this gives

$$\delta \tilde{H}^{(1)}_{0\alpha\beta} = \sum_{\mu} \left(\tilde{H}_{0\alpha\mu} C_{\beta\mu} - \tilde{H}_{0\mu\beta} C_{\mu\alpha} \right) = \left(\mathcal{E}_\alpha - \mathcal{E}_\beta \right) C_{\beta\alpha}$$

$$= -\tilde{V}_{\alpha\beta}\left(1 - \mathrm{e}^{i\omega_{\beta\alpha}t}\right) \xrightarrow[t\to\infty]{} -\tilde{V}_{\alpha\beta}, \qquad \alpha \neq \beta. \tag{11.40}$$

Thus, for $t \to \infty$ equation (11.28) is recovered, whereas for finite times

$$\delta\tilde{H}^{(1)}_{0\alpha\beta}(t) = \delta\tilde{H}^{(1)}_{0\alpha\beta} + \tilde{V}_{\alpha\beta} = \tilde{V}_{\alpha\beta}\mathrm{e}^{-i\omega_{\alpha\beta}t}, \qquad \alpha \neq \beta. \tag{11.41}$$

From equation (11.40) it also follows that for $\alpha = \beta$, $\delta\tilde{H}^{(1)}_{0\alpha\alpha} = 0$, in agreement with equation (11.31).

Now we are prepared to evaluate the terms entering in equation (11.33). For the first one we get

$$\left\langle \delta H_0^{(1)}; V\right\rangle_{\alpha\alpha} = \sum_\mu \left(\delta H^{(1)}_{0\alpha\mu}\delta\tilde{V}_{\mu\alpha} - \tilde{V}_{\alpha\mu}\delta H^{(1)}_{0\mu\alpha}\right)$$

$$= -\sum_\mu \tilde{V}_{\alpha\mu}\tilde{V}_{\mu\alpha}\left[\left(1 - \mathrm{e}^{i\omega_{\mu\alpha}t}\right) - \left(1 - \mathrm{e}^{i\omega_{\alpha\mu}t}\right)\right], \tag{11.42}$$

whereas for the second one,

$$\left\langle H_0; \delta V^{(1)}\right\rangle_{\alpha\alpha} = \sum_\mu \left(\tilde{H}_{0\alpha\mu}\delta\tilde{V}^{(1)}_{\mu\alpha} - \delta\tilde{V}^{(1)}_{\alpha\mu}H_{0\mu\alpha}\right) = \left(\mathcal{E}_\alpha - \mathcal{E}_\alpha\right)\delta\tilde{V}^{(1)}_{\alpha\alpha} = 0. \tag{11.43}$$

Equation (11.33) gives thus after some simple transformations

$$\delta\tilde{H}^{(2)}_{0\alpha\alpha} = \frac{i}{\hbar}\sum_{\mu\neq\alpha}\tilde{V}_{\alpha\mu}\tilde{V}_{\mu\alpha}\left(-\frac{1}{i\omega_{\alpha\mu}}\right)\frac{1}{2}\left(\left|1 - \mathrm{e}^{i\omega_{\alpha\mu}t}\right|^2 + \left|1 - \mathrm{e}^{-i\omega_{\alpha\mu}t}\right|^2\right)$$

$$= -\sum_{\mu\neq\alpha}\frac{\tilde{V}_{\alpha\mu}\tilde{V}_{\mu\alpha}}{\mathcal{E}_\alpha - \mathcal{E}_\mu}\left|1 - \mathrm{e}^{i\omega_{\alpha\mu}t}\right|^2 \xrightarrow[t\to\infty]{} -\sum_{\mu\neq\alpha}\frac{\tilde{V}_{\alpha\mu}\tilde{V}_{\mu\alpha}}{\mathcal{E}_\alpha - \mathcal{E}_\mu}. \tag{11.44}$$

For arbitrary times we get

$$\delta\tilde{H}^{(2)}_{\alpha\alpha}(t) = \delta\tilde{H}^{(2)}_{0\alpha\alpha} + \delta\tilde{V}^{(1)}_{\alpha\alpha}$$

$$= \frac{1}{2}\sum_{\mu\neq\alpha}\left[\tilde{V}_{\alpha\mu}C_{\alpha\mu}\left(1 + \mathrm{e}^{-i\omega_{\alpha\mu}t}\right) - \tilde{V}_{\mu\alpha}C_{\mu\alpha}\left(1 + \mathrm{e}^{-i\omega_{\mu\alpha}t}\right)\right], \tag{11.45}$$

whereas for $t \to \infty$,

$$\delta\tilde{H}^{(2)}_{\alpha\alpha} = \delta\mathcal{E}^{(2)}_\alpha = \sum_{\mu\neq\alpha}\frac{\tilde{V}_{\alpha\mu}\tilde{V}_{\mu\alpha}}{\mathcal{E}_\alpha - \mathcal{E}_\mu} = \frac{1}{2}\delta\tilde{V}^{(1)}_{\alpha\alpha}, \tag{11.46}$$

with $\delta\tilde{V}^{(1)}_{\alpha\alpha}$ given by equations (11.37) and (11.38) calculated for $t \to \infty$. Formulas (11.32) and (11.46) coincide with the respective expressions of conventional quantum perturbation theory written in terms of matrix elements.

11.1.3. THE MATRIX ELEMENTS AS RESPONSE AMPLITUDES

Using equation (11.16), one can rewrite the first-order correction to a dynamical variable $B(t)$ as

$$\delta B^{(1)}(t) = \frac{1}{i\hbar} \int_0^t dt' \left\langle B(t); H_1(t') \right\rangle , \tag{11.47}$$

where $H_1(t)$ is the perturbation Hamiltonian. A better feeling of the meaning of this formula is gained by recasting it in the language of linear-response theory. In such theory the perturbation Hamiltonian is assumed to be of the form $H_1(t) = -QF(t)$, where the coupling $Q = Q(x,p)$ is a given function of x and p (but not of time) and the excitation F is a function of time only.[3] With this form for the perturbation, one gets

$$\delta B^{(1)}(t) = -\frac{1}{i\hbar} \int_0^t dt' \left\langle B(t); Q(t') \right\rangle F(t') = - \int_0^t dt' \phi_{BQ}(t,t') F(t'), \tag{11.48}$$

where

$$\phi_{BQ}(t,t') = \frac{1}{i\hbar} \langle B(t); Q(t') \rangle \tag{11.49}$$

is the *after-effect* or *response function* of variable B at time t to a perturbation of the (stationary) state applied through Q at a time $t' < t$. Note that we are considering the quantity $\delta B^{(1)}$, rather than its average value as is normally done in linear-response theory. In particular, with $F(t) = F_0 \delta(t - t_1)$ one gets $\delta B^{(1)}(t) = (-F_0/i\hbar) \left\langle B(t); Q(t_1) \right\rangle \theta(t - t_1)$, which shows that the Poissonian (and thus also the commutator) can be considered to be essentially a response function.

For a given state α, the explicit form of the response function (11.49) in terms of matrix elements is, using (10.95),

$$\phi_{BQ\alpha\alpha}(t,t') = \frac{1}{i\hbar} \sum_{\mu} \left(\tilde{B}_{\alpha\mu} \tilde{Q}_{\mu\alpha} e^{i\omega_{\alpha\mu}(t-t')} - \text{c.c.} \right) . \tag{11.50}$$

Note, in particular, that if the coupling factor $Q(x,p)$ is a function of the Hamiltonian (and possibly other integrals of motion) and hence $\tilde{Q}_{\mu\alpha} = \tilde{Q}_{\mu\mu} \delta_{\mu\alpha}$, the response function ϕ is zero.

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[3] An introduction to linear-response theory can be seen, e.g., in Reichl (1980). The two most useful particular forms of excitation $F(t)$ are a periodic force of given frequency (to probe the frequency response), and an infinitely narrow pulse (to probe the transient behaviour of the system).

Let us apply these formulas to our problem and calculate the effect of the perturbation Hamiltonian (11.1) on the dynamical variables. For the calculation of the first-order correction one only needs to take into account the first term of this Hamiltonian. Working as before in the long-wavelength approximation, we set $Q = p_j$ and $F(t) = (-e/mc)A_j(t)$; selecting further $B = x_i$ and using equations (10.73), one gets

$$\delta x_i^{(1)} = -\frac{e}{mc}\int_0^t dt' \phi_{x_i p_j}(t,t') A_j(t'), \tag{11.51}$$

where a summation over the repeated index j is understood, with

$$\phi_{x_i p_j}(t,t') = \frac{1}{i\hbar}\langle x_i(t); p_j(t')\rangle = -\frac{2m}{\hbar}\delta_{ij}\sum_\beta \omega_{\alpha\beta}\left|\tilde{x}^i_{\alpha\beta}\right|^2 \cos\omega_{\alpha\beta}(t-t'). \tag{11.52}$$

For the last equality the field was assumed isotropic, and $\mathbf{p} = m\dot{\mathbf{x}}$ was used for the unperturbed variables, as was done in chapter 9. An integration by parts, with $\mathbf{E} = -(1/c)(\partial\mathbf{A}/\partial t)$, gives now

$$\delta x_i^{(1)}(t) = -\frac{2e}{\hbar}\sum_\beta \left|\tilde{x}^i_{\alpha\beta}\right|^2 \int_0^t ds E_i(t-s)\sin\omega_{\alpha\beta}s. \tag{11.53}$$

In the time-asymptotic limit the time integral is extended to infinity,

$$\delta x_i^{(1)}(t) = -\frac{2e}{\hbar}\sum_\beta \left|\tilde{x}^i_{\alpha\beta}\right|^2 \int_0^\infty ds E_i(t-s)\sin\omega_{\alpha\beta}s. \tag{11.54}$$

Hence within these approximations the system responds selectively — and again linearly— to the field modes of frequencies $\omega_{\alpha\beta}$, with the respective amplitudes proportional to $\left|\tilde{x}^i_{\alpha\beta}\right|^2$; this response may take the system to a different stationary state, as will be discussed in section 11.3. It is clear that, independently of its origin, any external perturbation $F(t)$ that oscillates with one of the characteristic frequencies $\omega_{\alpha\beta}$ can induce a resonant transition of this kind. Note that since these expressions involve the products $|\tilde{x}_{\alpha\beta}|^2 = \tilde{x}_{\alpha\beta}\tilde{x}^*_{\alpha\beta} = \tilde{x}^*_{\beta\alpha}\tilde{x}_{\beta\alpha} = |\tilde{x}_{\beta\alpha}|^2$, the two subindices appear on an equal footing, even if initially they seemed to play asymmetrical roles in equation (11.54). Indeed, the factor $\tilde{x}_{\alpha\beta}$ can be associated to the response that drives the system either from state α to state β, or from state β to state α. The same field modes are involved in both cases, the sign of $\omega_{\alpha\beta}$ being defined by Bohr's formula, equation (10.109), $\hbar\omega_{\alpha\beta} = \mathcal{E}_\alpha - \mathcal{E}_\beta$.

This discussion, which may sound somewhat trivial within the usual framework of the quantum description, is nevertheless important, because it helps us to understand a basic question related to the meaning of the present description, namely, why is it that the stationary states are described in terms of transition amplitudes? As we see, an answer is given by linear-response theory, according to which the response to an external perturbation of a system in equilibrium is described in terms of quantities that refer to the equilibrium situation. Numbers like $\tilde{x}_{\alpha\beta}$ are Fourier amplitudes associated with the resonant responses, which occur exactly at the characteristic frequencies of the previous chapter, and thus they are expected to be very large compared with the nonresonant amplitudes (which are entirely neglected in the quantum-regime description). Since the zeropoint field probes all frequencies, it is natural that the resonant amplitudes appear as the leading Fourier terms for the description of both the stationary state and the periodic perturbations, and that the corresponding frequencies are the ones of central interest. Thus, there is no need to think of the orbiting electron in state α as if it were jumping successively through all possible states λ, μ, ν, ..., to eventually arrive back at α.

11.2. Atomic stability

11.2.1. DETAILED ENERGY BALANCE

In quantum mechanics the existence of stable states is simply accepted as a matter of fact; they are predicted by the Schrödinger equation, and no further inquiry into their genesis is considered necessary. Let us now look at this problem from the viewpoint of linear SED; we shall demonstrate that detailed energy balance holds for the stationary states, thus verifying that the demand of nonrandom characteristic frequencies introduced in chapter 10 is equivalent to detailed balance. We start from the energy equilibrium condition to second order in e, equation (9.5),

$$-m\tau\left\langle\left|\ddot{\mathbf{x}}^{(0)}\right|^{2}\right\rangle+e\left\langle\delta\dot{\mathbf{x}}^{(1)}\cdot\mathbf{E}\right\rangle=0. \tag{11.55}$$

As discussed in §9.2.2, for this condition to hold the first-order contribution to $\langle\dot{\mathbf{x}}\cdot\mathbf{E}\rangle$ must be equal to zero, $\left\langle\dot{\mathbf{x}}^{(0)}\cdot\mathbf{E}\right\rangle=0$ (see equation (9.4)). But this equation is indeed verified, because $\mathbf{E}$ is essentially uncorrelated from the atomic motions, which means that its zeropoint component still corresponds to the free field, as discussed in section 11.1; this stresses the importance of equation (9.4). Coming back to our problem, we assume that the system is in a stationary state α. The calculation of the first term in equation (11.55) is straightforward, taking $B_\alpha = x^i_\alpha$ in equation (11.2); for the second term we use (11.54), and thus obtain for every Cartesian component

$$e\left\langle\delta\dot{x}_i^{(1)}E_i\right\rangle_\alpha=-\frac{4\pi^2e^2}{3\hbar}\sum_\beta\omega_{\alpha\beta}\rho(\omega_{\alpha\beta})\left|\tilde{x}^i_{\alpha\beta}\right|^2, \tag{11.56}$$

where $\rho(\omega)$ is the spectral energy density of the random field; of course, the spectral density is always a function of $|\omega|$, even if this is not usually explicitly stated. For the moment let us assume that all $\omega_{\alpha\beta}$ are negative (thus $\omega_{\beta\alpha}$ positive); the more general case will be considered in section 11.3. Then the energy-balance condition gives

$$\sum_{\beta}\left[-\frac{2e^2}{3c^3}\omega_{\alpha\beta}^4+\frac{4\pi^2e^2}{3\hbar}\left|\omega_{\alpha\beta}\right|\rho(\omega_{\alpha\beta})\right]\left|\tilde{\mathrm{x}}_{\alpha\beta}\right|^2=0. \tag{11.57}$$

Assume further, for simplicity, that the frequencies $\omega_{\alpha\beta}$ are nondegenerate. The equilibrium condition should be satisfied for the stationary state α of any system, or irrespectively of the coefficients $\tilde{x}_{\alpha\beta}$, which means that the expression between brackets must vanish separately for every $\omega_{\alpha\beta}$. This gives for the equilibrium field

$$\rho_0(\omega)=\frac{\hbar\omega^3}{2\pi^2c^3}, \tag{11.58}$$

which is just the spectral energy density of the zeropoint field. Whether or not there is degeneracy, when the zeropoint spectrum (11.58) is inserted into the equilibrium condition each term gives zero separately; hence detailed balance holds for any bounded system described by linear SED. This verifies that the Lorentz-invariant spectrum is the only one that guarantees detailed energy balance in accord with the requirement of universality demanded by Kirchhoff's law. Alternatively, the argument can be seen as a derivation of the zeropoint field spectrum from the requirement of detailed balance.

This result, which stands in sharp contrast with those of chapter 9, is a consequence of the characteristic response of linear SED. If instead of equations (11.51, 11.52) we had used the classical expression for $\delta x_i^{(1)}$,

$$\delta x_i^{(1)}(t)=-\frac{e}{mc}\int_0^t\frac{\partial x_i(t)}{\partial x_j(t')}A_j(t')dt', \tag{11.59}$$

with the propagator $\partial x_i(t)/\partial x_j(t')$ given by classical physics, we would have been led back to the Rayleigh-Jeans law for the equilibrium spectrum, as was the case with the calculations performed within standard SED. This verifies that detailed energy balance holds for the ω^3-spectrum due to the linear response to the field. The mixing of frequencies associated with a nonlinear response not only would destroy detailed balance; it would not allow the system to reach equilibrium with the zeropoint field, as was concluded in chapter 9.

11.2.2. GENERAL BALANCE CONDITION

The balance condition (11.55) that was derived from the average energy rate equation can be extended to more general integrals of the motion of the unperturbed system, as follows. Let ξ be any such integral; then classically to zero order

$$\frac{d\xi}{dt} = \frac{1}{m}\mathbf{p} \cdot \nabla\xi + \mathbf{F} \cdot \nabla_p \xi = 0. \tag{11.60}$$

Multiplying the equation of motion

$$m\ddot{\mathbf{x}} = m\tau \dddot{\mathbf{x}} + \mathbf{F}(\mathbf{x}) + e\mathbf{E}(t) \tag{11.61}$$

by $\nabla_p \xi$ one gets for the perturbed system, after averaging over the realizations of the field,

$$\left\langle \frac{d\xi}{dt} \right\rangle = \langle (m\tau\dddot{\mathbf{x}} + e\mathbf{E}) \cdot \nabla_p \xi \rangle. \tag{11.62}$$

Observe that $\varsigma \equiv \nabla_p \xi$ is itself a dynamical variable, so we assume that its components in a stationary state α can be written to zero order as $(\partial\xi/dp_i)_\alpha \equiv \varsigma^i_\alpha = \sum_\beta \tilde{\varsigma}^i_{\alpha\beta} a_{\alpha\beta}(t)$, with

$$\tilde{\varsigma}^i_{\alpha\beta} = \frac{1}{i\hbar} \langle x_i; \xi \rangle_{\alpha\beta} = \frac{1}{i\hbar} (\xi_\beta - \xi_\alpha) \tilde{x}^i_{\alpha\beta}. \tag{11.63}$$

Their first-order correction can now be computed using equations (11.48) and (11.50) with $Q = p_j$ and $F(t) = -(e/mc)A_j$,

$$\delta\varsigma^{(1)}_i(t) = -\frac{e}{mc} \int_0^t dt' \, \phi_{\varsigma_i p_j}(t, t') A_j(t'), \tag{11.64}$$

where

$$\phi_{\varsigma_i p_j}(t, t') = \frac{1}{i\hbar} \langle \varsigma_i(t); p_j(t') \rangle$$

$$= \frac{2m}{\hbar^2} \delta_{ij} \sum_\beta (\xi_\alpha - \xi_\beta) \left| \tilde{x}^i_{\alpha\beta} \right|^2 \frac{d}{dt'} \cos \omega_{\alpha\beta}(t - t'). \tag{11.65}$$

After an integration by parts one obtains, for $t \to \infty$,

$$\delta\varsigma^{(1)}_i(t) = -\frac{2e}{\hbar^2} \sum_\beta (\xi_\alpha - \xi_\beta) \left| \tilde{x}^i_{\alpha\beta} \right|^2 \int_0^\infty ds \, E_i(t - s) \cos \omega_{\alpha\beta} s. \tag{11.66}$$

Let us now calculate the right-hand side of (11.62) to second order in e. The first term gives for every Cartesian component

$$\langle m\tau \dddot{x}_i \, \varsigma_i \rangle = -\frac{m\tau}{\hbar} \sum_\beta \omega^3_{\alpha\beta} (\xi_\alpha - \xi_\beta) \left| \tilde{x}^i_{\alpha\beta} \right|^2. \tag{11.67}$$

The first-order contribution to the second term is $e\langle E_i\varsigma_i\rangle = e\sum_\beta E_{\alpha\beta}\tilde{\varsigma}^i_{\alpha\beta}$, with $E^i_{\alpha\beta} = (\pi\hbar\omega_{\alpha\beta}/V)^{1/2}$; this contribution goes to zero, as could be anticipated. The second-order contribution is, from (11.66),

$$e\left\langle E_i\delta\varsigma_i^{(1)}\right\rangle = -\frac{2e^2}{\hbar^2}\sum_\beta(\xi_\alpha - \xi_\beta)\left|\tilde{x}^i_{\alpha\beta}\right|^2$$

$$\int_0^\infty ds\ \langle E_i(t)E_i(t-s)\rangle\cos\omega_{\alpha\beta}s. \tag{11.68}$$

After performing the integration and summing over the Cartesian index one gets finally for the equilibrium condition

$$\sum_\beta\left[-\frac{m\tau\omega^3_{\alpha\beta}}{\hbar} - \frac{4\pi^2e^2}{3\hbar^2}\rho(\omega_{\alpha\beta})\right](\xi_\alpha - \xi_\beta)\left|\tilde{\mathrm{x}}_{\alpha\beta}\right|^2 = 0. \tag{11.69}$$

For $\xi = H$ and $\omega_{\alpha\beta} < 0$ this result reduces of course to equation (11.57). For any other constant of the motion for which $\xi_\alpha - \xi_\beta \neq 0$, this leads to the same equilibrium spectral energy density $\rho_0(\omega)$, equation (11.58). Therefore, the zeropoint field with spectral density $\rho_0(\omega)$ guarantees not only detailed energy balance, but more generally, detailed balance of the integrals of motion.

11.2.3. CAUSAL VERSION OF THE SED EQUATION

The SED dynamical equation used in the foregoing sections as a starting point to derive the detailed-balance condition, is plagued with the difficulties associated with the Abraham-Lorentz equation of classical electrodynamics, which give rise to the noncausal behaviour discussed in detail in section 3.3. There it was found that the problem of the noncausality can be solved by replacing the $\dddot{x}$-term with a retarded expression for the self-force that is explicitly causal, of the form

$$m\tau\ \dddot{x}\ (t) \rightarrow \int_0^t dt'\ G(t-t')\dot{x}(t'), \tag{11.70}$$

with an appropriate Green function $G(t)$. If, however, this is done in equation (11.61), the detailed-balance condition (11.69) cannot any longer be derived from it. As is evident from (11.69), the spectrum of the random background field required for detailed balance is determined by the structure of the dissipative force. This seems to imply that a change of the type (11.70) with the purpose of recovering causality would lead either to a loss of detailed balance, or to a modification of the spectrum of the field required to maintain this balance, neither of which seem to be very appealing prospects.

To find a solution to this dilemma, let us repeat here the procedure that was used in chapter 3 in deriving equations (3.128)-(3.130); except for small changes in the notation we have for the self-force

$$\mathbf{F}_{\text{self}} = -\frac{2m\tau}{\pi}\int_0^\infty d\omega\,\omega^2\int_0^t dt'\dot{\mathbf{x}}(t')K(\omega s)\cos\omega(t-t'), \tag{11.71}$$

$$K(\omega s) = -3\left[\frac{\cos\omega s}{(\omega s)^2} - \frac{\sin\omega s}{(\omega s)^3}\right], \tag{11.72}$$

with $s \equiv |\mathbf{x}(t) - \mathbf{x}(t')|/c$. We recall that the dipole approximation implies $s = 0$, whence $K(0) = 1$ and equation (11.71) reduces to the usual form $\mathbf{F}_{\text{self}} = m\tau\,\dddot{\mathbf{x}}$ plus a mass correction. As discussed in section 3.3, by fixing s at a minimum (effective) value $s_0 > 0$ an effective structure of size $s_0 c$ is assigned to the particle, which thus ceases to respond to the modes of frequencies higher than $2\pi s_0^{-1}$. All taken together, this means that one should rewrite equation (11.71) by inserting an *effective form factor*

$$\mathbf{F}_{\text{self}} = -\frac{2m\tau}{\pi}\int_0^\infty d\omega\,\omega^2\int_0^t dt'\dot{\mathbf{x}}(t')K_{\text{eff}}(\omega s_0)\cos\omega(t-t'), \tag{11.73}$$

given by

$$K_{\text{eff}}(\omega s_0) \equiv K(\omega s_0)C(\omega s_0) = -3\left[\frac{\cos\omega s_0}{(\omega s_0)^2} - \frac{\sin\omega s_0}{(\omega s_0)^3}\right]C(\omega s_0) \tag{11.74}$$

with $C(\omega s_0)$ a cutoff function having value 1 for $\omega < 2\pi s_0^{-1}$ and decreasing rapidly enough for $\omega > 2\pi s_0^{-1}$ (one can for instance use the step function $\theta(1 - \omega s_0)$ or a negative exponential function). Equation (11.73) can be rewritten as

$$\mathbf{F}_{\text{self}} = \int_0^t dt' G(t-t')\dot{\mathbf{x}}(t'), \tag{11.75}$$

with

$$G(t) = -\frac{2m\tau}{\pi}\int_0^\infty d\omega\;\omega^2 K_{\text{eff}}(\omega s_0)\cos\omega t. \tag{11.76}$$

Equations (11.73, 11.74) express the effect of the spatial structure of the self-field on the dynamics of the particle.

Now also the random field has a spatial structure, and for consistency it must be assumed that this structure has its consequences on the (effectively) extended particle, whence it is illegitimate to use the dipole approximation for $\mathbf{E}(\mathbf{x}, t)$. Therefore, to keep in line with the above philosophy the term $\mathbf{E}(\mathbf{x}, t)$ will be replaced in the equation of motion by an equivalent field $\mathbf{E}_{\text{eff}}(t)$ with *effective* spectral density given by

$$\rho_{\text{eff}}(\omega) = \rho(\omega)K_{\text{eff}}(\omega s_0), \tag{11.77}$$

where $\rho(\omega)$ is the spectral energy density of the field $\mathbf{E}(\mathrm{x}, t)$. The equation becomes thus

$$m\ddot{\mathrm{x}} = \mathbf{F} + \int_0^t dt' G(t-t')\dot{\mathrm{x}}(t') + e\mathbf{E}_{\text{eff}}(t) \tag{11.78}$$

with $G(t)$ given by (11.76) and with $\mathbf{E}_{\text{eff}}(t)$ having the spectral density (11.77). This is a self-consistent, simplified version of the original Braffort-Marshall equation that is explicitly causal and does not lead to divergent integrals in ω. Note that the random background field has still the spectral density ρ; the expression $\rho_{\text{eff}}(\omega)/\rho(\omega)$ is simply a measure of the effect that this field has on the extended particle.

Now a detailed-balance condition can be derived from this modified equation. For simplicity we refer only to the equation for the energy rate, which is obtained by multiplying (11.78) by $\dot{\mathrm{x}}(t)$ and taking the statistical average; the result is the energy-balance condition

$$\int_0^t dt' G(t-t') \left\langle \dot{\mathrm{x}}^{(0)}(t)\cdot\dot{\mathrm{x}}^{(0)}(t')\right\rangle + e\left\langle \delta\dot{\mathrm{x}}^{(1)}(t)\cdot\mathbf{E}_{\text{eff}}(t)\right\rangle = 0, \tag{11.79}$$

instead of equation (11.55). Using equations (11.76) and (11.54) with $\mathbf{E}$ replaced by $\mathbf{E}_{\text{eff}}$ one obtains on integrating over t' and ω in the time-asymptotic limit

$$\sum_\beta \left[-m\tau\omega_{\alpha\beta}^4 - \frac{4\pi^2 e^2}{3\hbar}\left|\omega_{\alpha\beta}\right| \rho(\omega_{\alpha\beta})\right] K_{\text{eff}}(\omega_{\alpha\beta}s_0)\left|\tilde{\mathrm{x}}_{\alpha\beta}\right|^2 = 0. \tag{11.80}$$

Hence the spectral energy density of the field required to maintain detailed balance is still given by equation (11.58), corresponding to the zeropoint field. Since s_0^{-1} is certainly much larger than the usual atomic frequencies $\omega_{\alpha\beta}$, in practice one may use in equation (11.80) the approximate value $K_{\text{eff}}(\omega_{\alpha\beta}s_0 << 1) \simeq 1$, which leads us back to equation (11.57). The conclusion is that also in the quantum world, the harm due to the noncausal structure of the Abraham-Lorentz equation is of theoretical concern rather than of practical importance.

11.2.4. DETAILED FLUCTUATION-DISSIPATION RELATIONS

An interesting extension of the previous discussion is to the case where there are material objects in the vicinity of the particle, that may affect the structure of the surrounding radiation field. Let us once more assume that the particle is in a stationary state α. We start by constructing a stress tensor D_{ij} that is a generalization to coordinates i, j of the last term of equation (11.55), representing the power absorbed by the particle.

From equations (11.48) and (11.49) we have (the index α is omitted and a summation over the repeated index k is understood)

$$D_{ij} = e\langle E_i(t)\delta p_j^{(1)}(t)\rangle = -e^2 \int_0^t dt' \phi_{p_j x_k}(t,t')\langle E_i(t)E_k(t')\rangle \tag{11.81}$$

with

$$\phi_{p_j x_k}(t,t') = \frac{1}{i\hbar}\langle p_j(t); x_k(t')\rangle. \tag{11.82}$$

In the Fokker-Planck equation this tensor D is identified as the pp—diffusion coefficient, as follows from equation (7.46).

Now we use the general development given in equation (3.37), to allow for the presence of macroscopic bodies and corresponding boundary conditions, so that the zeropoint electric field has the form

$$\mathbf{E}(\mathbf{x},t) = i\sum_\beta \sqrt{\pi\hbar\omega_\beta}a_\beta(t)\mathbf{G}_\beta(\mathbf{x}) + \text{c.c.} \tag{11.83}$$

and the two-time correlation is

$$\langle E_i(t)E_k(t')\rangle = \pi\hbar\sum_\beta \omega_\beta G_{i\beta}(\mathbf{x})G^*_{k\beta}(\mathbf{x})\exp i\omega_\beta(t-t') + \text{c.c.}, \tag{11.84}$$

assuming that the particle does not move appreciably in the time interval $(t-t')$, so that $\mathbf{x}(t') \simeq \mathbf{x}(t)$; this approximation holds as long as the wavelengths of the relevant field modes are large compared with $|\mathbf{x}(t') - \mathbf{x}(t)|$. Notice that even when the geometrical distribution of the field modes changes, their average energy is still given by $\mathcal{E}_\beta = \frac{1}{2}\hbar\omega_\beta$, as can be ascertained by integrating equation (11.84) for $t = t'$ (and a similar expression for the square average of the magnetic field component) over all space.

With (11.84), equation (11.81) takes the form

$$D_{ij} = -\pi\hbar e^2\sum_\beta \omega_\beta G_{i\beta}(\mathbf{x})G^*_{k\beta}(\mathbf{x})\int_0^t \phi_{p_j x_k}(t,t')\exp i\omega_\beta(t-t')dt' + \text{c.c.} \tag{11.85}$$

A similar description can be used for the field radiated by the particle, since it is subject to the same boundary conditions; as in equation (7.166) we write

$$E_i^{\text{rad}} = -2\pi e\sum_\beta G_{i\beta}(\mathbf{x})G^*_{k\beta}(\mathbf{x})\int_0^t \dot{x}_k(t')\exp i\omega_\beta(t-t')dt' + \text{c.c.} \tag{11.86}$$

To the present order of approximation one can use for $\dot{x}_k(t')$ the zero-order solution, so that

$$\dot{x}_k(t') = \frac{1}{i\hbar}\langle x_k(t'); H\rangle = \frac{1}{i\hbar}\langle x_k(t'); p_j\rangle\frac{p_j}{m} + \frac{1}{i\hbar}\langle x_k(t'); x_j\rangle\frac{\partial V}{\partial x_j}$$

$$= -\phi_{p_j x_k}(t,t')\dot{x}_j(t) + \phi_{x_j x_k}(t,t')m\ddot{x}_j(t). \tag{11.87}$$

Introducing this into the previous expression one arrives at the following result for the Lorentz electric force due to self-radiation:

$$eE_i^{\text{rad}} = -\gamma_{ij}\dot{x}_j - \delta m_{ij}\ddot{x}_j, \tag{11.88}$$

where

$$\gamma_{ij} = -2\pi e^2 \sum_\beta G_{i\beta}(\mathrm{x})G_{k\beta}^*(\mathrm{x}) \int_0^t \phi_{p_j x_k}(t,t')\exp i\omega_\beta(t-t')dt' + \text{c.c.}, \tag{11.89}$$

$$\delta m_{ij} = 2\pi m e^2 \sum_\beta G_{i\beta}(\mathrm{x})G_{k\beta}^*(\mathrm{x}) \int_0^t \phi_{x_j x_k}(t,t')\exp i\omega_\beta(t-t')dt' + \text{c.c.} \tag{11.90}$$

The δm_{ij} represent an electromagnetic mass correction, whereas the γ_{ij} are the components of a dissipation tensor; for the harmonic oscillator they coincide with equations (7.168). There is a striking similarity between equations (11.85) and (11.89), which suggests writing

$$\gamma_{ij} = \sum_\beta \tilde{\gamma}_{ij}(\omega_\beta, t), \quad D_{ij} = \sum_\beta \tilde{D}_{ij}(\omega_\beta, t), \tag{11.91}$$

with

$$\tilde{\gamma}_{ij}(\omega_\beta, t) = -2\pi e^2 G_{i\beta}(\mathrm{x})G_{k\beta}^*(\mathrm{x}) \int_0^t \phi_{p_j x_k}(t,t')\exp i\omega_\beta(t-t') + \text{c.c.}, \tag{11.92}$$

whence

$$\tilde{D}_{ij}(\omega_\beta, t) = \frac{1}{2}\hbar\omega_\beta\tilde{\gamma}_{ij}(\omega_\beta, t) \tag{11.93}$$

or, inserting $\mathcal{E}(\omega) = \hbar\omega/2$,

$$\tilde{D}_{ij}(\omega) = \mathcal{E}(\omega)\tilde{\gamma}_{ij}(\omega). \tag{11.94}$$

Here we have a detailed fluctuation-dissipation relation similar to the one met before in chapter 7, equation (7.171) and in chapter 8, equation (8.83), i.e., a relation of proportionality between the diffusion coefficient $\tilde{D}_{ij}$ and the friction coefficient $\tilde{\gamma}_{ij}$ for each frequency ω, with the factor of proportionality given in this case by the average energy of the corresponding random field mode.[4] As can be seen from its derivation, a similar result holds for classical linear systems; however, in classical theory energy equipartition holds, so that $\mathcal{E} = kT$ for any ω. Equation (11.94) reads then $\tilde{D}_{ij} = kT\tilde{\gamma}_{ij}$, so that one can sum over the frequency, which leads to the well-known Einstein relation for the tensors D and γ, $D = \gamma kT$ [see, e.g., Papoulis 1965, chapter 15].

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[4] The theory of the fluctuation-dissipation theorem can be found, e.g., in Reichl (1980), chapter 15. The general relations were introduced in Callen and Welton (1951), although particular instances were known already in the theory of Brownian motion since the early work of Einstein in 1905. Similar results are known in nonrelativistic QED [see, e.g., Milonni 1988 or 1994, section 7.3].

A most interesting feature of equation (11.93) is that it holds whatever the configuration; the boundary conditions on the field affect the structure of both the damping and the diffusion terms precisely in the same way, so that the equilibrium condition is maintained. This result implies that the steady-state solutions of linear SED in the radiationless approximation —which are equivalent to the stationary solutions of quantum mechanics— are stable against environmental perturbations. This is an extension of the similar result found above and of the one found in connection with the harmonic oscillator in §7.4.4, where its importance and meaning were discussed. It is evident from equation (11.93) that what is crucial for detailed balance to hold is not the ω^3-spectrum of the zeropoint field, but its energy per normal mode $\sim \omega$. In free space these two conditions are equivalent; in a cavity the density of modes is of course modified and becomes in general inhomogeneous, but the energy per mode is still $\frac{1}{2}\hbar\omega$, as long as only the zeropoint field is present.

To complete the section we add a couple of simple results, this time related to the xp–diffusion coefficient (see equation (7.47)),

$$d_{ij} = e\langle E_i x_j\rangle = -e^2 \int_0^t dt' \phi_{x_j x_k}(t,t')\langle E_i(t)E_k(t')\rangle \tag{11.95}$$

the trace of which represents the average work done on the particle by the random force. The calculation is similar to that of D_{ij} and gives

$$d_{ij} = -\pi\hbar e^2 \sum_\beta \omega_\beta G_{i\beta}(\mathrm{x})G^*_{k\beta}(\mathrm{x}) \int_0^t \phi_{x_j x_k}(t,t') \exp i\omega_\beta(t-t')dt' + \text{c.c.} \tag{11.96}$$

Let us now write, in analogy with (11.91),

$$\delta m_{ij} = \sum_\beta \delta\tilde{m}_{ij}(\omega_\beta, t), \qquad d_{ij} = \sum_\beta \tilde{d}_{ij}(\omega_\beta, t); \tag{11.97}$$

then a comparison of equations (11.90) and (11.96) gives

$$\tilde{d}_{ij} = -\frac{1}{2}\hbar\omega\frac{\delta\tilde{m}_{ij}}{m} = -\mathcal{E}(\omega)\frac{\delta\tilde{m}_{ij}}{m}. \tag{11.98}$$

The coefficients of the mass correction terms are therefore proportional to those of the cross-diffusion tensor, the factor of proportionality being again the average energy per mode. At the end of next section a further relation involving the tensor $\tilde{d}$ will be derived, in connection with the Lamb shift.

11.3. Radiative transitions

According to the results of the foregoing sections, the average power exchanged between the particle in a state α and the radiation field is given (in free space, for simplicity) by

$$\left\langle \frac{dH}{dt} \right\rangle = -m\tau \sum_{\beta} \left[\omega_{\alpha\beta}^4 + \frac{2\pi^2 c^3}{\hbar} \omega_{\alpha\beta}\rho(\omega_{\alpha\beta}) \right] |\tilde{\mathrm{x}}_{\alpha\beta}|^2 . \tag{11.99}$$

It was seen, in particular, that equilibrium is maintained if the spectral energy density ρ is that of the zeropoint field. To arrive at this conclusion only negative values of $\omega_{\alpha\beta}$ were included in the sum, to guarantee that the second term within the parentheses of equation (11.99) is always negative and cancels the first term, which is always positive. Since the sign of $\omega_{\alpha\beta}$ is determined by the difference of energies of states α and β according to Bohr's law, we see that this assumption implies $\mathcal{E}_\beta > \mathcal{E}_\alpha$ for all β; in other words, the system must be in its ground state for detailed energy balance to hold. In the next two sections we shall study what happens when these conditions are lifted.

11.3.1. EINSTEIN *A* AND *B* COEFFICIENTS

In this section the application of equation (11.99) is extended in two directions. On the one hand the possibility of positive values of $\omega_{\alpha\beta}$ is considered, with the purpose of studying the energy balance when the system is in an excited state. Secondly, it is assumed that the system is subject to an arbitrary external electromagnetic field in addition to the zeropoint field, so that the total spectral energy density is $\rho = \rho_0 + \rho_e > \rho_0$. The results of these calculations represent therefore a generalization of those obtained in §7.5.1 for the harmonically bound particle, to a general bounded problem with external excitation. At first sight a derivation of the Einstein coefficients could be seen as superfluous, because the *B* coefficients can be obtained from quantum mechanics, which is already part of linear SED anyway, and the *A* coefficient for spontaneous emission can be obtained simultaneously with the Lamb shift formula, as will be shown in the next section. However, we prefer to perform the derivation to stress the fact that they are a natural part of the theory and are applicable to any bound system.

We start by recasting equation (11.99) in the slightly different form

$$\left\langle \frac{dH}{dt} \right\rangle = -\frac{4\pi^2 e^2}{3\hbar} \sum_{\beta} \omega_{\alpha\beta} \left[\rho + \operatorname{sign}(\omega_{\alpha\beta})\, \rho_0\right] |\tilde{\mathrm{x}}_{\alpha\beta}|^2 , \tag{11.100}$$

and separate positive from negative components, adding a superindex $(\pm)$ to $\tilde{x}_{\alpha\beta}$ to keep track of this sign,

$$\left\langle \frac{dH}{dt} \right\rangle = -\frac{4\pi^2 e^2}{3\hbar} \sum_{\beta} \omega_{\alpha\beta} \left[(\rho - \rho_0) \left| \tilde{\mathrm{x}}_{\alpha\beta}^{(-)} \right|^2 + (\rho + \rho_0) \left| \tilde{\mathrm{x}}_{\alpha\beta}^{(+)} \right|^2 \right]. \quad (11.101)$$

This can be rewritten as

$$\left\langle \frac{dH}{dt} \right\rangle = W_{\mathrm{ab}} - W_{\mathrm{em}}, \quad (11.102)$$

where

$$W_{\mathrm{ab}} = \frac{4\pi^2 e^2}{3\hbar} \sum_{\omega_{\alpha\beta}<0} |\omega_{\alpha\beta}| (\rho - \rho_0) \left| \tilde{\mathrm{x}}_{\alpha\beta}^{(-)} \right|^2 \quad (11.103)$$

and

$$W_{\mathrm{em}} = \frac{4\pi^2 e^2}{3\hbar} \sum_{\omega_{\alpha\beta}>0} \omega_{\alpha\beta} (\rho + \rho_0) \left| \tilde{\mathrm{x}}_{\alpha\beta}^{(+)} \right|^2 \quad (11.104)$$

are the contributions of the absorptions and emissions to the energy change. From (11.103) it is clear that absorptions occur only for $\rho - \rho_0 = \rho_e > 0$, which means that only absorptions induced by an external field can take place, and one can write

$$W_{\mathrm{ab}}^{\mathrm{ind}} = \frac{4\pi^2 e^2}{3\hbar} \sum_{\omega_{\alpha\beta}<0} |\omega_{\alpha\beta}| \rho_e \left| \tilde{\mathrm{x}}_{\alpha\beta}^{(-)} \right|^2. \quad (11.105)$$

In particular, we see that if only the zeropoint field is present there are no absorptions; in other words, 'spontaneous absorptions' do not occur,

$$W_{\mathrm{ab}}^{\mathrm{spont}} = 0. \quad (11.106)$$

On the other hand, equation (11.104) gives for the emissions stimulated by the external field, with $\rho = \rho_0 + \rho_e$,

$$W_{\mathrm{em}}^{\mathrm{ind}} = \frac{4\pi^2 e^2}{3\hbar} \sum_{\omega_{\alpha\beta}>0} \omega_{\alpha\beta} \rho_e \left| \tilde{\mathrm{x}}_{\alpha\beta}^{(+)} \right|^2, \quad (11.107)$$

and for the spontaneous emissions,

$$W_{\mathrm{em}}^{\mathrm{spont}} = \frac{8\pi^2 e^2}{3\hbar} \sum_{\omega_{\alpha\beta}>0} \omega_{\alpha\beta} \rho_0 \left| \tilde{\mathrm{x}}_{\alpha\beta}^{(+)} \right|^2. \quad (11.108)$$

Since each of the individual absorptions and emissions implies a change from state α to a state β and a concomitant gain or loss of energy given by equation (10.109), $\Delta\mathcal{E}_{\alpha\beta} = \mathcal{E}_\alpha - \mathcal{E}_\beta = \hbar\omega_{\alpha\beta}$, we can introduce the familiar relations

$$W_{\rm ab} = \sum \Delta\mathcal{E}\left(B^{(-)}\rho_e + A^{(-)}\right), \tag{11.109}$$

$$W_{\rm em} = \sum \Delta\mathcal{E}\left(B^{(+)}\rho_e + A^{(+)}\right), \tag{11.110}$$

and thus obtain by comparison with equations (11.105)-(11.108) the following formulas for the Einstein coefficients:

$$B^{(-)}_{\alpha\beta} = \frac{4\pi^2 e^2}{3\hbar^2}\left|\tilde{\mathrm{x}}^{(-)}_{\alpha\beta}\right|^2, \quad B^{(+)}_{\alpha\beta} = \frac{4\pi^2 e^2}{3\hbar^2}\left|\tilde{\mathrm{x}}^{(+)}_{\alpha\beta}\right|^2; \tag{11.111}$$

$$A^{(-)}_{\alpha\beta} = 0, \quad A^{(+)}_{\alpha\beta} = \frac{8\pi^2 e^2}{3\hbar^2}\rho_0\left|\tilde{\mathrm{x}}^{(+)}_{\alpha\beta}\right|^2. \tag{11.112}$$

Note that if the causal expressions (11.80) had been used, these coefficients would all be multiplied by the correction factor $K_{\rm eff}(\omega_{\alpha\beta}s_0)$, meaning that the atomic systems do not emit or absorb radiation of very high frequencies, $\omega_{\alpha\beta} > 2\pi s_0^{-1}$.

These results are an appropriate generalization to arbitrary bound systems of the results obtained for the harmonic oscillator in §7.5.1, so that we refer the reader to the general comments and conclusions contained in that section. Let us just add a couple of remarks. Firstly, just as was the case with the harmonic oscillator and occurs also in QED [see, e.g., Milonni 1976, 1994 and references therein], the factor 2 that appears in equation (11.108) is due to two effects contributing equally to spontaneous emission, namely, the fluctuations due to the zeropoint field and the radiation reaction. In the formula for $W_{\rm ab}^{\rm spont}$ these two terms appear with opposite signs and cancel each other; the system is stable against spontaneous absorptions, as mentioned above. Equation (11.103) —or (11.106)— provides an answer to the two basic questions concerning atomic stability, namely: i) why does the atomic electron not collapse toward the nucleus?, and ii) why is the atom not excited by the fluctuations of the zeropoint field? The two questions have a common answer: atom and zeropoint field reach a stable situation in which the average effects of the fluctuations and those of dissipation counterbalance each other exactly.

To give an estimate of the order of magnitude of the results, observe that the natural width Γ of an excited atomic level can be written in terms of the leading contribution to (11.112) as $\Gamma \sim \tau\omega^2(2ma^2\omega/\hbar)$, where $\omega = \omega_{\alpha\beta}$ and $|\tilde{x}^{(-)}_{\alpha\beta}|^2 = a^2$. Usually, for not too highly excited states the factor $2ma^2\omega/\hbar$ is of order unity (for instance, for the harmonic oscillator $2ma^2\omega/\hbar = n$, where n indicates the level of the excited state), so that $\Gamma \sim \tau\omega^2$. Hence in a time interval T of the order of $(\tau\omega^2)^{-1}$ a spontaneous transition occurs with non-negligible probability. Considering $2\pi/\omega$ as a representative period of the orbital motions, we find that a large number $N \sim \omega/\tau\omega^2 = 1/\tau\omega$ of orbital periods are contained in T; for radiation in the visible zone of the spectrum one has $N \sim 10^8$. Hence the atomic stationary states are quite stable under such conditions, having lifetimes of millions of such atomic periods. This fits quite well with the observation that the stationary states correspond to (relatively) highly stable solutions. In other words, the excited atoms embedded in the zeropoint field may undergo spontaneous transitions to the less excited levels, but this process is so slow that such states can be considered as 'truly' stationary in a zero order approximation (the one used in quantum mechanics). Of course, for highly excited bound systems having a very short natural lifetime, this language of transitions between stationary states may not be so appropriate anymore.

As a final point, recall that in chapter 5 the Planck distribution was shown to be derivable either from the condition of energy conservation, or from linear momentum conservation, because both demands lead to the same expressions for the Einstein coefficients. In fact, we could have started from a more general balance equation than (11.99) and followed the procedure used above, to study the average exchange of a dynamical variable ξ during the transitions, where ξ is any constant of the motion of the unperturbed stationary states. By writing instead of equations (11.109) and (11.110)

$$\left\langle \frac{d\xi}{dt} \right\rangle_{\mathrm{ab}} = \sum \Delta\xi \left(B^{(-)} \rho_e + A^{(-)} \right), \tag{11.113}$$

$$\left\langle \frac{d\xi}{dt} \right\rangle_{\mathrm{em}} = \sum \Delta\xi \left(B^{(+)} \rho_e + A^{(+)} \right), \tag{11.114}$$

one obtains for the A and B coefficients the results already reported in equations (11.111), (11.112). This shows that the Einstein coefficients can be derived, for instance, from a study of the equilibrium of the average square of the angular momentum in central problems.

11.3.2. DETAILED BALANCE FOR AN EXCITED SYSTEM

It was mentioned at the beginning of this section that the condition of detailed balance holds only for a system in its ground state, because only then is the second term within the parentheses of equation (11.99) negative and can counterbalance the first term for each and every frequency. This is true in general; now let us take a closer look at equation (11.99), with the purpose of inquiring about the possibility of existence of detailed balance for excited systems under specific conditions. With this aim we rewrite the equation in the equivalent form

$$\left\langle \frac{dH}{dt} \right\rangle_\alpha = \frac{4\pi^2 e^2}{3\hbar} \sum_\beta |\omega_{\alpha\beta}| \left[\rho_e \left| \tilde{\mathrm{x}}_{\alpha\beta}^{(-)} \right|^2 - (2\rho_0 + \rho_e) \left| \tilde{\mathrm{x}}_{\alpha\beta}^{(+)} \right|^2 \right]. \qquad (11.115)$$

This expression vanishes under any one of the following three conditions.

a) $\rho_e = 0$ and $\left| \tilde{\mathrm{x}}_{\alpha\beta}^{(+)} \right| = 0$ for all β; both particle and field are in their ground states, as has been already discussed, and there are no transitions at all.

b) The particle is in an excited state α such that to every term in the sum with $\omega_{\alpha\beta}^{(+)} > 0$ there corresponds a term with $\omega_{\alpha\beta}^{(-)} < 0$, with the same absolute value for $\omega_{\alpha\beta}$, and with

$$\rho_e \left| \tilde{\mathrm{x}}_{\alpha\beta}^{(-)} \right|^2 = (2\rho_0 + \rho_e) \left| \tilde{\mathrm{x}}_{\alpha\beta}^{(+)} \right|^2. \qquad (11.116)$$

In this case there are upward and downward transitions, with the same frequency. Observe, however, that for (11.116) to be fulfilled, the condition $\left| \omega_{\alpha\beta}^{(+)} \right| = \left| \omega_{\alpha\beta}^{(-)} \right|$ must hold for every α and β, which means that $|\omega_{\alpha\beta}|$ must be constant, so that the energy levels of all excited states accessible via a transition must be equally spaced. This is known to happen only for the harmonic oscillator, for which $|\omega_{\alpha\beta}| = \omega_0$. For the one-dimensional oscillator the nonzero matrix elements are

$$\left| \tilde{x}_n^{(-)} \right|^2 = \frac{\hbar(n+1)}{2\omega_0}, \qquad \left| \tilde{x}_n^{(+)} \right|^2 = \frac{\hbar n}{2\omega_0}, \qquad (11.117)$$

whence condition (11.116) reduces to

$$\rho = \rho_0 + \rho_e = (2n+1)\rho_0. \qquad (11.118)$$

According to this result, detailed balance can exist between an oscillator in the excited state n and a field with spectral density given by $(2n+1)\rho_0$. However, in contrast to case (a), here there are transitions to other states. To get a clearer image of the situation, consider an ensemble of oscillators, all initially in state n. Some oscillators will make an upward transition to state $n+1$, while others will go to state $n-1$, and since neither of these states is balanced by the field described by (11.118), the equilibrium breaks down. In other words, no system in an isolated excited state can be maintained in equilibrium with the help of any radiation field.

The only way of maintaining equilibrium is by having a distribution of equivalent systems in different energy states, embedded in the appropriate field so that upward and downward transitions can take place from the various levels at a constant rate. The condition of detailed balance means then that for every pair of levels (α, β) connected by a transition, the rate of transitions from α to β must be equal to the rate of transitions from β to α. As is well known, for a canonical ensemble of systems, which represents a distribution in thermodynamical equilibrium, this condition leads to Planck's formula for the radiation spectrum. This is easily checked by considering equation (11.115) applied to just these two levels, with relative populations $C_\alpha = \exp(-\mathcal{E}_\alpha/kT)$ and $C_\beta = \exp(-\mathcal{E}_\beta/kT)$; choosing $\mathcal{E}_\alpha < \mathcal{E}_\beta$ and multiplying (11.115) alternately by C_α and C_β, one gets from equating the rates of upward and downward transitions between these two states

$$e^{-\mathcal{E}_\alpha/kT}\rho_e \left|\tilde{\mathrm{x}}_{\alpha\beta}\right|^2 = e^{-\mathcal{E}_\beta/kT}(2\rho_0 + \rho_e)\left|\tilde{\mathrm{x}}_{\beta\alpha}\right|^2 \tag{11.119}$$

and since $\left|\tilde{\mathrm{x}}_{\alpha\beta}\right|^2 = \left|\tilde{\mathrm{x}}_{\beta\alpha}\right|^2$, this leads to the blackbody formula

$$\rho = \rho_0 + \rho_e = \rho_0 \cosh(\hbar\omega_{\beta\alpha}/2kT) \tag{11.120}$$

for the equilibrium spectral density. In conclusion, the only ensemble of excited systems in equilibrium for which detailed balance can be maintained, is a canonical ensemble embedded in the blackbody radiation field.

c) A very singular form of maintaining a system in an excited state α is of course to enclose it in a cavity that has just the right geometry and dimensions to exclude all field modes of the frequencies corresponding to the downward transitions, so that the system cannot reach (via a radiative transition, at least) the lower-lying states. In equation (11.115) this situation would be expressed by the exclusion from the sum of the modes corresponding to downward transitions $(\omega_{\alpha\beta} > 0)$, so that α appears as an 'effective' ground state, and the system can be maintained in equilibrium with the 'enclosed' zeropoint field.

11.4. Lamb shift of atomic levels

Let us now apply the perturbative treatment of section 11.1 to the study of the radiative corrections to the atomic energy levels. The perturbation Hamiltonian H_1 is given by equation (11.1), and contains a first-order and a second-order term. Hence to first order in e the correction to the average energy of state α is given according to equation (11.32) by

$$\delta\tilde{H}^{(1)}_{\alpha\alpha} = \tilde{V}_{\alpha\alpha} = -\frac{e}{mc}\left(\mathbf{p}\cdot\mathbf{A}\right)_{\alpha\alpha}. \tag{11.121}$$

As was already discussed, the zero-order components of the momentum and the perturbing zeropoint field are uncorrelated, so that for every Cartesian component $\langle A_i p_i\rangle_{\alpha\alpha} = 0$, exactly as happened in §11.2.1 in connection with the first-order contribution to the energy-balance equation. As a result we get $\delta\tilde{H}^{(1)}_{\alpha\alpha} = 0$, and we must go over to a second-order calculation.

From equations (11.1), (11.23) and (11.46) one has to second order in e

$$\delta\tilde{H}^{(2)}_{\alpha\alpha} = \frac{1}{2}\delta\tilde{V}^{(1)}_{\alpha\alpha} + \frac{e^2}{2mc^2}\left\langle \mathbf{A}^2 \right\rangle . \tag{11.122}$$

As already discussed in chapter 7, the contribution of the second term is independent of the atomic state, hence it does not affect the frequency spectrum, although it does contribute to the self-energy of the atomic electron, just as is the case in QED. For the calculation of the first term we use equation (11.37) and write

$$\delta\tilde{V}^{(1)}_{\alpha\alpha} = \frac{1}{i\hbar}\left(\frac{e}{mc}\right)^2 \sum_{\mu} \int_0^t dt' \langle A_i(t)A_j(t')\rangle$$

$$\times \left(\tilde{p}^{i}_{\alpha\mu}\tilde{p}^{j}_{\mu\alpha} \exp i\omega_{\alpha\mu}(t-t') - \text{c.c.}\right) . \tag{11.123}$$

For the zeropoint field we know that

$$\langle A_i(t)A_j(t')\rangle = \frac{2\hbar}{3\pi c}\delta_{ij}\int_0^\infty d\omega\omega\cos\omega(t-t'), \tag{11.124}$$

so that

$$\delta\tilde{V}^{(1)}_{\alpha\alpha} = \frac{4e^2}{3\pi m^2c^3}\sum_{\mu}|\tilde{\mathbf{p}}_{\alpha\mu}|^2\int_0^t dt'\int_0^\infty d\omega\,\omega\cos\omega(t-t')\sin\omega_{\alpha\mu}(t-t'). \tag{11.125}$$

To define the time integrals for large times we add a factor $\exp(-\epsilon t)$ to the integrand, as was done before (§11.1.2); at the end we take the limit $\epsilon \longrightarrow 0$. The double integral then becomes, for $t \longrightarrow \infty$,

$$I \equiv \frac{1}{2}\int_0^\infty d\omega\,\omega\left(\frac{1}{\omega+\omega_{\alpha\mu}} - \frac{1}{\omega-\omega_{\alpha\mu}}\right)$$

$$= -\frac{1}{2}\omega_{\alpha\mu}\int_0^{\omega_c} d\omega\left(\frac{1}{\omega+\omega_{\mu\alpha}} + \frac{1}{\omega-\omega_{\mu\alpha}}\right). \tag{11.126}$$

The cutoff has been introduced in the last integral to avoid a logarithmic divergency. The real part of the integral gives

$$I = -\omega_{\alpha\mu}\ln\left|\frac{\omega_c}{\omega_{\mu\alpha}}\right| ; \tag{11.127}$$

the imaginary part, which is related to the Einstein A coefficient, is calculated below. Thus,

$$\delta\tilde{V}^{(1)}_{\alpha\alpha} = -\frac{4e^2}{3\pi m^2c^3}\sum_{\mu}|\tilde{\mathbf{p}}_{\alpha\mu}|^2\,\omega_{\alpha\mu}\ln\left|\frac{\omega_c}{\omega_{\mu\alpha}}\right| . \tag{11.128}$$

To simplify this expression a standard procedure suggested by Bethe (1947) can be used; it consists in substituting the logarithmic term by its average value (to be calculated numerically), $\ln|\omega_c/\omega_{\mu\alpha}| \longrightarrow \ln|\omega_c/\bar{\omega}|$, which can then be extracted from the sum. This leads to

$$\delta\tilde{V}^{(1)}_{\alpha\alpha} = -\frac{4e^2}{3\pi\hbar m^2 c^3}\ln\left|\frac{\omega_c}{\bar{\omega}}\right|\sum_\mu |\tilde{\mathbf{p}}_{\alpha\mu}|^2 (E_\alpha - E_\mu)\,. \tag{11.129}$$

One thus gets for the first right-hand-side term of equation (11.122) the well-known Bethe formula

$$\delta E_{\text{Lamb}} = \frac{1}{2}\delta\tilde{V}^{(1)}_{\alpha\alpha} = -\frac{2e^2}{3\pi\hbar m^2 c^3}\ln\left|\frac{\omega_c}{\bar{\omega}}\right|\sum_\mu |\tilde{\mathbf{p}}_{\alpha\mu}|^2 (E_\alpha - E_\mu)\,. \tag{11.130}$$

As was the case with the equivalent calculation for the harmonic oscillator, the only divergence that arises in this calculation (before the introduction of the cutoff) is the logarithmic one; the absence of stronger divergences is due to the fact that the two terms contributing to the integral in equation (11.126) cooperate to transform the variable factor ω in the numerator of the integrand into the constant $\omega_{\alpha\mu}$. This simplification does not occur in the usual nonrelativistic QED calculation, where it is necessary to subtract the free-particle contribution to get rid of the linear divergence. All this is already clearly discussed by Bethe (1947).

To avoid an oversimplification, instead of the result given in equation (11.127) one should write $\ln(\omega_c/\omega_{\mu\alpha}) = \ln[|\omega_c/\omega_{\mu\alpha}|\exp(-i\pi)]$ for $\omega_{\mu\alpha} < 0$ (or $\omega_{\alpha\mu} > 0$), which means that there is an imaginary contribution to $\delta\tilde{V}^{(1)}_{\alpha\alpha}$ that was omitted in writing equation (11.127). This gives[5]

$$\operatorname{Im}\left(\delta\tilde{E}_{\alpha\alpha}\right) = \operatorname{Im}\left(\frac{1}{2}\delta\tilde{V}^{(1)}_{\alpha\alpha}\right) = \frac{2e^2}{3m^2c^3}\sum_\mu \left|\tilde{\mathbf{p}}^{(+)}_{\alpha\mu}\right|^2 \omega_{\alpha\mu}. \tag{11.131}$$

Hence in the expression $B_\alpha = \sum_\beta \tilde{B}_{\alpha\beta}a^0_{\alpha\beta}\exp[(iE_\alpha + (iE_\beta)^*)t/\hbar]$, the diagonal elements $\tilde{B}_{\alpha\alpha}$ become multiplied by $\exp[-2\operatorname{Im}(\delta\tilde{E}_{\alpha\alpha})t/\hbar]$, which means that their values decay spontaneously at a rate given by $\Gamma = 2\operatorname{Im}(\delta\tilde{E}_{\alpha\alpha})/\hbar$. This gives for the Einstein A coefficient corresponding to the $\alpha \longrightarrow \mu$ spontaneous transition the expression

$$A^{(+)}_{\alpha\mu} = \frac{4e^2}{3\hbar c^3}\left|\tilde{\mathbf{x}}^{(+)}_{\alpha\mu}\right|^2 \omega^3_{\alpha\mu}, \tag{11.132}$$

which coincides with the previous result, second equation (11.112) with ρ_0 given by equation (11.58).

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[5] The corresponding quantum calculation can be found in Sakurai (1967).

Before closing this section let us go back to equation (11.123) and perform a double integration by parts on t', to get

$$\delta\tilde{V}^{(1)}_{\alpha\alpha} = -\frac{e^2}{i\hbar}\sum_{\mu}\int_0^t dt' \langle E_i(t)E_j(t')\rangle \left(\tilde{x}^i_{\alpha\mu}\tilde{x}^j_{\mu\alpha}\exp i\omega_{\alpha\mu}(t-t') - \text{c.c.}\right)$$

or, using equation (11.49),

$$\delta\tilde{V}^{(1)}_{\alpha\alpha} = -e^2\int_0^t dt' (\phi_{x_i x_j})_{\alpha\alpha}(t,t')\langle E_i(t)E_j(t')\rangle, \tag{11.133}$$

where a summation over the indices i, j is understood. A comparison of this result with (11.95) shows that

$$\delta V^{(1)} = e\langle \mathbf{E}\cdot \mathrm{x}\rangle = \mathrm{tr} d, \tag{11.134}$$

where d is the xp—diffusion tensor and $e\langle \mathbf{E}\cdot \mathrm{x}\rangle$ is the average work done on the particle by the random force; hence,

$$\delta E_{\text{Lamb}} = \tfrac{1}{2}\delta V = \tfrac{1}{2}\mathrm{Tr} d. \tag{11.135}$$

This extends the result derived in chapter 7 for the harmonic oscillator, equation (7.77), to more general bound systems [de la Peña and Cetto 1978]. The formula allows to understand the Lamb shift as the mean work performed by the vacuum field (all of it, not only the 'relevant' modes) on the atomic electron, as was discussed in detail in §7.3.2 in relation to Welton's semiquantitative interpretation. Again the correct quantum prediction is obtained here, despite the fact that the zeropoint field is not second quantized. Contrary to the extended belief that spontaneous emission and the Lamb shift must be described in terms of a quantized radiation field, to avoid conflicts with experiment [see, e.g., Milonni 1994, p. 79], one concludes from the present analysis that they should be considered effects of the random zeropoint field on already quantized matter.

11.5. A glance at chaos

As we know by construction, the bound SED systems described by the Heisenberg equations have the remarkable property of being quite insensitive to the fluctuations of the zeropoint field and are notably less random than their standard stochastic counterparts (when they exist), so they correspond to quite stable motions. Further, such quantum solutions do not include any more the transient terms, and so their description is highly insensitive to initial conditions. A clear consequence of all this is that the solutions x_α, p_α, etc., represent for each realization of the k-averaged field a regular, periodic or quasiperiodic motion in one or more dimensions, when the characteristic frequencies are related by rational numbers. It is even possible to state this in more general terms and omit the last requirement, because the quantum evolution can be shown to be generated by a unitary transformation which, for bound systems, is multiply-periodic [Percival 1961].

This conclusion can be reinforced by translating into linear SED a procedure recently suggested for the study of chaos in bound quantum mechanical systems [Hossein Partovi 1992]. As a starting point, we consider a measure of sensitivity used in classical dynamics and constructed in terms of the matrix of partial derivatives

$$T_{ij}(t) = \frac{\partial \xi_i(t)}{\partial \xi_j(0)}, \tag{11.136}$$

where the components of the vector $\boldsymbol{\xi}$ are the N pairs of canonical variables $\xi_i = q_i$, $\xi_{N+i} = p_i$. Specifically, the logarithm of the eigenvalues of $\lim_{t\to\infty}(T^\dagger T)^{1/2t}$ can be identified with the Lyapunov characteristic exponents for the system [Eckmann and Ruelle 1985]. Hossein Partovi (1992) has extended this definition to the quantum case by introducing the *sensitivity matrix* $\mathcal{T}$, such that $\mathcal{T}_{ij}(t) = \partial\xi_i(t)/\partial\xi_j(0)$ with $\xi_i(t) = \mathrm{tr}\hat{\xi}_i(t)\hat{\rho}$ and establishing a generalized Heisenberg inequality in terms of the variances $\delta\xi_i(t)$,

$$|\mathcal{T}_{ij}(t)| \leq \frac{2}{\hbar}\delta\xi_i(t)\delta\xi_{j\pm N}(0). \tag{11.137}$$

According to this inequality, there cannot be an unbounded growth of the sensitivity matrix for a bound quantum state (i.e., a state for which all variances $\delta\xi_i(t)$ are bounded); hence such a quantum state cannot be chaotic.[6]

Using the methods developed in chapter 10 it is a simple matter to construct the SED counterpart of T; one gets for the stationary state α

$$\begin{aligned} T_{ij}(t) \equiv \frac{\partial \xi_i(t_0+t)}{\partial \xi_j(t_0)} &\Rightarrow \frac{1}{i\hbar}\langle \xi_i(t_0+t); \xi_{j\pm N}(t_0)\rangle \\ &= \frac{1}{i\hbar}\sum_\beta \left(\tilde{\xi}^i_{\alpha\beta}\tilde{\xi}^{j\pm N}_{\beta\alpha} e^{i\omega_{\alpha\beta}t} - \mathrm{c.c.}\right). \end{aligned} \tag{11.138}$$

From this result and the Schwartz inequality it follows that

$$|T_{ij}(t)| \leq \frac{2}{\hbar}\delta\xi_i(t)\delta\xi_{j\pm N}(0), \tag{11.139}$$

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[6] A positive Lyapunov exponent is a necessary condition for the existence of chaos, because then nearby trajectories separate exponentially with the evolution of the system. The condition is not sufficient, however, because mixing is also needed for chaos to occur [see, e.g., Lichtenberg and Lieberman 1983, section 5.2; Rañada 1990, chapter 16].

where $\delta\xi_i$ is the variance of ξ_i in state α, $|\delta\xi_i|^2 = \sum_\beta \left|\tilde{\xi}^i_{\alpha\beta}\right|^2$, in coincidence with equation (11.137). For example, for $\xi_i = \xi_{j+Ni} = x$ one gets

$$|T_{ii}(t)| = \frac{2m}{\hbar}\left|\sum_\beta \omega_{\alpha\beta}|\tilde{x}^i_{\alpha\beta}|^2 \cos\omega_{\alpha\beta}t\right| \leq \frac{2m}{\hbar}\left|\sum_\beta \omega_{\alpha\beta}|\tilde{x}^i_{\alpha\beta}|^2\right| = 1. \quad (11.140)$$

In the last step we used equation (10.94), the SED equivalent of the Thomas-Reiche-Kuhn sum rule.

In this form we arrive at a conclusion already reached by other authors, namely that bound systems in the quantum regime described by stationary states are not chaotic [see, e.g., Ingraham et al. 1990; Hossein Partovi 1992]. Other related questions were touched upon in chapter 10, particularly in §10.5.1.

The need for an improved description

We have found that some of the limitations of the radiationless description of the quantum regime provided by linear SED can be remedied by reintroducing the zeropoint field as a perturbation, just as is done in nonrelativistic QED with respect to quantum mechanics. With this procedure one is able to reproduce important results of QED without any need to renounce to the random nature of the zeropoint field. However, the approach can be considered at most a partial corrective to the original radiationless description, since it demands the simultaneous use of two different descriptions of the equilibrium zeropoint field. From a physical perspective this simply means that the corrections are mainly due to those 'noisy' (taken previously as 'irrelevant') components of the field that were neglected in the linear SED approach; but of course the procedure is clearly unsatisfactory from a mathematical perspective.

The fact that the situation is conceptually not much better in QED, because in this theory the (quantized) vacuum field is responsible for the corrections only, the quantum-mechanical behaviour of matter being taken for granted and remaining unexplained, is of little help from the point of view of principle. In chapter 14, after having had the opportunity to develop other aspects of the theory, we shall come back to these matters and discuss a possible solution to this schizophrenic situation.

CHAPTER 12

THE WAVE PROPERTIES OF MATTER

> *Après ma thèse, on a souvent interpreté faussement mes idées en disant que, d'après moi, l'électron etait une onde, ce qui escamotait la particule.* [de Broglie 1973]

With the theory thus far developed it has been possible to address two of the most conspicuous properties of the quantum domain, namely, the random behaviour of matter and the quantization phenomenon, both of them efficiently and succinctly (though quite cryptically) expressed by the Heisenberg equations of motion augmented with the quantum rule. However, there is a third and most remarkable quantum feature that has been practically ignored in all our previous considerations, namely, the undulatory behaviour of matter. Of course, with formal manipulations of the results obtained so far it would be possible to ascertain the wave content of quantum mechanics; but this would appear to reduce it to a mere accident due to mathematical coincidences, and it would add little if anything to our physical understanding of the related phenomena. This is clearly an unsatisfactory situation, considering the fundamental nature that the wave properties of quantum systems are considered to have.

The wave aspects of matter not only played a major historical role during the heroic times of the quantum pursuit when they gave rise to a whole *wave mechanics*; they continue to be indispensable for the understanding of many experiments in which interference and diffraction with corpuscles take place. For instance, in recent years important technological breakthroughs have allowed to transform into very fine real ones some Gedankenexperimente by means of the neutron interferometer.[1] With this device, experiments related to the Schrödinger cat and other quantum delicacies have become feasible, and they have consistently confirmed the theoretical predictions in detail. However, all this language and conceptual framework is absent from SED as presented up to this point, indicating an essential incompleteness of the theory.

To explore how SED can thus be complemented is the aim of the present chapter. Needless to say, the undulatory nature of the zeropoint field, which is a basic constituent of the SED system, is expected to play an important role in this task. Having seen that some of the most basic quantum properties of matter can be understood as acquired by its interaction with the vacuum field, it would be even surprising if the field did not in some form impress also its wave properties on the particles. Trivial as it sounds, this simple consideration should be expected to take us very far in understanding some of the most mysterious aspects of quantum systems.

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[1] This device and some of the many significant results obtained with its help are briefly discussed in §12.4.3.

The study of the wave properties of matter is a relatively new subject within SED, and is still in an early stage of evolution, in which the ideas and proposals are still under scrutiny and to a large extent require further elaboration. In such circumstances it is unavoidable to resort here and there to some hand waving in trying to develop a detailed account. Although the authors have made an effort to reduce such moments to a minimum, they apologize in advance for what inevitably remains.

12.1. A revision of de Broglie's undulatory hypothesis

We recall that in de Broglie's theory of matter waves,[2] a *wave* of Compton's frequency $\omega_C = m_0c^2/\hbar$ is associated with the point electron at rest, and by a Lorentz transformation the phase of this wave for a moving electron is shown to have de Broglie's wavelength. This, which becomes the central object of the theory, is considered to be a physical wave, although the nature of it remains unspecified.[3]

The scheme that will be developed here is somewhat different, though close in spirit to de Broglie's theory.[4] We start by recalling that in chapter 3 the radiation reaction of the particle was shown to generate oscillations of a very high frequency, estimated to be of the order of the Compton frequency ω_C. Along with this it was demonstrated that the particle acquires an effective structure whose radius cannot in principle be reduced to zero. Similar results were obtained for both a point particle described by the causal version of the Abraham-Lorentz equation studied in section 3.3, and for the extended charge studied in section 3.4. Other instances of the same phenomenon appear elsewhere, as in §7.2.4, where it is found that the selfcorrelation of the position coordinate of the harmonic oscillator contains a permanent oscillatory contribution of a frequency determined by the cutoff (see equation (7.101)), and not too far from the Compton frequency.

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[2] A detailed, first-hand exposition of de Broglie's theory can be found in de Broglie (1956). A modern presentation by one of its distinguished advocates is given in Selleri (1990), whereas a couple of the many discussions of de Broglie's thesis can be found in MacKinnon (1976) and Schlegel (1977). A short all-embracing discussion is presented by Lochak (1993). A most elaborate development of a variant of de Broglie's theory for the relativistic electron is the *geometrical mechanics* developed by Synge (1954).

[3] In Surdin (1979a, 1985b) it is proposed to consider that de Broglie's wave is of electromagnetic nature and is in some way associated with the electromagnetic zeropoint field.

[4] Previous versions of the material presented in this chapter can be found in de la Peña and Cetto (1993b, 1994a) and Cetto and de la Peña (1995a, b).

In a classical context, these high-frequency oscillations are normally transient, because they come in association with short-lived processes. However, as was mentioned at the end of chapter 3, when the particle is interacting with a *permanent* random radiation field, as is the case in SED, things change essentially, because the electromagnetic environment not only puts the particle into resonance, but is acting on it constantly, so that the oscillations induced by radiation reaction become continuously renewed and acquire a permanent character, though certainly with stochastic properties.

Hence we conclude that the electron subject to the zeropoint field performs, in addition to the low-frequency motions of the type described by linear SED or usual quantum mechanics, a sustained oscillation of a very high frequency that can reasonably be estimated to be of the order of the Compton frequency $\omega_C = mc^2/\hbar$. From the same arguments it also follows that the particle should be considered as an effectively extended object.

In §3.2.3 it is shown that the electron with an effective structure of size a uncouples from the components of the radiation field with wavelengths smaller than a, i.e., with frequencies larger than $\omega_c \sim c/a$; this is, then, the cutoff frequency. Any specific model for the charge structure would be arbitrary at this stage, but also unnecessary, since for the purpose of taking it into account it suffices to introduce the appropriate cutoff. The most delicate point is the determination of the frequency of vibration, identified with the cutoff frequency ω_c —as in equation (7.101)—; however, since we know that it must be close to the Compton frequency ω_C, we write for the time being (when there is no risk of confusion the electron rest mass is denoted by m)

$$\omega_c = \frac{1}{\mu}\frac{mc^2}{\hbar} = \frac{\omega_C}{\mu}, \tag{12.1}$$

and consider the numerical constant μ (with $\mu \leq 1$), as a parameter to be determined.

Further, as suggested in chapter 3, it seems reasonable to identify these fine oscillations with the zitterbewegung,[5] of which we here only have a nonrelativistic (and certainly, poor) rendering. The recognition of this phenomenon is the point of departure for our construction of the SED version of de Broglie's theory.[6]

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[5] We recall that the frequency of zitterbewegung is twice the Compton frequency, and its amplitude is of the order of the Compton wavelength, so that the associated velocities are luminal. Therefore any reasonable quantitative theory aimed at modelling this phenomenon should be relativistic. Indeed, the zitterbewegung appears in connection with the Dirac theory of the electron, and is absent from nonrelativistic quantum mechanics.

12.2. Vibrations induced by the zeropoint field

12.2.1. PARTICLE WITHIN CONDUCTING WALLS

Before developing de Broglie's model on the basis of the above ideas, it is useful to reinforce the previous conclusions by making a concrete calculation to show how the introduction of an effective structure for the particle embedded in the zeropoint field gives rise to the high-frequency oscillations. Of course, an accomplished theory should be relativistic due to the high velocities involved in these oscillations; however for the purposes of a heuristic treatment such as the present one, it seems reasonable that we continue to use a nonrelativistic description. The example has the extra value of stressing the importance of the spatial structure of the field modes; this means, according to our discussion in section 10.2, that we will actually override the limits of the quantum mechanical description which was obtained in chapter 10 in the coarse-grain average, long-wavelength approximation, and proceed to a finer description. We shall come back to this point at the end of §12.3.1.

Our model consists of a particle with effective extension (taken into account through the cutoff), moving in a zeropoint field subject to boundary conditions. For the boundary conditions we use the simplest ones, namely, those imposed by a planar surface made of a perfectly conducting material, as was done in connection with the theory of dispersion forces studied in chapter 6. The particle is close to a perfectly conducting wall but is otherwise free, which means that no other interaction between it and the wall is being considered; the only force acting on it is the Lorentz force due to the vacuum and to its own radiation field. Therefore, we start from the equation

$$m\ddot{\mathbf{x}} = e\mathbf{E}_{\mathrm{rr}} + e\mathbf{E}, \tag{12.2}$$

where $\mathbf{E}$ is the electric component of the vacuum field and $\mathbf{E}_{\mathrm{rr}}$ is the field radiated by the particle. In the absence of boundary conditions the latter term has the familiar form $m\tau\,\dddot{\mathbf{x}}$, but in the present case it assumes a more complicated structure. The field can be written as usual in terms of normal modes; since it is more convenient to work with sums than with integrals, we temporarily assume the existence of a second wall at a distance L which in due time is taken as infinite. Then the zeropoint field is given by equation (6.14) in terms of an appropriate orthonormal basis $\mathbf{G}_{\mathbf{n}\sigma}(\mathbf{x})$, and the radiated field can be calculated as in section 3.2 (see also equation (7.182)),

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[6] Other mechanisms to generate zitterbewegung-like oscillations have been explored in the literature on relativistic models of the electron. For instance, by considering within SED that the structure of the particle is related to the difference between the centers of inertia and charge of the particle, it has been shown that the electron responds to the random field by performing a zitterbewegung, although with a wide spectrum of frequencies [Rueda 1993a]. Normally these oscillations are related to models of the spin; for instance, a fine old example is Huang (1952). For many years, Cavalleri has insisted that zitterbewegung should be considered a fundamental property lying at the basis of quantum phenomena; see, e.g., Cavalleri (1985).

$$\mathbf{E}_{\mathrm{rr}}(t) = -2\pi e \sum_{\mathbf{n},\sigma} \mathbf{G}_{\mathbf{n}\sigma}(\mathbf{x})) \int_0^\infty ds\, \mathbf{G}^*_{\mathbf{n}\sigma}(\mathbf{x}(t-s)) \cdot \dot{\mathbf{x}}(t-s) e^{i\omega_n s} + \text{c.c.} \quad (12.3)$$

The particle is assumed to move slowly enough along its trajectory, so that

$$\left|\dot{\mathbf{G}}_{\mathbf{n}\sigma}\right| \sim \omega_n \left|\frac{\dot{x}}{c}\right| |\mathbf{G}_{\mathbf{n}\sigma}| \ll \omega_n |\mathbf{G}_{\mathbf{n}\sigma}|, \quad (12.4)$$

in due accord with the nonrelativistic treatment; this allows one to take $\mathbf{G}^*_{\mathbf{n}\sigma}(\mathbf{x}(t-s))$ out of the integral. Neglecting the initial transient term, one is left with a stationary solution of equation (12.2) of the form

$$\dot{\mathbf{x}} = \mathbf{v} + \sum_{\mathbf{n},\sigma} (f_n \mathbf{G}_{\mathbf{n}\sigma} a_{\mathbf{n}\sigma} + \text{c.c.}), \quad (12.5)$$

with $\mathbf{v}$ a systematic (constant) velocity,

$$f_n = \frac{e\,(\pi\hbar\omega_n)^{1/2}}{m\omega_n\,(1+i\delta_n)} \quad (12.6)$$

and

$$\delta_n = \frac{4\pi e^2}{3m\omega_n} \sum_{\mathbf{m},\lambda} |\mathbf{G}_{\mathbf{m}\lambda}|^2 \int_0^\infty ds\, e^{i\omega_n s} \cos\omega_m s. \quad (12.7)$$

The average kinetic energy associated with the stochastic motion is, from equations (12.3)-(12.7),

$$\langle T_s \rangle = \frac{1}{2} m \left\langle (\dot{\mathbf{x}} - \mathbf{v})^2 \right\rangle = \frac{\pi\hbar e^2}{m} \sum_{\mathbf{n},\sigma} \frac{|\mathbf{G}_{\mathbf{n}\sigma}|^2}{\omega_n |1+i\delta_n|^2}. \quad (12.8)$$

For a further simplification it can be assumed that the walls are made of a perfect conductor, up to a certain (very high) frequency ω_c at which they become transparent. More realistic electric or magnetic properties of the material would lead to a modification of the $\mathbf{G}_{\mathbf{n}\sigma}$, and the calculations would become more involved. However, the end effect would be substantially the same, because the cutoff frequency will be selected without assuming it to characterize any specific conductor up to the frequency ω_c. For a field confined between two vertical, parallel conducting plates of area S, separated by a distance L one has (see equations (6.15) and (6.16))

$$\mathbf{G}_{n1} = \sqrt{\frac{2}{LS}} e^{i\mathbf{k}_\rho \cdot \rho} \left(\hat{\mathbf{k}}_\rho \times \hat{\mathbf{x}} \right) \sin \xi, \tag{12.9}$$

$$\mathbf{G}_{n2} = \sqrt{\frac{2}{LS}} e^{i\mathbf{k}_\rho \cdot \rho} \left(\frac{k_x}{k_n} \hat{\mathbf{k}}_\rho \sin \xi + i \frac{k_\rho}{k_n} \hat{\mathbf{x}} \cos \xi \right), \tag{12.10}$$

with ρ lying in the present case on the yz-plane of the plates and $\hat{\mathbf{x}}$ perpendicular to it; x is the distance of the particle to the left plane, and index $\sigma = 1, 2$ has been chosen to represent waves of linear polarization. The wave vector is $\mathbf{k}_n = \mathbf{k}_\rho + \mathbf{k}_x$, with $k_x = \pi n_x / L$, n_x an integer, and $\xi = k_x x$. From equations (12.9-12.10) one gets

$$\sum_\sigma |\mathbf{G}_{\mathbf{n}\sigma}|^2 = \frac{2}{LS} \left(1 - \frac{k_x^2}{k_n^2} \cos 2\xi \right). \tag{12.11}$$

As the area of the walls is made infinite, the triple sum over the indices of the normal modes (n_x, n_y, n_z) transforms into

$$\frac{1}{S} \sum_{\mathbf{n}} \underset{S \to \infty}{\longrightarrow} \frac{1}{4\pi} \sum_{n_x}{}' \int_{k_x^2}^{k_c^2} dk_n^2, \tag{12.12}$$

where $\sum'$ is a sum over $n_x \geq 0$ with the first term multiplied by $\frac{1}{2}$, and $k_c = \omega_c / c$ is the upper limit for the wave number. The expression for the kinetic energy becomes then, from equations (12.8) and (12.11) and with $\omega = ck$,

$$\langle T_s \rangle = \frac{e^2 \hbar}{2mc^2 L} \sum_{n_x}{}' \int_{\omega_x}^{\omega_c} d\omega_n^2 \frac{1 - \frac{\omega_x^2}{\omega_n^2} \cos 2\xi}{\omega_n \left| 1 + i\delta_n \right|^2}, \tag{12.13}$$

with δ_n determined from equations (12.7) and (12.11) as

$$\delta_n = \frac{2e^2}{3mc^2 \omega_n L} \sum_{m_x}{}' \int_{\omega_x}^{\omega_c} d\omega \omega \left(1 - \frac{\omega_x^2}{\omega_n^2} \cos 2\xi \right)$$

$$\times \left(\pi \delta \left(\omega - \omega_n \right) + 2i\omega_n \mathrm{P} \frac{1}{\omega_n^2 - \omega^2} \right), \tag{12.14}$$

where $\omega_x = \pi m_x c / L$. The real part of this integral gives

$$\mathrm{Re}\delta_n = \frac{\pi \tau c}{L} \sum_{m_x}{}' \left(1 - \frac{\omega_x^2}{\omega_n^2} \cos 2\xi \right). \tag{12.15}$$

The result of the summation can be expressed as

$$\mathrm{Re}\delta_n = \tau_x \omega_n, \tag{12.16}$$

where τ_x depends on x but is always of the order of τ (in the absence of boundary conditions, $\tau_x = \tau = 2e^2/3mc^3$); explicit evaluation of this term is not necessary, as will be seen below (see comments to equation (12.25)). The calculation of the imaginary part of δ_n is more cumbersome, but one can see that it gives a result of the order of $\tau\omega_c$. Since this term represents a radiative correction to the mass of the particle, one may include it in the renormalized mass and thus write

$$\langle T_s \rangle = \frac{e^2\hbar}{mc^2 L} \sum_{n_x}{}' \int_{\omega_x}^{\omega_c} d\omega \frac{1 - \frac{\omega_x^2}{\omega_n^2}\cos 2\xi}{1 + \tau_x^2 \omega_n^2}. \tag{12.17}$$

After integrating one obtains

$$\langle T_s \rangle = \frac{e^2\hbar}{mc^2 \tau_x L} \sum_{n}^{N}{}' F(n), \tag{12.18}$$

with

$$F(n) = \arctan \eta_c - \arctan \varepsilon n + \frac{\varepsilon^2 n^2}{\eta_c} (1 + \eta_c \arctan \eta_c) \cos \zeta n$$
$$- \varepsilon n \cos \zeta n - \varepsilon^2 n^2 (\arctan \varepsilon n) \cos \zeta n \tag{12.19}$$

and the abbreviations

$$\eta_c = \tau_x \omega_c, \quad \varepsilon = \frac{\pi c \tau_x}{L}, \quad N = \frac{L\omega_c}{\pi c} = \frac{\eta_c}{\varepsilon}, \quad \zeta = \frac{2\pi x}{L}. \tag{12.20}$$

N is the number of smallest wavelengths that fit between the two walls; the maximum value attained by εn is $\varepsilon N = \eta_c$. Since ω_c is very large, N has a huge value, whereas the parameter η_c is much smaller than one. For example, for $\omega_c \sim \omega_C$, $\eta_c = \tau_x \omega_c \sim e^2/\hbar c = \alpha$; observe, incidentally, that the mass correction referred to above becomes also of order α. To lowest order in η_c one gets

$$F(n) = \eta_c \left(1 - \frac{n}{N} - \frac{n}{N} \cos \zeta n + \frac{n^2}{N^2} \cos \zeta n \right). \tag{12.21}$$

The sums can be performed with the help of formulas taken from §1.35 of the Gradshteyn-Ryzhik Tables, to get

$$\sum_{n=0}^{N}{}' F(n) = \frac{\eta_c N}{2} \times \left(1 - \frac{\sin (N+1)\zeta \cos \frac{\zeta}{2}}{2N^3 \sin^3 \frac{\zeta}{2}} \right.$$

$$+\frac{N+N\cos\zeta N+\cos\left(N+1\right)\zeta+\cos\zeta N}{2N^3\sin^2\frac{\zeta}{2}}\Bigg). \tag{12.22}$$

Very close to any of the walls this expression reduces to

$$\sum_{n=0}^{N}{}'F(n)\underset{\zeta\to 0,2\pi}{\longrightarrow}\frac{1}{3}N\eta_c, \tag{12.23}$$

whereas in the space between them,

$$\sum_{n=0}^{N}{}'F(n)=\eta_c\frac{N}{2}\left(1+\frac{\cos^2\frac{N\zeta}{2}}{N^2\sin^2\frac{\zeta}{2}}\right),\quad \zeta\neq 0,2\pi \tag{12.24}$$

and the average kinetic energy becomes, by equation (12.18),

$$\langle T_s\rangle=\frac{\alpha}{2\pi}\frac{\hbar^2\omega_c^2}{mc^2}\left(1+\frac{\cos^2 k_c x}{N^2\sin^2(\pi x/L)}\right)\equiv\langle T_1\rangle+\langle T_2\rangle,\quad x\neq 0,L. \tag{12.25}$$

Note that the dependence on τ_x has cancelled out. The first term $\langle T_1\rangle$ in equation (12.25) is a (mass) correction that we have met on several occasions; it is independent of the wall separation and of the position of the particle, and exists even in the absence of any body. On the other hand, the second term $\langle T_2\rangle$ oscillates with the high wave number k_c, even for large separations L, so that in the presence of a single wall it attains the value

$$\langle T_2\rangle\underset{L\to\infty}{\longrightarrow}\frac{\alpha}{2\pi}\frac{\hbar^2}{mx^2}\cos^2 k_c x. \tag{12.26}$$

In a strict sense this is not the expected result, since instead of an oscillation in time with frequency ω_c we got a spatial vibration with wave number $k_c=\omega_c/c$. What we have shown here is nevertheless that the cutoff or, equivalently, the finite (maybe effective) size of the particle, leads naturally to these vibrations with wave number k_c. Now, for a relativistic vibration with velocity close to c, both results would be equivalent and would correspond to the general predictions of the theory.[7]

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[7] The assumption that the vibrations of very high frequencies occur with velocities close to the velocity of light is not as strange as it may look at a first glance. To begin with, this is just the case with the usual interpretation of zitterbewegung; also, jerky movements of the electron with velocity c have been considered in relativistic models of the spinning electron [see, e.g., Corben 1968 and references therein] and other instances, such as relativistic treatments of the electron as a stochastic particle. Two examples of the latter kind of theories, widely different in their conceptual framework, but both involving motions with speeds arbitrarily close to c, are given in Lehr and Park (1977) and Gaveau et al. (1984).

12.2.2. VIBRATIONS OF AN ELECTRIC DIPOLE

Similar calculations can be performed for a small, electric vibrating dipole, leading essentially to the same result, namely, an average energy that oscillates with the high wave number k_c. Hence the whole discussion applies not only to charged particles, but also to uncharged particles having electromagnetic interactions; this is an important point for SED, since the theory must of course be also applicable to neutral particles.

To study the motion of an electric dipole moment $\mathbf{q}$, we take equation (12.2) (multiplied by e) with an additional harmonic force term and make the substitution $e\mathrm{x} \longrightarrow \mathbf{q}$,

$$m\ddot{\mathbf{q}} = -m\omega_0^2\mathbf{q} + e^2\mathbf{E}_{\mathrm{rr}} + e^2\mathbf{E}. \tag{12.27}$$

In equation (12.3), $e\dot{\mathrm{x}}$ must be simultaneously replaced by $\dot{\mathbf{q}}$. Writing the stochastic part of the solution of equation (12.27) as previously,

$$\mathbf{q} = \sum_{\mathbf{n},\sigma} \left(g_n \mathbf{G}_{\mathbf{n}\sigma} a_{\mathbf{n}\sigma} + \text{c.c.} \right), \tag{12.28}$$

one obtains

$$g_n = \frac{-ie^2 \left(\pi\hbar\omega_n\right)^{1/2}}{m\Delta_n}, \tag{12.29}$$

with $\Delta_n = \omega_n^2 - \omega_0^2 - i\tau_x\omega_n^3$. The function τ_x is defined in equations (12.15), (12.16). For the calculation of the average force of the background field on the dipole we use

$$\langle \mathbf{F} \rangle = \left\langle \left(\mathbf{q}\cdot\boldsymbol{\nabla}\right)\mathbf{E} + \frac{1}{c}\dot{\mathbf{q}} \times \mathbf{B} \right\rangle$$

$$= -\frac{\pi e^2\hbar}{m} \sum_{\mathbf{n},\sigma} \frac{\omega_n}{\Delta_n} \left(\left(\mathbf{G}_{\mathbf{n}\sigma}\cdot\boldsymbol{\nabla}\right)\mathbf{G}^*_{\mathbf{n}\sigma} + \mathbf{G}_{\mathbf{n}\sigma} \times \left(\nabla \times \mathbf{G}^*_{\mathbf{n}\sigma}\right)\right) + \text{c.c.} \tag{12.30}$$

With $\mathbf{G}_{\mathbf{n}\sigma}$ given by equations (12.9, 12.10), this leads after some simplifications to

$$\langle \mathbf{F} \rangle = -\nabla \left\langle U(\mathrm{x}) \right\rangle, \tag{12.31}$$

with

$$\langle U \rangle = -\frac{e^2\hbar}{mc^2L} \sum_{\mathbf{n},\sigma}{}' \int_{\omega_x}^{\omega_c} d\omega \frac{\omega_x^2(\omega^2 - \omega_0^2)\cos 2\xi}{(\omega^2 - \omega_0^2)^2 + \tau_x^2\omega^6}, \tag{12.32}$$

where $\xi = \omega_x x/c, \omega_x = \pi n_x c/L$, and the sum extends to $N = L\omega_c/\pi c$. In the limit $\omega_0 = 0$, equation (12.32) coincides with the expression (12.17) for $\langle T_s \rangle$, and hence we obtain the same end result, equation (12.25). It can be readily seen that for usual values of ω_0, which are much smaller than ω_c, the additional (ω_0-dependent) contribution to U is negligible. Hence the electric dipole shows essentially the same oscillatory phenomenon as the charged particle.

Incidentally, note that instead of following the above procedure one could have chosen to calculate the average energy of the dipole

$$\langle T_s + V_s \rangle = \frac{m}{2e^2}\left\langle \dot{\mathbf{q}}^2 + \omega_0^2 \mathbf{q}^2 \right\rangle, \tag{12.33}$$

with $\mathbf{q}$ given by equation (12.28), in the same approximation $\omega_0 \ll \omega_c$. By this method one arrives at the result obtained above, plus the constant ground-state energy of the oscillator $\frac{3}{2}\hbar\omega_0$ due to the contribution of the integrand at the pole, $\omega = \omega_0$.

12.2.3. THE VIBRATING ELECTRON

Inspection of the calculations leading to equation (12.26) shows that it is mainly the field modes of wave numbers close to k_c that are effective in producing the fine vibrations of the particle, the action of the low-frequency field being irrelevant in this respect. This has two important consequences. In the first place, it means that the same phenomenon can be induced on the particle by any object (of whatever size and shape) capable of appropriately affecting the structure of these higher-frequency modes in the close vicinity of the particle. This object may be the particle itself, if it is assumed that it imposes some sort of boundary conditions on the field; as is manifest in equation (12.26), it is precisely in the close vicinity of the boundary ($x \to 0$) where the modifications of the field are most effective. Under this assumption the presence of a second body is no longer necessary to induce the fine vibrations of the particle.

In the second place, observe from (12.8) that $\langle T_s \rangle$ does not depend on the state of motion, represented by the first term on the right-hand side of (12.5); it depends only on intrinsic properties of the particle and the zero-point field. Indeed, the result (12.26) is valid as long as (12.4) holds, which means that it applies to any particle moving with a nonrelativistic systematic velocity which may itself be a function of time as a result of the action of an external force. This conclusion is confirmed by a simple observation: for an order-of-magnitude calculation one can model an external binding force with a term of the form $-m\omega_{\text{eff}}^2 x$, with ω_{eff} appropriately defined, but always much smaller than ω_c. Now for the jiggling motion discussed here, with frequency ω_c, $|\ddot{x}| \simeq |\omega_c^2 x| \gg |\omega_{\text{eff}}^2 x|$; hence the free particle is an appropriate model for the study of the high-frequency motion, and the details of the specific dynamics are irrelevant.

This observation shows that it is possible to formally consider the motions of high and low frequency as independent of one another, although soon we will find that there is a profound relationship between them, even if a very indirect one. Under the term 'low frequencies' we are referring of course to the relevant frequencies $\omega_{\alpha\beta}$ of chapters 10 and 11, whereas the 'high frequencies' are those of order ω_C, their ratio being of order (typical atomic binding energy/mc^2) $\sim \alpha^2$. A large gap separates these two bands, but both are needed to 'complement' the quantum description; the low frequencies for the mechanical and spectral properties characteristic of atoms and other bound systems, whereas the high frequencies for the oscillations of the atomic constituents themselves, whether bound or not. In what follows we intend to show that these latter oscillations are dynamically important, even if at a first glance they may appear to be irrelevant.

According to the foregoing discussion, a simple model for the motion of the free particle is a superposition of a constant (systematic) velocity and the jittering movement. The very fine jiggle may be considered as a (refined) nonrelativistic version of the zitterbewegung,[8] which is superposed to an approximately smooth motion involving frequencies that are smaller by several orders of magnitude. The constant velocity term may seem to contravene the stochastic nature of the problem; however, already in standard SED we encountered a similar situation, in terms of p/m (see §8.1.1), and also linear SED (if extended to the free particle) makes a similar prediction. Further, the description is in agreement with what is observed in experimental situations; consider, for instance, the high precision attained in modern electron beam technology, where velocity deviations of less than one part in 10^6 are normal.

12.3. Wave mechanics

12.3.1. GENESIS OF DE BROGLIE'S WAVELENGTH

Our purpose in this section is to understand why the vibrations with frequency ω_c just explained play such a very special role for the electron, as was proposed by de Broglie. In a recent precursor of the treatment to follow, Kracklauer (1992) proposed that the electron is tuned to a wave originating in the high-frequency modes of the zeropoint background field, which in the electron's frame is a standing wave, but viewed from the laboratory is a modulated wave that moves along with the particle.[9] Here a slightly modified form of this proposal will be presented, following a kinematic analysis parallel to that of de Broglie's, although the conceptual approach is essentially that of Kracklauer; we will deal with waves of the zeropoint electromagnetic field (that fill all the space), whereas in de Broglie's theory a 'clock' wave of unspecified nature is assumed to exist only in the neighborhood of the particle and associated to it. Since the calculations involved are simple and well known, unnecessary details will be skipped.

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[8] In the previous discussions on the nonrelativistic version of zitterbewegung as furnished by standard SED (mainly in section 8.3), oscillations of arbitrary frequency were being considered; the model here discussed refers to frequencies of the right order of magnitude for the motion to be indeed classified as zitterbewegung.

[9] Other recent discussions, from very different points of view, on the need of a wave field as a fundamental element for an understanding of quantum mechanics, can be found in Tiwari (1986) and Ferrero and Santos (1994).

Consider for simplicity the case of a free particle, say an electron, moving with velocity v.[10] Now since the electron is performing fine oscillations with frequency ω_c, it radiates at this frequency. In a state of equilibrium, such radiation must be compensated by the interaction with the vacuum field. This means that the particle interacts intensely with the modes of frequency ω_c, as measured in its proper frame, and that these modes sustain the jitter. The specific mechanism of this interaction is irrelevant for the kinematics; what is important is that the particle interacts selectively with just these and nearby modes in the band of very high frequencies.

Now suppose that the electron moves with respect to laboratory and along the $+x$ direction, and consider the two waves of the zeropoint field having a frequency ω_c in the electron's proper frame, propagating along the $+x$ and $-x$ directions; their frequencies in the laboratory frame are respectively [see, e.g., Jackson 1975, section 11.3]

$$\omega_+ = \gamma\omega_c(1+\beta), \quad \omega_- = \gamma\omega_c(1-\beta), \tag{12.34}$$

with $\gamma = (1-\beta^2)^{-1/2}$ and $\beta = v/c$. In a more convenient notation we write

$$\omega_\pm = \omega_{\tilde{z}} \pm \omega_B; \quad \omega_{\tilde{z}} = \gamma\omega_c; \quad \omega_B = \gamma\beta\omega_c, \tag{12.35}$$

where $\gamma\omega_c$ is a shifted frequency associated to the Lorentz contraction of time and ω_B is a Doppler shift. The wave number k_c transforms accordingly into

$$k_\pm = \gamma k_c \pm k_B, \quad k_B = \gamma\beta k_c. \tag{12.36}$$

As seen from the reference frame of the electron, the two waves form a standing wave of wavelength $\lambda_c = 2\pi/\omega_c$; seen from the laboratory, they still propagate in opposite directions but with frequencies $\omega_\pm$ and wave numbers $k_\pm$. To describe the ensuing travelling wave, we write the random mode amplitudes a_n^0 of the vacuum field in the form $a_n^0 = e^{i\theta_n}$, with uniformly distributed random phases θ_n such that $\left\langle e^{i(\theta_n - \theta_{n'})}\right\rangle = \delta_{nn'}$; the magnitude of the amplitude is taken as constant only for simplicity, since it is of little importance in the present context. The superposition of the waves φ_+, φ_- of frequencies ω_+ and ω_- travelling in opposite directions gives

$$\varphi(v) = \varphi_+ + \varphi_- = e^{i(\omega_+ t - k_+ x + \theta_+)} + e^{i(\omega_- t + k_- x + \theta_-)} + \text{c.c.}$$

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[10] As discussed above, one can actually consider the more general case in which the velocity is slowly varying; also, there is no problem in considering a dipole instead of a charged particle.

$$= 4\cos\left(\omega_z t - k_B x + \theta_1\right)\cos\left(\omega_B t - \gamma k_c x + \theta_2\right), \qquad (12.37)$$

where $\theta_{1,2} = \frac{1}{2}\left(\theta_+ \pm \theta_-\right)$, are two statistically independent random phases. For $v = 0$ the above expression reduces to

$$\varphi(0) = 4\cos\left(\omega_c t + \theta_1\right)\cos\left(k_c x - \theta_2\right), \qquad (12.38)$$

representing the original standing wave; equation (12.37), instead, represents a travelling wave that is temporally and spatially modulated. Keeping x fixed in equation (12.37) one sees that the carrier amplitude is modulated by a factor that oscillates in time with the Doppler frequency ω_B. On the other hand, keeping t fixed one sees that the fine oscillations (of wave number γk_c) are spatially modulated, k_B being the corresponding wave number; hence there is a modulation of wavelength

$$\lambda_B = \frac{2\pi}{k_B} = \frac{\lambda_c}{\gamma\beta} = \frac{m_0 \lambda_c c}{p}, \qquad (12.39)$$

where $p = m_0 \gamma v = m_0 c \gamma \beta$ is the linear momentum of the particle. (An index 0 is temporarily added to the rest mass to distinguish it from $m = \gamma m_0$). Note that λ_B originates in the Doppler shift of the frequency; for $v = 0$, there is no Doppler shift and λ_B becomes infinite. It is clear that these results remain valid when the velocity v is a slowly changing function of time or position, as long as the fractional changes of β with t or x are small with respect to $2\pi/\omega_c$ or λ_c, respectively; these conditions are well satisfied for nonrelativistic motions.[11]

Although only waves along the direction of the particle's motion were considered in equation (12.37), it is easy to show that waves propagating orthogonally —or, for that matter, in any direction— give rise to the same phenomenon. Consider for instance two waves of frequency ω_c travelling along the $+y$ and $-y$ directions, in the particle's frame; their parameters in the laboratory frame are $\omega' = \gamma\omega_c$, $k'_x = \gamma\beta k_c$ and $k'_y = k_c$, and the superposition of these waves gives

$$\varphi(v) = 4\cos(\gamma\omega_c t - \gamma\beta k_c x + \theta_1)\cos(k_c y - \theta_2). \qquad (12.40)$$

The second factor represents the short-wavelength stationary wave along the y direction; the first factor is again as in equation (12.37), with frequency ω_z and wave number k_B given respectively by (12.35) and (12.36).

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[11] Here a conceptual problem seems to arise, as remarked in Ferrero and Santos (1994). Since p is normally understood in de Broglie's theory as the canonical momentum, which can be arbitrarily changed when there is present an electromagnetic field by a gauge transformation of the field, the ontological status of λ_B corresponds to that of the electromagnetic potential, usually considered to lack of direct physical meaning. However, the modulation wave refers to a real process, so its phase in equation (12.37), and hence p in (12.39), should be defined so as to be gauge independent.

Let us take a closer look at the first factor in equations (12.37) or (12.40). It represents a wave with frequency ω_z given by the dispersion relation

$$\omega_z^2 = \omega_c^2 + c^2 k_B^2, \tag{12.41}$$

and its group velocity is therefore equal to the velocity of the particle,

$$v_g = \frac{\partial \omega_z}{\partial k_B} = \frac{c^2 k_B}{\omega_z} = v. \tag{12.42}$$

Its phase velocity u is given, as follows from equations (12.35) and (12.36), by

$$u = \frac{\omega_z}{k_B} = \frac{c^2}{v}. \tag{12.43}$$

Further, the phase of the modulation factor in (12.37) travels with the velocity of the particle, as follows from (12.35),

$$u_m = \frac{\omega_B}{\gamma k_c} = \frac{\beta \omega_c}{k_c} = v. \tag{12.44}$$

Thus the velocity of the particle determines the basic features of the carrier wave, namely, the group velocity v and the phase velocity u, as well as the wavelength of the modulation, which is given by

$$\lambda_B = \frac{m_0 c \lambda_c}{p} = 2\pi \frac{m_0 c^2}{\omega_c} \frac{1}{p}. \tag{12.45}$$

Note that of all these properties only λ_B depends on the value of the cutoff frequency. Assume provisionally that ω_c is eliminated in what follows by means of equation (12.1),

$$\frac{m_0 c^2}{\omega_c} = \mu \hbar, \tag{12.46}$$

with μ as the parameter to be fixed; many of the important ensuing results will then contain μ. On the other hand, these results must be compatible with the rest of SED developed so far, which does not contain μ (nor ω_c) at all; this means that the value of the parameter should be furnished by the theory itself, without recourse to any extra hypothesis, in contrast to de Broglie's theory. It can be anticipated that the outcomes of the present treatment will be at variance with those previously obtained in SED unless we set $\mu = 1$. This is a general rule; for example, for the energy of the ground state of the harmonic oscillator one gets $\frac{1}{2}\mu\hbar\omega$ instead of $\frac{1}{2}\hbar\omega$; a more general example is given below in relation to equation (12.54). Thus, the demand of internal consistency of SED implies setting $\mu = 1$ in all cases. Lacking for the moment of elements to determine this parameter from first principles, from now on we take this value for granted, which means that[12]

$$\omega_c = \frac{m_0 c^2}{\hbar} = \omega_C. \tag{12.47}$$

Although this result still awaits a satisfactory explanation, it certainly fits well with all that is already known. This value of ω_c assigns to the electron an effective radius of the order of the Compton wavelength, $a \simeq c/\omega_C = \hbar/m_0 c = \lambda_C$ (see §3.2.3), which is also equal to the characteristic amplitude of zitterbewegung (see, e.g., Milonni 1994, section 9.8). Since the particle is localized within a volume of radius $\Delta x \sim c/\omega_C$ and the characteristic velocity of the jitter is $\Delta v \sim \omega_C \Delta x \sim c$, the involved vibrations are essentially relativistic and conform to the usual quantum mechanical rules, $\Delta x \Delta p \gtrsim \hbar$.

In conclusion, from equations (12.45) and (12.47) the wavelength of the modulation wave is given by the familiar de Broglie formula

$$\lambda_B = \frac{2\pi\hbar}{p} = \frac{h}{mv}, \tag{12.48}$$

which constitutes the basic principle of wave mechanics.

The foregoing discussion assigns a physical meaning to de Broglie's wave: it is the *modulation* of the wave formed by the Lorentz-transformed, Doppler-shifted superposition of the whole set of random stationary electromagnetic waves of frequency ω_C with which the electron interacts selectively. The properties of this modulation depend of course on the relative velocity v of the electron in the laboratory. Although the jiggling frequency ω_c and the frequency shift $\gamma\omega_c$ are of relativistic nature, the modulation arises essentially as a consequence of Doppler shifts on the component waves, and its motion can be studied safely in the nonrelativistic approximation, as long as $v \ll c$. The requirement that the wave number of the modulation must be much smaller than that of the carrier wave, $k_B \ll \gamma k_C$, for the whole picture to be valid, actually implies that we are working within the nonrelativistic limits. The modulated wave is of course as real as the component waves that give rise to it, which are part of the zeropoint field;

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[12] This is very probably more than just a fortuitous relationship. It may be interesting to observe that in classical mechanics, the mass alone is not sufficient to define an energy; the introduction of the universal constant c in relativity allows however to construct the energy mc^2. On the other hand, the introduction of $\hbar$ in quantum mechanics allows for the construction of the energy $\hbar\omega$. Observe that equation (12.47) represents the equality of this two fundamental relations. Indeed, this is the point of departure of de Broglie's (1956) theory, where the equality was taken as a 'natural' assumption.

hence it should exhibit all phenomena typical of waves, such as interference, diffraction, etc., as remarked in Dolin (1976) and Kracklauer (1992).

Nonrelativistic quantum mechanics corresponds to a description over distances several orders of magnitude larger than Compton's wavelength and times much longer than the inverse of the Compton frequency. But although the frequency associated with a typical (e.g. atomic) λ_B is of the same order of magnitude as the atomic transition frequencies, the elementary oscillations that lie at the basis of the wavelike behaviour and give rise to de Broglie's wavelength occur on the much finer scale Compton scale; λ_B appears thus as a large-scale manifestation of a phenomenon that lies outside nonrelativistic quantum mechanics. According to this scheme, any attempt to derive the wavelike behaviour from first principles within the limits of quantum mechanics should be useless.

12.3.2. THE WAVE EQUATION

We have seen that the motion of a particle with velocity v through the zeropoint field gives rise to a modulated wave, in the laboratory frame; in a stationary state of motion this wave, in its turn, acts on the particle that is (selectively) coupled to it. Hence any influence of the surroundings on this wave can in principle have an effect on the particle. It is therefore clear that a more detailed description of the wave associated with the particle is required.

To construct such description, we start by observing that the first factor in equation (12.37) depends on both particle and wave properties; in particular it has a group velocity equal to v and a phase velocity $u = c^2/v$. This information can be used to construct a wave equation for the amplitude ϕ associated with this factor. Note that since we are isolating one of the factors that constitute the physical wave $\varphi(v)$, the amplitude ϕ *will not* in itself represent an electromagnetic wave. We start from the corresponding (three-dimensional) wave equation,

$$\nabla^2\phi - \frac{1}{u^2}\frac{\partial^2\phi}{\partial t^2} = 0. \tag{12.49}$$

From equation (12.37) it follows that $\partial^2\phi/\partial t^2 = -\omega_z^2\phi$, with ω_z given by equation (12.35), so that one gets

$$\nabla^2\phi + \frac{\omega_c^2}{c^2}\gamma^2 v^2\phi = 0, \tag{12.50}$$

or, by equation (12.47),

$$\nabla^2\phi + \frac{p^2}{\hbar^2}\phi = 0. \tag{12.51}$$

Since relativistic expressions have been used to arrive at this result, one can use the relativistic formula for the momentum of the particle in terms of the external potential V; the above equation transforms then into

$$\nabla^2\phi + \frac{1}{\hbar^2 c^2}\left[(\mathcal{E} - V)^2 - m_0^2 c^4\right]\phi = 0, \tag{12.52}$$

where $\mathcal{E}$ is the relativistic energy of the particle. Equation (12.52) can be identified as the stationary Klein-Gordon equation for the *particle*, despite the fact that it has been derived to describe a *wave* amplitude; this important dichotomy will be discussed below.

In the nonrelativistic regime, where the excess energy $E = \mathcal{E} - m_0c^2$ is much smaller than the proper energy m_0c^2, the identity

$$(\mathcal{E} - V)^2 - m_0^2c^4 = \left(E - V + m_0c^2\right)^2 - m_0^2c^4 = 2m_0c^2(E-V)\left[1 + \frac{E-V}{2m_0c^2}\right] \tag{12.53}$$

can be approximated by its leading term, and equation (12.52) takes then the form of the Schrödinger equation,

$$\nabla^2\phi + \frac{2m_0}{\hbar^2}(E - V)\phi = 0, \tag{12.54}$$

where the non-relativistic expression $E = p^2/2m + V$ must be used, for consistency. The same result can of course be obtained by introducing this nonrelativistic approximation directly in equation (12.51). Note that equations (12.52) and (12.54) are expressed in terms of mechanical parameters, as the energy of the particle, although they have been derived for a wave amplitude.

This is one of the instances referred to above in relation with the need to set $\mu = 1$ in equation (12.46) to guarantee internal consistency of the theory. If μ had been left unspecified, the factor $(\mu\hbar)^{-2}$ would have appeared instead of $\hbar^{-2}$ in equation (12.54), and the solution of this equation would eventually lead to contradictions with the rest of SED for any value of μ other than 1.

12.3.3. THE TWO FACES OF THE SCHRÖDINGER EQUATION

In section 10.4.3 it was shown that a statistical description for the particle is provided by the Schrödinger equation for $\psi(x,t)$,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi, \tag{12.55}$$

with

$$\rho = \psi^*\psi \quad \text{and} \quad \mathrm{v} = \frac{i\hbar}{2m}\left(\frac{\nabla\psi^*}{\psi^*} - \frac{\nabla\psi}{\psi}\right), \tag{12.56}$$

under the following minimal set of requirements: i) the conservation of the number of particles, ii) an average form of Newton's Second Law (or rather, of the Braffort-Marshall equation, with the radiative corrections neglected), and iii) a constant universal value for the parameter $\hbar$. Under these conditions Schrödinger's equation appeared as a natural means to perform a statistical description of the SED system in the quantum regime. Now let us consider equation (12.55) in the stationary case for conservative problems, for which the energy is given by $E = \langle(\mathbf{p}^2/2m) + V\rangle$,

$$E\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \tag{12.57}$$

and compare with equation (12.54): it is clear that the two equations coincide. Further, since the wave amplitude ϕ and the probability amplitude ψ must satisfy the same general mathematical requisites such as continuity, square integrability, and so on, we see that (at least in the stationary case) $\phi \propto \psi$, so that the probability density for the particles $\rho = \psi^*\psi$ becomes proportional to the wave intensity $\phi^*\phi$. This conclusion is strongly reminiscent of de Broglie's (1956) principle of the guiding wave.

The parallelism discloses two faces of the Schrödinger equation: it describes simultaneously the *modulation wave* associated with a particle in terms of a wave amplitude ϕ, and the configuration-space distribution for an *ensemble of particles* in terms of the probability amplitude ψ. It appears that in the long discussions about the nature of the object described by the Schrödinger equation, whether waves or particles or any other entity enjoying properties of both,[13] everybody has his (or her) share of truth. However, the present answer is not a mere reiteration of the conventional view, according to which the quantum object is or behaves as either a particle or a wave, in correspondence with the specificities of the setup. On the contrary, both wave and particle have their place in nature and in theory, without excluding each other; they are clearly distinct and *coexisting* entities, each one with its own life and fate. We are in presence of two alternative and complementary (partial) views of the same system (as has happened since the almost simultaneous emergence of Schrödinger's, wave, and Heisenberg's, matrix, original theories), and though each of these two

[13] Examples of such entities are Eddington's (1928) 'wavicles', or Bunge's (1967, 1973) 'quantons', or Barut's 'wavelets'. The term wavelet refers to localized nonspreading solutions of massless wave equations that move like massive quantum particles. Wavelets are seen as a bridge between classical point particles and the waves of quantum mechanics; the mass of the particle is determined by the internal frequency of the wavelet. For details see Barut (1993) and references therein.

A few recent examples, taken from among so many others, of the kind of contrasting points of view to which the consideration of the wave properties of matter is prone to lead, may be seen in Agazzi (1988) and Diner et al. (1983); see also Combourieu and Rauch (1992).

views puts its accent on a different feature, they share the fundamental equation, namely the Schrödinger equation, which becomes thus the formal unifying element, even if retaining the two readings.[14]

The particle remains localized somewhere within the wave, and moves in space following a certain trajectory, roughly represented by $\mathbf{v}(\mathbf{x})$. The associated wave, on the other hand, can have a considerable spatial extension; indeed, for a free electron of well-defined velocity the wave train is in principle infinitely long, in the absence of barriers, according to the above equations. Even in a more realistic picture that describes an ensemble of electrons having nearly the same velocity $\mathbf{v}$, both the length and the width of the wave train extend for a good number of wavelengths, as was already inferred by Thomson (1930) from the results of his early diffraction experiments. With the present-day techniques used in high-resolution electron microscopy for the production of monochromatic electron beams, it is possible to obtain a lateral coherence typically of 10^3 wavelengths or more, and wave packets much longer than this [see, e.g., Spence 1988, chapter 4; Ohtsuki and Zeitler 1977]; but it is important to note that these practical limits are due to the angular spread and the energy dispersion of the actual electron beam (represented by the corresponding wave packet). In other words, at the present degree of precision the wave associated with a free electron has indeed an infinite length, in agreement with the description provided by equations (12.52) or (12.54).

12.4. The wave properties of particles

12.4.1. WAVE STATIONARITY AND PHASE QUANTIZATION

It still seems amazing that the wave equation derived to describe the modulation amplitude fully coincides with the Schrödinger equation that describes the behaviour of an ensemble of particles. Let us dig a bit more into the heuristics of such marvelous 'coincidence' and establish contact with some aspects of usual quantum mechanics, with the help of a few simple problems. We shall see, in particular, how the quantization rules emerge in this picture, which affords an intuitive understanding of this most basic phenomenon. The discussion that follows is, apparently, well known; however, there are subtle differences of interpretation with the more conventional expositions, so a reasonably detailed analysis seems to be in place, even at the risk of seeming repetitive.

Consider first a particle and its associated wave, in an infinite one-dimensional potential well of width L. The particle moves with a systematic velocity $\mathbf{v}$; since the walls are perfectly rigid, it bounces back and forth between them at constant speed. In a stationary situation the associated zeropoint modulation will form a stationary pattern, which according to equations (12.37) or (12.40) is of the form $\sim \cos(k_B x)$. Now since the particle is interacting with the modulated wave, it will remain in its stationary state of motion only if it encounters this wave in the same condition after each complete cycle, or after a displacement $\Delta x = 2L$. Now, it will be absurd to demand stationarity of the carrier wave itself; this condition would be just impossible to satisfy, due to the very small wavelengths involved, of the order of the Compton wavelength. In other words, the condition of stationarity can refer only to the modulation. With the phase of the stationary modulation given by $k_B x$, this condition of stationarity means then

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[14] Recall that a similar conclusion was reached in chapter 5 in connection with Einstein's fluctuations formula.

$$k_B \Delta x = 2Lk_B = 2\pi n, \tag{12.58}$$

with n an integer, or (we exclude $n = 0$ because it means $v = 0$)

$$n\lambda_B = 2L, \qquad n = 1, 2, 3, \ldots \tag{12.59}$$

With λ_B given by equation (12.48), this result reads (in the nonrelativistic approximation, and dropping the now unnecessary subindex 0 from the rest mass)

$$p = mv = \frac{nh}{2L}, \quad \text{or} \quad E = \frac{p^2}{2m} = \frac{h^2 n^2}{8mL^2}, \tag{12.60}$$

which means that only for certain discrete values of v does the condition of stationarity hold; the corresponding allowed values of E coincide with the result obtained in quantum mechanics for the energy of a particle in an infinite potential well.

This simple example points to a dynamical role played by the stationary modulation wave: it stabilizes the particle's motion around certain values of the velocity (or the energy), which are determined by the geometry of the accessible space or the boundaries of the system. Also remarkable is the fact that the particle's states of motion are determined (causally, of course) nonlocally by the whole setup, with the radiation field acting as the intermediary. Since a stationary situation is being considered, in which particle and modulation field are already in equilibrium, the radiation field cannot be the free zeropoint field, but an appropriate equilibrium field, that includes the effects of radiation and the appropriate boundary conditions. Qualitatively, this coincides with the point of view developed in chapter10.

A heuristic analysis similar to the above one can be performed in other simple cases in which the particle describes a closed orbit with constant velocity v. Take for instance the circular motion with radius r about a fixed axis. The periodicity condition on the wave reads, from equation (12.58),

$2\pi r k_B = 2\pi n$, and with $k_B = mv/\hbar$ by equation (12.48) one gets the quantization of angular momentum for circular motion

$$l = mvr = n\hbar. \tag{12.61}$$

The picture suggested by this calculation is not far from the popular image that has emerged from de Broglie's theory, of a standing wave filling the circumference of the orbit with exactly an integer number of half-waves. However, here we are referring to the modulation wave, and not to a wave associated to the particle.

Quantization as a result of the demand of phase stationarity appears whenever the particle performs a periodic motion in phase space, even if this motion proceeds with a variable velocity $v(x)$. This is because for such cases one can generalize approximately the above results by imposing the appropriate requisite of stationarity on the modulation. Indeed, for motions with variable velocity one has $k_B = k_B(x)$, and the phase increment

$$\Delta\phi = \int dx\, k_B = \frac{1}{\hbar}\int dx\, p = \frac{m}{\hbar}\int dx\, v \tag{12.62}$$

along one orbit is $\Delta\phi = \hbar^{-1}\oint dx\, p(x)$; hence the condition of quantization becomes, by equation (12.48),

$$\oint p\, dx = m\oint v\, dx = nh. \tag{12.63}$$

This formula reproduces the original Wilson-Sommerfeld quantization rule, which gives correct results only in some particular instances, notably when the particle is reflected head-on by sharp boundaries and v is constant along the trajectory. One should expect equation (12.63) to be a good approximation whenever the smoothed-out path of the particle coincides with the path of the corresponding wave. When the particle moves in a shallow potential well, things become complicated because the partial penetration of the modulated wave into a region where $p = \sqrt{2m(E-V)} < 0$ can contribute a considerable amount to its phase increment $\Delta\phi$, whereas it does not contribute to the (classical) phase integral $\int p\, dx$. Moreover, variations of $V(x)$ may contribute also to alter the amplitude of the modulated wave, an effect that is not taken into account in writing equation (12.37), though it is of course properly described by equation (12.54).

For a general analysis of the quantization of a bounded, periodic or nonperiodic motion in more than one dimension, it would be necessary to consider the complete set of zeropoint waves interacting with the moving particle and to proceed with the corresponding derivation of the modulated wave and its phase quantization. Unfortunately the problem becomes too involved and it is not possible to extract detailed conclusions without a close scrutiny and detailed development of the wave description. [15] Nevertheless, recently a renewed interest in this kind of analysis has appeared, in connection with studies on mesoscopic phenomena for which the semiclassical approximation becomes a necessity.

12.4.2. NONLOCAL WAVE EFFECTS

As already discussed, the wave corresponding to a single particle can extend for a good number of de Broglie wavelengths, spanning all the accessible space, whereas the particle remains essentially localized, with an effective size of the order of a Compton wavelength. This effective size divides the configuration space available to the electron into 'cells' of volume $\sim \lambda_C^3$. Analogously, the time axis is subdivided into intervals of value $2\pi/\omega_C$, or, which amounts to the same, the momentum space is subdivided into cells of linear size $\sim m_0\lambda_C/(2\pi/\omega_C) = m_0c$, whence the whole phase space is structured into cells of volume $\sim \lambda_C^3(m_0c)^3 = h^3$. Thus the convention that quantization means a subdivision of phase space into cells of volume h^3, which can be occupied at most by a single electron, may be interpreted as the result of the electron continuously performing the oscillation of frequency ω_C (or zitterbewegung).

The particle can be effectively localized anywhere in the region where the wave amplitude has a significant value, according to the rule $|\phi|^2 \sim |\psi|^2 = \rho$, valid at least in the stationary case, when ϕ and ψ obey the same equation. This means that there is a tendency from the side of the particles to concentrate in the regions where the modulated carrier has its maximal intensity, or antinodes. Where exactly a specific particle will be found in each instance is a matter of chance; therefore only a statistical description for the particles can match the wave description.

Now, through the spatial extension of the modulation wave, the influence of a given setup on the associated particle will be very different in general from the influence this setup would have on the particle alone. In the foregoing section some examples of this difference were given, referring to simple cases of stationary states for bound particles and leading to the quantization of energy.[16] More generally speaking, seemingly nonlocal effects arise, which become apparent through the influence of the modified wave on the particle. Consider, for instance, Young's double slit experiment performed with electrons. The wave corresponding to particles with a certain velocity is diffracted and interferes with itself after passing through the slits. After having gone through either one of the slits, each particle enters the interference region, where the field exerts an influence on it, and finally it lands at some specific point along the screen. If this experiment is performed with a collimated, monochromatic beam of (many noninteracting) electrons, sent either one by one or simultaneously, but without changing the setup, the coherence of the associated wave is not lost, and thus eventually the well-known electron interference pattern is built up on the screen. If one ignores the role played by the background field in these processes, one can be led to conclude that each electron 'interferes with itself', which seems to be just the right language to conceal the real phenomenon.

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[15] A very suggestive illustration of the potentialities of this approach, presented in a relativistic context, is given in Synge's (1954) monograph, where it is succesfully applied to the study of the Dirac equation.

[16] Another remarkable example that deserves serious consideration is provided by the tunnel effect; by appealing to the influence of the modulation wave on the associated particle, it should be possible in principle to understand the partial penetration of particles through classically forbidden zones and the very 'wavelike' formulas obtained for the reflection and transmission coefficients, etc.

It would be appropriate to remark here that the kind of nonlocalities that we are here discussing, which are due to the interaction with the modulation wave, and thus entirely causal and intelligible, are of a different kind from the nonlocalities discussed in connection with the Bell inequalities (through distant entangled pairs), which will be discussed in the next chapter. Nevertheless, in the quantum-mechanical formalism they all appear simply as nonlocalities, without any further qualification.

12.4.3. INTERFERENCE OF PARTICLES

A variety of very fine Gedankenexperimente related to interference of particles have been actually performed in recent years, thanks to the development of the single-crystal interferometer.[17] From these and related experiments one learns that there are circumstances in which only a full account of the wave aspects of the behaviour of matter allows for an understanding of the facts. We shall approach some of the simpler instances of this complex problem from within the framework of the present theory, in an attempt to gain more intuition on such phenomena.

Let us first describe very schematically a modern particle interferometer. Three slabs are cut from a silicon monocrystal of around 10 cm in length, and aligned with an amazing precision of about 1 part in 10^9, as illustrated in figure 12.1. A beam of neutrons of wavelength of order $1\mathring{A}$ incident with certain inclination on the first slab at A, is split by Bragg scattering into a series of coherent beams; beams I and II are then redirected by a second scattering as they traverse the second slab, to recombine and

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[17] The first interferometer of this kind was used with X rays by Bonse and Hart (1965a, b), and Rauch et al. (1974) were the first to apply it to neutrons. A detailed review, with references to other overviews, is given in Greenberger (1983).

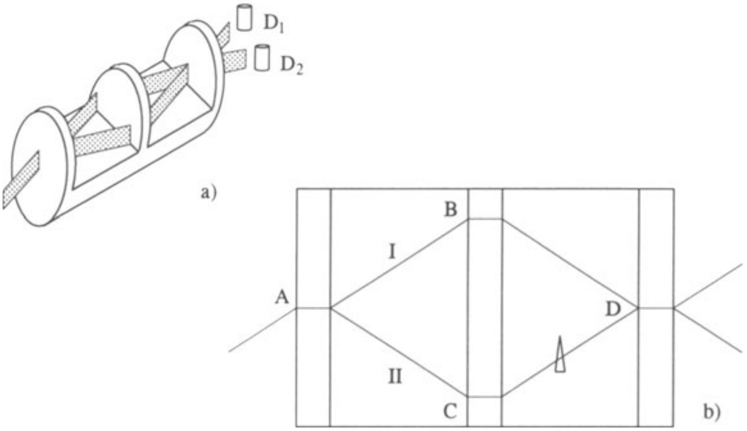

Figure 12.1. Diagram of a single-crystal interferometer. The beam is split at A and recombines at D, giving rise to an interference pattern. In b), a device has been introduced to produce a phase shift on beam II.

interfere at D on the third slab. The phases of beams I and II can each be changed independently at will by various means, as will be seen below, thus affecting the interference pattern at D; the relative phase at D determines the counting rates at the detectors D_1 and D_2 situated beyond the posterior face of the crystal. The neutrons have a relatively low velocity, of the order of 10^5 cm/s, which makes it easy to control electronically any measurement that may be required. Note, incidentally, that although the experiments are performed with particles (normally with neutrons), the wave terminology is freely used to describe what happens in the interferometer. Originally the particles move between the slabs with an essentially constant velocity $v_0 = p_0/m = \sqrt{2E/m}$. Now assume that the upper beam is subject to an external field that varies locally within the interferometer, and call $V(x)$ the corresponding potential. Since the modulation wave is affected by the changes in the momentum of the particle according to equation (12.62), this produces a relative phase shift of the associated wave given by

$$\Delta\phi = \frac{1}{\hbar}\int dx(p - p_0) = \frac{\sqrt{2m}}{\hbar}\int dx\,(\sqrt{E-V} - \sqrt{E}). \tag{12.64}$$

Under normal experimental conditions, $|V(x)| \ll E$ (typically $|V|/E \sim 10^{-7}$); hence one can altogether neglect the variation of the wave amplitude, and for the phase shift one gets

$$\Delta\phi \simeq -\frac{1}{\hbar}\int dx\,\frac{V(x)}{v(x)} = -\frac{1}{\hbar}\int dt\,V, \tag{12.65}$$

where the integration is to be performed along the path of the perturbed beam.

Several different experiments have been performed to which this result is applicable. For instance, with the interferometer in a vertical position, beam II is subject to a slightly smaller gravitational force, a difference in height of 2 cm between the two beams amounting to a difference in potential energy of about 10^{-9} eV. The direct mechanical effect of this difference on the neutrons is minute and totally negligible; but the phase shift that can be produced by it over a distance of about 10cm is of the order of 100 rad! [Colella et al. 1975].

In a different experiment [Werner et al. 1975, Rauch et al. 1975] one of the beams traverses a uniform magnetic field **B** which causes the neutron spin to precess around the direction $\hat{\mathbf{n}}$ of the field. The phase shift is in this case given by equation (12.65) with

$$V = -(e/mc)\mathbf{S}\cdot\mathbf{B} = -2\omega_L \mathbf{S}\cdot\hat{\mathbf{n}}, \tag{12.66}$$

where $\mathbf{S} = (\hbar/2)\hat{\boldsymbol{\sigma}}$ is the spin angular momentum and $\omega_L = eB/2mc$ is the Larmor frequency, as in §8.3.4; hence

$$\Delta\phi = \frac{2\omega_L}{\hbar}\,(\mathbf{S}\cdot\hat{\mathbf{n}})\,t = \omega_L\,(\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{n}})\,t. \tag{12.67}$$

On the other hand, during the same time interval t the neutrons rotate about the axis $\hat{\mathbf{n}}$ through an angle given by $\theta = \omega_B t$, where the frequency of rotation is $\omega_B = eB/mc = 2\omega_L$, hence equation (12.67) can be written as

$$\Delta\phi = \frac{1}{2}\,(\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{n}})\,\theta, \tag{12.68}$$

where $\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{n}} = \pm 1$. Note that the beam does not have to be spin-polarized for this shift to take place, since the spins are always set to precess around $\hat{\mathbf{n}}$ by the external magnetic field. An interesting feature of this phenomenon is that the phase shift is given by half the rotation angle; hence a spin rotation of 4π is needed to produce a phase shift of 2π (and thus recover the original interference pattern), as is well known from the theory of spin.

In an elegant variation of the above experiment [Summhammer et al. 1982], a spin-polarized beam (say, along the $+z$ axis) is sent to the interferometer, and a spin-flip coil rotates the particles in beam II to the spin-down direction, so that the two beams have opposite spin projections. When a detector that registers only spin-up or spin-down particles is used, no diffraction pattern is seen; however when it is rotated to detect the spin projection in the x-direction, the expected interference pattern appears. These experiments illustrate clearly the sort of nonlocal influences that the setup can have on the particles (a spin flip performed on only one of the beams has effects on both, and so on). Such nonlocal influences can be intuitively understood by recalling that any alteration of the particle's state of motion (such as a perturbative force, or a spin flip) changes also its corresponding wave; in its turn, any change in the geometry of the setup (such as the closing of a slit, or the rotation of a detector) affects the background waves and thus has an indirect influence on the associated particles.

A related effect that has been extensively discussed in the literature, and tested in the laboratory, is of course the Aharonov–Bohm effect [Aharonov and Bohm 1959]. In this case the perturbation is produced by a magnetic vector potential $\mathbf{A}(\mathrm{x})$, which can give rise to a phase shift of the wave, $\Delta\phi$, even in regions where the corresponding magnetic field is zero, $\nabla \times \mathbf{A} = 0$, so that there is no direct mechanical effect on the particles. The phase shift can again be obtained from equation (12.62), by considering that the velocity of the particles in terms of the linear momentum changes from $m\mathbf{v} = \mathbf{p}$ to $m\mathbf{v} = \mathbf{p} - (e/c)\mathbf{A}$, whence

$$\Delta\phi = -\frac{e}{\hbar c}\int d\mathrm{x}\cdot\mathbf{A} \tag{12.69}$$

If the two beams are in a region where $\mathbf{A}(\mathrm{x}) \neq 0$, the total relative phase shift is simply given by the integral over the closed circuit $\Delta\phi = (-e/\hbar c)\oint d\mathrm{x}\cdot\mathbf{A} = -\,e\Phi/\hbar c$, with Φ the magnetic flux within the contour. From this example, as from the previous ones, one sees that in an interference experiment the particles can be significantly deviated from their path by a mere phase shift, even when there are no external forces present, to the extent that a phase shift of 180° causes the particles to hit the screen on regions where originally no particles landed. This seems to be a particularly impressive effect of the action of the modulation wave on the particles.

12.4.4. REFLECTIONS ON THE WAVE FUNCTION

Suppose that in the double-slit experiment, neutrons are being sent one at the time, and at some point one decides to intervene with the purpose of determining which slit a certain particle passed through. If this cannot be done without affecting its state of motion, and consequently also its corresponding wave, this particular neutron will not contribute to the interference pattern that has been building up, which does not mean, however, that the whole of this pattern is necessarily destroyed. In fact, it shouldn't; if the experimenter has been careful enough not to upset the setup, he may abandon his pretension to determine the precise path of the particles and resume his original experiment with the neutrons contributing to the formation of the interference pattern. The pattern will be formed by those particles that belong to the coherent beam; particles hitting the screen without being part of the coherent beam will only add noise to the interference fringes and reduce the visibility. If, however, it becomes possible to determine the path of every single particle without significantly altering its state of motion and destroying the coherence of the beam, an interference pattern with good fringe visibility can be obtained whereas at the same time it is clear that every neutron has travelled through only one of the two branches of the interferometer.

Let us now try to focus on the meaning of the wave function, drawing from all what has been said above, and considering first one-particle stationary situations, such as those of the bound-particle problems or the conventional interference experiments just mentioned, for which ψ is the solution of the stationary Schrödinger equation (12.57) with appropriate boundary conditions. In previous paragraphs we have indistinctly referred to a representative particle or to an ensemble of them, all subject to the same experimental conditions, as having the same modulated wave; a statistical meaning is of course assigned thus to the wave function. In addition, however, ψ is not expected to refer to the particle (or the ensemble) alone, but in a given setup. It would therefore be more correct to say that ψ refers to a *situation*; any particle (or any number of independent particles, whether a whole beam, or just one, or none at all) that is found in a given situation will be correctly described by the same ψ, appropriately normalized. Conversely, any particle that ceases to be in that situation (for instance, by being absorbed by a detector) does not form part any more of the ensemble considered; another ψ (an atomic wave function, say) will be necessary to refer to its new situation.

This view helps to understand apparently paradoxical questions such as that raised by Dicke (1981): can ψ be reduced by recording the *absence* of a particle? For instance, consider a neutron that has entered the interferometer and is not absorbed by a detector placed in beam II; does this reduce the ψ for that neutron to beam I? This question seems to imply that the experimenter can change the situation of a neutron without interfering with it. However, no single outcome changes the ψ describing a given experimental situation. Experiments with absent particles are actually quite common in neutron interferometry; they can be performed by placing an absorbing material in one of the beams. Assume a fraction η of the neutrons of beam II are absorbed; these of course will not contribute to the interference pattern. If the original wave function referring to beam II is ψ_{II}, the introduction of the absorber transforms it into $\psi'_{II}(x) = \sqrt{1-\eta}e^{i\delta}\psi_{II}(x)$ beyond the absorber (δ representing any possible phase shift generated by the absorber); it is this wave function with reduced amplitude that is superposed to $\psi_I(x)$ producing the interference pattern at D.

The extension of this simple analysis to systems of two or more particles is not trivial, as is well known, and new phenomena appear, which are completely unknown in the classical domain, such as those related to the statistics of quantum particles and notably the Pauli exclusion principle. We may attempt an heuristic description of the situation as follows. The most dramatic feature that emerges in the passage from a system of one particle to one formed by a pair of quantum particles, is the possibility of entangled states, which were discussed at length in chapter 1. The present viewpoint suggests that an entangled state of two particles is produced when both particles sit on the same carrier wave and together contribute to create a common modulation wave. Of course, this same idea can be extended to the case of several particles. Under such circumstances one may expect the particles to become maximally correlated (or anticorrelated) pairwise, even without a direct classical interaction between them. One can envisage that this situation occurs only when the particles are close enough to each other as to constitute a physical unit together with the carrier wave —as may happen with an atom or a molecule—, so that the stationary states of such complex systems would be the natural place to look for this phenomenon, which is what apparently lies at the root of the exclusion principle. Thus, the essence of the quantum statistics would appear as the result of an indirect interaction mediated by (some components of) the equilibrium zeropoint field; this would explain why in a nonseparable state no particle is in a single-particle state.

CHAPTER 13

STOCHASTIC OPTICS

In this chapter we introduce the reader to the relatively new branch of SED called stochastic optics, whose main subject of study is the origin and meaning of the quantum properties manifested by light in optical experiments. The discussion of this subject provides us with a valuable opportunity to discuss a fundamental problem of much concern to SED, namely, the Bell inequalities and their optical tests.

13.1. The purpose of stochastic optics

After having learned that the quantization of matter can be understood as a result of its permanent interaction with the zeropoint field, we should ask ourselves to what extent also the quantum properties of radiation can be understood as a consequence of this permanent matter-field interaction. Stated in such terms, the question has received almost no attention in SED. Taking into account its fundamental importance, the final chapter of this book has been reserved for a discussion of the possible alternatives that can be envisaged for an answer to it within the framework of SED. This chapter, on the other hand, will be dedicated at length to a somewhat simpler, related question, namely, to what extent some (or all?) quantum properties of the radiation field can be understood as a result of the active presence of the zeropoint field, independently of the quantum properties of matter, or at least without an obvious link with them. For the radiation phenomena dealt with up to now (such as spontaneous emission or the Lamb shift studied in chapter 11) it has been sufficient to consider second-order correlations, which have been shown to be correctly described by resorting to the stochastic nature of the zeropoint field, at least in the nonrelativistic description; on the other hand, the study of higher-order field correlations becomes unavoidable for phenomena which require the field quantization for an explanation.

It is in quantum optics where normally such studies are performed, particularly in relation with the so-called *nonclassical light* and some of its most peculiar expressions, such as the squeezed and number states that have become highly popular in recent years. The existence of nonclassical light is usually considered definitive evidence of the unavoidable quantum (photon) nature of light (which explains the radical name used to refer to it), because it is characterized by having (fourth-order) correlations that arguably cannot be obtained from any classical probability theory. Indeed, they require the use of negative probabilities, which are of course classically unacceptable, but supposedly harmless in quantum theory.[1]

The branch of SED developed to examine how much of the known nonclassical behaviour of light can be understood by assuming a classical description of the propagation of light through macroscopic bodies (mirrors, lenses, polarizers, etc.), but taking the zeropoint field explicitly into account, is known as *stochastic optics*. It turns out that with a few simple hypotheses referring to the interaction of light with macroscopic bodies, nonclassical states of light find a satisfactory explanation within stochastic optics; the set of the reinterpreted phenomena contains all known processes associated with nonclassical light, namely, anticorrelations in a beam splitter, coincidence counting rates in two-photon atomic cascades, antibunching in resonance fluorescence, vacuum squeezing, and so on. The purpose of stochastic optics is not (at least as interpreted by the present authors) to develop a semiclassical theory of the random-field variety, of the kind discussed in section 2.3, but to study under which conditions, if any, the presence of the zeropoint field may help us to understand the particle-like behaviour of light.

Stochastic optics has been under study for almost a decade; it started at a time when SED faced the acute problems with atomic systems referred to in chapter 9, and has grown mainly as a joint effort of T. W. Marshall, E. Santos and a reduced number of collaborators. Even though few authors have contributed to its development, the literature on stochastic optics is far from meagre. A comprehensive overlook of it is still waiting, but several of the published papers contain partial reviews, notably Marshall and Santos (1988a, b) and (1989b), from which we are heavily drawing here.

The theory emerged in an attempt to analyze important problems that belong to SED in a broad sense, but focussing on the behaviour of light; as a practical measure, the difficulties with the description of the interaction with atoms were by-passed with the help of phenomenological models. Thus, the place of a detailed description of matter-field interaction is occupied by a couple of postulates of an empirical flavour, but consistent with the physics suggested by SED, mostly by taking into account atomic fluctuations. However, the theory has been evolving in almost total independence from the more recent developments in SED. It should be borne in mind that rather than being a finished theory, stochastic optics is still in a stage of construction and development.

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[1] More precisely, the nonclassical states of light are those not having a positive Glauber-Sudarshan (P) representation; any classical state of light has a positive P-representation, which can be interpreted without difficulty as the probability distribution of the amplitudes of its normal modes; see §13.3.1.

There are a number of textbooks on quantum optics dealing in detail with the topics relevant to this chapter; among them we cite Loudon (1983), Louisell (1973), Peřina (1984, 1985) and Walls and Milburn (1994). An introduction to some topics of quantum optics, including the theory of detection, is given in Ballentine (1989), chapter 19.

A fundamental motivation for stochastic optics has been the desire to surmount the unrealistic viewpoint of quantum optics on some phenomena, by constructing a local realistic alternative description. As an illustration, consider the well-known experiment in which 'single photons' [2] are sent to a beam splitter, as shown in figure 13.1; in the two channels of the beam splitter the incoming photons are registered by detectors, and anticoincidences are observed. According to the usual description in terms of photons, as one of them is detected a collapse of the entangled state occurs, whereby the photon is localized in one or the other of the two widely separated detectors. This brings into the description all the nonlocalities and problems associated with the collapse, as was discussed in chapter 1, and creates an unconvincing picture from the standpoint of a local realistic conception of nature.[3]

The observed fact that only one of both detectors is normally activated in each case is considered in quantum optics as evidence of the corpuscular (localized, indivisible) properties that can be revealed by the photon under appropriate conditions. The low number of coincidences also observed is explained as due to the accidental presence of more than one photon in the instrument at a given time. In the view of stochastic optics the explanation is entirely another, and totally in wave terms. The electromagnetic packet constituting the photon is effectively divided by the beam splitter, but simultaneously some relevant modes of the zeropoint field are also affected (as illustrated in figure 13.5); depending on the relative phases of the two field components (the packet and the zeropoint beam), in one channel there is constructive interference giving rise to a detection, while in the other one destructive interference takes place by necessity (due to conservation of energy) and no detection occurs. Hence the field is always a field, and no nonlocal effects are required.

Since the descriptions afforded by the two theories differ widely (as the example just discussed makes it clear), one being in terms of noncommuting field amplitudes and the other in terms of a stochastic c-number field, discrepancies between their respective predictions should be expected, particularly in higher-order phenomena. This is certainly the case for several of the reinterpreted problems, but it is also certain that there is agreement between the predictions of both theories and all experiments performed until now, within the experimental errors. So for the time being stochastic optics affords realistic explanations consistent with experiment, even if they may disagree with those of QED. In an attempt to arrive at a more definitive answer, experiments aimed to discriminate between the two formulations have been proposed by Hardy (1991) and by Marshall and Santos in several places.

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[2] Although the term 'photon' is being used here freely, it should be taken with a grain of salt; as will become clear along the discussion, the term refers to elementary processes of absorption or emission of radiation rather than to indivisible and localized corpuscles, or any other specific model. The issue is precisely to find out if SED (or, rather, its optical branch) can help us gain a clearer picture of this uttermost elusive entity of modern physics.

[3] The precise meaning of these terms is discussed in section 13.4.

13.2. Wave and particle properties of light

13.2.1. OBSERVED FACTS AND THEIR INTERPRETATION

We start by recalling some experiments designed to observe both the wave and the particle-like properties of light. As another by-product of his studies on the EPR problem (the best known is the cat paradox), Schrödinger suggested the convenience of carrying out an experimental study of the correlations in the counting of photoelectrons in the two channels (reflected and transmitted) of a beam splitter. The first such experiment was performed by Adam et al. (1955a, b) in Hungary, with an arrangement similar to that shown in figure 13.1. Assuming that the source generates isolated photons, their indivisibility implies the absence of coincident countings. By contrast, a classical analysis in terms of waves (without the zeropoint field, for the moment) predicts beams of equal intensity in the transmitted and reflected arms, and thus a high rate of coincidences. So the experiment is suitable for revealing the quantum behaviour of photons, and indeed this was the result reported by the experimenters, who assumed that the observed coincidences were accidental. However, the conclusion was incorrect

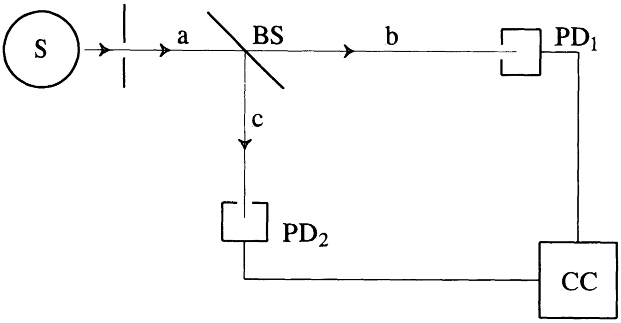

Figure 13.1. Anticorrelation experiment to show the particle-like behaviour of light. S=source; BS=beam splitter; PD=photodetectors; CC=counter and correlator.

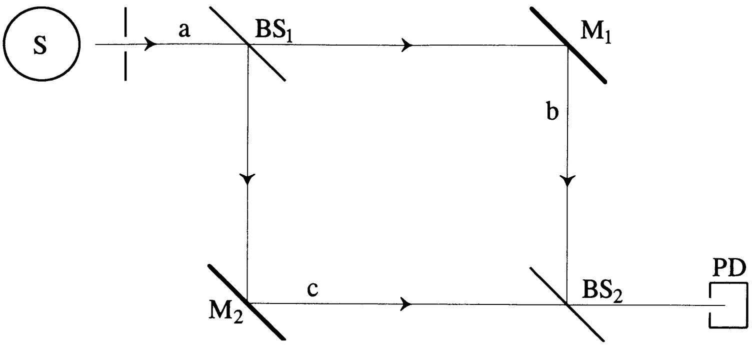

Figure 13.2. Diagram of a recombination experiment to show the wave-like behaviour of light. M=mirrors.

as we know now, because a thermal, and thus, chaotic source of radiation was used, and for chaotic light classical and quantum theories predict just the same correlations. Chaotic light behaves statistically as waves do, at all intensities.[4] For the observation of the particle-like properties, more 'quantal' sources of light are required, a matter to which we turn below.

The apparatus shown in figure 13.1 can be transformed into a recombination device by adding an extra beam splitter and mirror as shown in figure 13.2; detection in a single channel is then enough, since the two channels are physically equivalent. The light beams in the transmitted and reflected channels of the beam splitter BS$_1$ recombine at the second beam splitter BS$_2$, so that by varying the optical length of any of the two arms (or adding a variable phase shifter) the interference pattern observed at the photodetector is modified. Of course, if light is understood as a wave phenomenon this result seems trivial; but in quantum theory, for which light is composed of photons, it requires an explanation. The usual one is that somehow the photon chooses each time from the two possible exclusive behaviours —either undivided particle going through a single arm or divided waves going simultaneously through both arms— according to the setup. This of course brings into play the problems of locality and collapse associated with an entangled state, that were discussed in chapter 1.[5]

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[4] The experience was repeated in the following year by Hanbury Brown and Twiss (1956), still with chaotic light since no other sources were by then available. In a variant of the experiment, photon-number correlations are measured as a function of an arbitrary (but small) delay τ between the registrations of the photodetectors. Small delays are more frequent for chaotic light than for coherent (laser) light; this is the so-called ***photon bunching*** phenomenon, which was first observed experimentally by Twiss et al. (1957). For details see, e.g., Loudon (1973), section 6.5. See also §13.2.3.

[5] Assuming that the source delivers a single photon, after the first beam splitter the field state is described by $|\Psi\rangle = (1/\sqrt{2})\,[|0\rangle\,|1\rangle + |1\rangle\,|0\rangle]$, the entangled state of vacuum

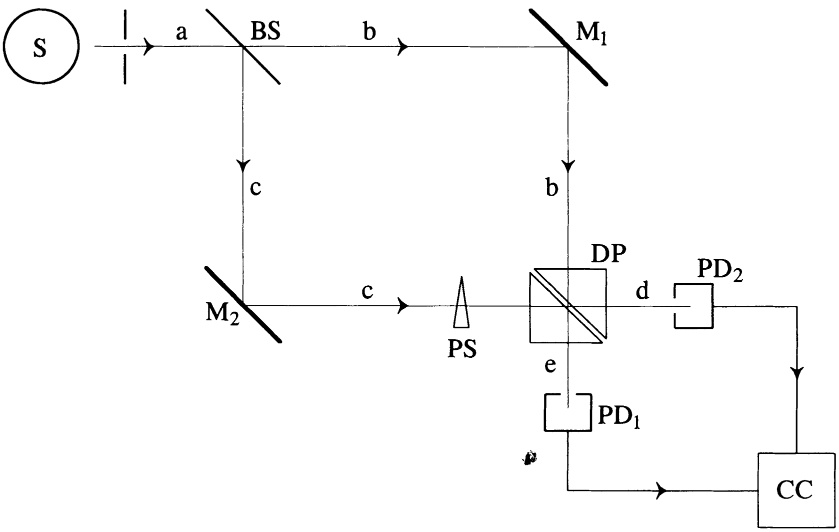

Figure 13.3. Experiment to display wave- and particle-like behaviour of light. PS=phase shifter; DP=double prism.

To illustrate the kind of difficulties involved, assume that the photon was emitted by a quasar millions of light-years away, and replace the beam splitter BS_1 with a galaxy acting as a gravitational lens. Now, the photon must have decided either to come through one or the other arm of the instrument (in case we decide to perform the anticorrelation experiment) or to go through both (if we decide to perform the interference experiment), millions of years before the experiment was actually set up.

This is but an extreme form of a *delayed-choice experiment*,[6] the terrestrial version of which has been used in the last years to test the violation of Bell's inequalities by quantum systems. It is possible to transform the interference experiment sketched above into a delayed-choice version [Ghose et al. 1991, 1992; Mizobuchi and Ohtaké 1992], by replacing the second beam splitter with a couple of prisms DP as shown in figure 13.3. When the prisms are in good contact, light is transmitted; beams b and e are then the same and so are c and d. When the prisms are well separated, each one acts by total reflection as a mirror; thus, beam b goes into d, and c into e. In both cases one gets an anticorrelation (particle-like) result. However, for a certain separation between them, the prisms act as a beam splitter, and the setup corresponds thus to an interference experiment. One can ac-

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plus single–photon states.

[6] Delayed-choice experiments, as they were called by Wheeler, are briefly discussed in Wheeler and Zurek (1983), p. 778, where some pertinent bibliography is given.

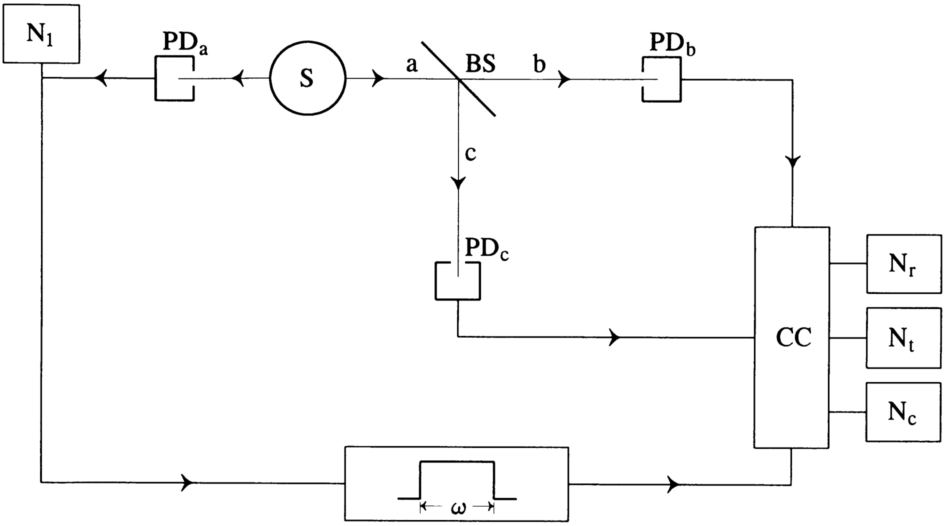

Figure 13.4. Triggered coincidence experiment. Atoms in S produce pairs of cascade photons. Detection at N_l triggers a gate of width w, during which the photodetectors PD$_b$ and PD$_c$ are activated to count the single rates N_r, N_t and coincidences N_c.

tually make it change from particle to wave behaviour, with intermediate situations along the way. It has been possible to verify that the delay in the choice does not have any effect on the outcome [Alley 1983, Hellmuth et al. 1987].

As mentioned above, attention must be paid to the source of light. The advent of the laser opened the way to interference experiments with coherent light, of which the first one was made by Pfleegor and Mandel (1967). Two decades afterwards Grangier et al. (1986) used for this purpose atomic cascades with two emissions, the first one of which triggers a detection window activating both photodetectors for the observation of the cascade partner, as shown in figure 13.4. A similar upgrading was later incorporated into the arrangement of figure 13.3.

In an anticorrelation experiment (e.g., figure 13.3 with no PS and perfect transmission at DP) one measures the counting rates $N_e = N_b$ and $N_d = N_c$ and the coincidence counts $N_{bc} = N_{de}$; these are used to calculate the normalized fourth-order correlation[7]

$$\alpha = \frac{p_{bc}}{p_b p_c}, \tag{13.1}$$

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[7] Note that the quantities of interest are indeed of fourth-order with respect to the field intensities (amplitudes). However, they are of second-order with respect to the counts, and some authors [as, e.g., Walls and Milburn 1994] refer to them as of second order.

assuming that the detection probabilities p_b, etc. are proportional to the corresponding counting rates.

For completely independent events, the joint probability is clearly $p_{bc} = p_b p_c$ so that $\alpha = 1$. For classical light, made of waves whose intensities are divided by the beam splitter, it follows by applying the Cauchy-Schwartz inequality that $\alpha \geq 1$. In contrast, if the signal consists of single photons, their indivisibility implies total anticorrelation, i.e. $p_{bc} = 0$, and thus $\alpha = 0$; any value of $\alpha < 1$ is thus considered a signature of nonclassical light. In the quantum language p_{bc} is proportional to (the ± signs denote the positive- or negative-frequency parts of the fields)

$$\left\langle \hat{E}_b^{(-)} \hat{E}_c^{(-)} \hat{E}_c^{(+)} \hat{E}_b^{(+)} \right\rangle \sim \left\langle \hat{a}_b^\dagger \hat{a}_c^\dagger \hat{a}_c \hat{a}_b \right\rangle = \langle \hat{n}_b \hat{n}_c \rangle . \tag{13.2}$$

because of the commutation between operators referring to the different channels. In this language, for a signal consisting of one photon only, necessarily $\langle \hat{n}_b \hat{n}_c \rangle = 0$, hence $\alpha = 0$.

In an interference experiment the probabilities p_d and p_e depend on the position of the phase shifter (assuming 50-50 division at DP), so that it is conventional to define a fringe *visibility* v as

$$v = \frac{p_{\max} - p_{\min}}{p_{\max} + p_{\min}} . \tag{13.3}$$

The value of v falls between 1 (maximum possible interference) and 0 (no interference at all). In the experiment of Grangier et al. (1986) the results were $\alpha \sim 0.1$ and $v = 0.98$, clearly confirming the existence of an interference pattern in the single-photon states.[8]

13.2.2. HEURISTIC DESCRIPTION ACCORDING TO STOCHASTIC OPTICS

Let us present now a heuristic description of the phenomena described above from the standpoint of stochastic optics. One expects the incorporation of the zeropoint radiation field to produce qualitative changes in the behaviour of the system with respect to its classical (i.e., no zeropoint field) counterpart, hopefully bringing it closer to the quantum behaviour. The discussion that follows will be reduced to its simplest possible terms, since the intention is to help us grasp the consequences of the presence of the vacuum field.

We start by considering the action of a beam splitter.[9] The usual (deterministic) account given in classical optics is illustrated in figure 13.5(a).

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[8] These figures were taken from Marshall and Santos (1988b).

[9] We follow the expositions given in Marshall and Santos (1988a, b) and Marshall (1988c).

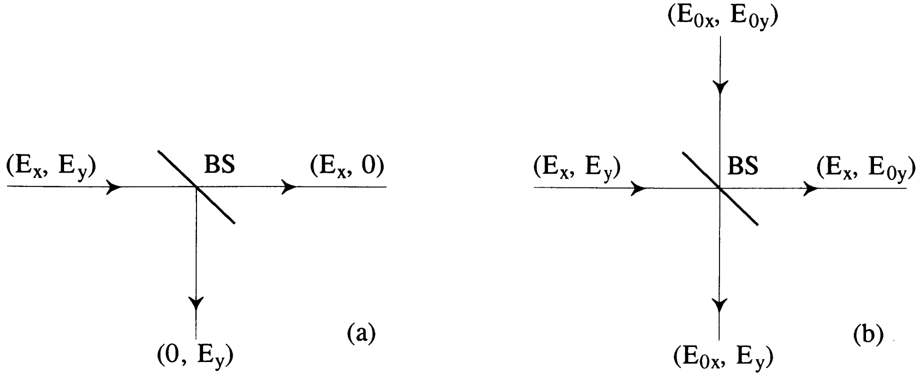

Figure 13.5. Beam splitter according to: a)classical optics, b)stochastic optics.

In stochastic optics the same treatment is given to the incident beam, but the existence of a relevant mode of the zeropoint field is taken into account, so that the transmitted and reflected beams are written as a superposition of components coming from both fields (represented by their transverse electric field components),

$$E_t = \frac{1}{\sqrt{2}}(E + iE_0)\,; \qquad E_r = -\frac{1}{\sqrt{2}}(E_0 + iE)\,. \tag{13.4}$$

It is convenient to write the zeropoint field in the Stokes form, i.e., as a two-dimensional vector perpendicular to the propagation vector, with components

$$E_0 = \mathrm{Re} E_0 e^{i(\omega t + \chi_0)} \begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 e^{i\psi_0} \end{pmatrix}. \tag{13.5}$$

For simplicity the amplitude E_0 is taken fixed. The angles $\lambda_0 \equiv (\phi_0, \psi_0, \chi_0)$ are the Stokes parameters of classical optics [Born and Wolf 1964, p. 30; Peřina 1985, chapter 6]; they play here the role of hidden parameters, and are random variables whose distribution can be determined from the requisite of rotational invariance. Thus, the field is distributed according to

$$\rho(\phi, \psi, \chi) = \frac{\sin 2\phi}{4\pi}, \qquad 0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \psi, \chi < 2\pi. \tag{13.6}$$

A most important assumption is now made, namely, that the detection probability in a given channel has the form

$$P = \xi w(2\overline{E^2} - \gamma)_+. \tag{13.7}$$

Here ξ depends on the detector efficiency and the geometry, w is the time window and $2\xi w\overline{E^2}$ is the usual semiclassical expression for the activation probability of a signal of amplitude E [see, e.g., Mandel 1976]. A threshold parameter γ has been introduced to subtract the contribution of the zeropoint field, for which the detection probability is zero; the symbol $(\cdot)_+$ means that P is zero if the quantity within parentheses is negative.

Consider now a coincidence experiment, as illustrated in figure 13.6, which is figure 13.3 with the relevant modes of the incident fields explicitly indicated. An analysis shows that the relative intensity of the signal in the transmitted and reflected channels lies between $(\beta-1)^2/2$ and $(\beta+1)^2/2$, where $\beta = |E/E_0|$. The sum of these intensities is β^2+1, hence for $\gamma > \frac{1}{2}(\beta^2+1)$ at most one detector can be activated by a single-atom signal. Further, in order that activation of the detector occurs for some values of the relative phases between the original signal and the zeropoint component, the condition $\gamma < \frac{1}{2}(\beta+1)^2$ should also be satisfied. It follows that the description gives the same statistics for the correlation as quantum optics, provided

$$(\beta+1)^2 > 2\gamma > \beta^2+1. \tag{13.8}$$

Indeed, a more detailed calculation shows that with quite reasonable values for β and γ, for a perfect beam splitter the anticorrelation parameter α acquires values in the region of 0.1, which though not coinciding with those of quantum optics, fits the experimental data available as well as quantum optics does. It is important to notice that this model, without making use of a quantum field, gives $\alpha < 1$, which as stated above, is normally regarded as definitive evidence of the quantum nature of light, and has even been considered definitive evidence of quantum nonlocality [Chubarov and Nikolayev 1985]. This result, added to those of chapter 11 (and others to be added below) show how hazardous it can be to use terms such as 'definitive' when applied to our present understanding of the quantum world.

A very satisfying property of the model is that the recombination experiment and the observed visibility of 100% can be easily explained, despite the presence of the random field. Indeed, as seen from figure 13.6, the second beam splitter recombines the signals E_r and E_t from the first beam splitter and produces the transmitted signal

$$E_t' = \frac{1}{\sqrt{2}}\left[E_t + ie^{ikl}E_r\right], \tag{13.9}$$

where l represents the difference between the path-lengths, controlled by PS. From equations (13.4) and (13.9) we get

$$E_t' = \frac{1}{2}\left[(1+e^{ikl})E + i(1-e^{ikl})E_0\right] = \left\{ \begin{array}{ll} E, & l=0 \\ iE_0, & l=\pi/k \end{array} \right., \tag{13.10}$$

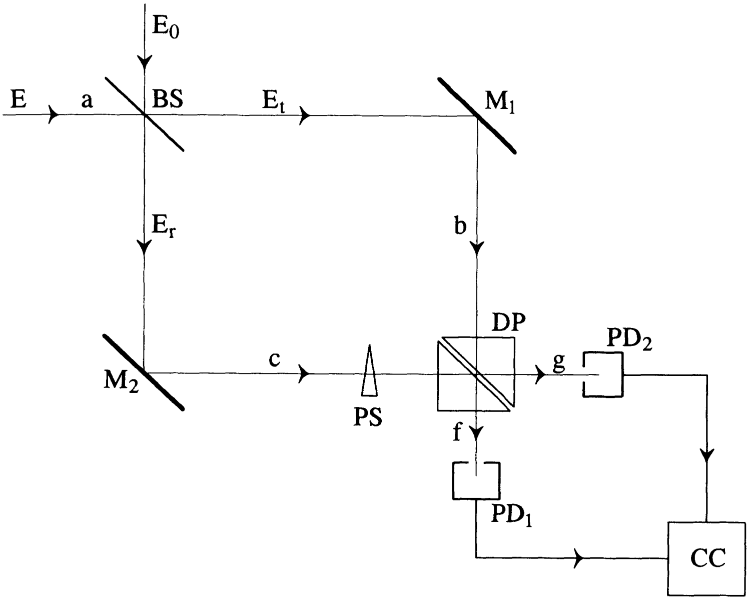

Figure 13.6. Recombination (interference) experiment according to stochastic optics.

so when the path-lengths differ by an integral number of half-wavelengths, the signal E and the noise E_0 are perfectly reproduced in the outgoing channels. Since the threshold is above the zeropoint level ($\gamma > 1$) the fringe visibility is 100%, just as in the quantum (and the classical) case. Further, this behaviour is not affected by a delayed decision about the insertion of the second beam splitter.

Although the above heuristic arguments give a satisfactory qualitative explanation of the wave and particle behaviour of light signals, the actual coincidence rates predicted from a more detailed analysis are not identical with those of quantum optics. More specifically, there is even a range of values of the parameters of the zeropoint field for which the beam splitter gives signals of around the same intensity in both channels, resulting in a simultaneous activation of both detectors, which is quite contrary to the description in terms of photons. This effect is observable in principle, but so small that the existing data are not enough to discriminate between the two theories.

Such slight differences are characteristic and inevitable in stochastic optics, even when more detailed models are used, so that there is room for future experiments to decide about the legitimacy of the present point of view.

13.2.3. HEURISTICS OF PHOTON ANTIBUNCHING

The analysis of fourth-order correlations can be easily extended to study the time dependence, i.e., the statistics of the activation times of the detectors. As an interesting example consider the two-time correlations in photon emissions from a single atom. In such experiments the light emitted by the excited atom is sent through a beam splitter toward two photodetectors and the number $N(\tau)$ of delayed coincident counts is measured, where τ is the delay between the activation of the first and the second detector (the interval τ is discretized in small units). Quantum optics predicts [Carmichael and Walls 1976] and experiment shows [Dagenais and Mandel 1978] that $N(\tau)$ increases from its initial value $N(0)$ up to a maximum, and then decreases, sometimes in an oscillatory way, towards its asymptotic value $N(\infty)$. What is of interest for us here is the increase of $N(\tau)$ for small delays, which is interpreted conventionally as a tendency of the emitted photons to separate from each other in time; this phenomenon is known as *photon antibunching*, and belongs to the category of nonclassical light phenomena.

In quantum optics the problem is solved by constructing the normalized correlation function [Walls and Milburn (1994), section 11.3]

$$g^{(2)}(\tau) = G^{(2)}(\tau) / \left\langle n(t)^2 \right\rangle, \tag{13.11}$$

$$G^{(2)}(\tau) = \left\langle \hat{E}^{(-)}(t)\hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t)\hat{E}^{(+)}(t+\tau) \right\rangle = \langle : \hat{n}(t)\hat{n}(t+\tau) : \rangle , \tag{13.12}$$

where, as usual, the dots indicate normal ordering, and showing that for a single atom $g^{(2)}(0) = 0$, whereas $g^{(2)}(\tau) > 0$ for $\tau > 0$. For classical fields, on the other hand, assuming a 50-50 division of the signal by the beam splitter, the correlation is given in terms of the corresponding light intensities by

$$g^{(2)}_{\text{cl}}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle \langle I(t+\tau)\rangle}, \tag{13.13}$$

so that $g^{(2)}(0) \geq 1$; further, $g^{(2)}(0) > g^{(2)}(\tau)$ for $\tau > 0$, hence there is no antibunching.

However, the above conclusion requires reconsideration in the light of the present theory; firstly because according to our previous discussion only the intensity above threshold can trigger detection events; secondly, because the division of the signal by the beam splitter into transmitted and reflected beams is a stochastic process, so that instead of (13.13) one should write

$$g^{(2)}_{\text{s}}(\tau) = \frac{\langle n_r(t)n_t(t+\tau)\rangle}{\langle n_r(t)\rangle \langle n_t(t+\tau)\rangle}, \tag{13.14}$$

which is a direct generalization of equation (13.1) to the time-dependent case. To get a qualitative idea of the kind of predictions that can be derived from this expression, we make the approximation

$$n_r(t) + n_t(t) \simeq n(t), \tag{13.15}$$

and write

$$g_s^{(2)}(\tau) = \frac{\langle n(t)n(t+\tau)U(t)\,(1-U(t+\tau))\rangle}{\langle n(t)U(t)\rangle\,\langle n(t+\tau)\,(1-U(t+\tau))\rangle}, \tag{13.16}$$

with $U(t) = n_r(t)/n(t)$. A few extra simplifying assumptions can be made. Firstly, $n(t)$ and $U(t)$ can be considered statistically independent stochastic processes, with $\langle U(t)\rangle = 1/2$, so that the above expression reduces to

$$g_s^{(2)}(\tau) = \frac{\langle n(t)n(t+\tau)\rangle}{\langle n(t)\rangle\,\langle n(t+\tau)\rangle}\,[2 - 4\,\langle U(t)U(t+\tau)\rangle]\,. \tag{13.17}$$

From the above discussion we know that for low intensities $U(t)$ is essentially a process switching between 0 and 1. For large enough delays between countings, assuming that the memory of the phases of the noise is lost, the number of switchings follows a Poisson distribution; however, for small delays a better approximation would be [Marshall and Santos 1988b][10]

$$\langle U(t)U(t+\tau)\rangle = \frac{1}{4}\left(1 + Ce^{-\alpha\tau^2}\right), \qquad C \lesssim 1 \tag{13.18}$$

so that

$$g_s^{(2)}(\tau) = \left(1 - Ce^{-\alpha\tau^2}\right)\frac{\langle n(t)n(t+\tau)\rangle}{\langle n(t)\rangle\,\langle n(t+\tau)\rangle}. \tag{13.19}$$

This expression shows that for small delays, $g_s^{(2)}(\tau)$ may indeed attain values close to zero; the model serves thus to explain the presence of antibunching. A more detailed analysis shows that equation (13.19) fits quite well the experimental data [Marshall and Santos 1988b]. The particle-like behaviour is generated in the present description by the interplay of the zeropoint field and the beam splitter, through the stochastic process $U(t)$.

13.3. The Wigner representation in stochastic optics

13.3.1. THE QUANTUM DISTRIBUTIONS

In quantum optics it is usual to describe the radiation in terms of some privileged representation, the most popular ones being the Glauber-Sudarshan

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[10] Equation (13.18) gives $\langle U^2\rangle - \langle U\rangle^2 = \langle U\rangle C/2$, so that for $C < 2$, $\sigma_U^2 < \langle U\rangle$ and the related process is sub-Poissonian. We recall that the squeezed states are also sub-Poissonian and that the near-number states of quantum optics are strongly sub-Poissonian.

P (normally ordered), the Husimi Q (antinormally ordered) and the Wigner W (symmetrically ordered) representations.[11] For every single field mode, the corresponding functions $P(\alpha)$, $Q(\alpha)$ and $W(\alpha)$ are defined in terms of appropriate characteristic functions $\chi_N(\xi)$, $\chi_A(\xi)$, $\chi_W(\xi)$ of the density operator $\hat{\rho}(\hat{a},\hat{a}^\dagger)$, where $\hat{a},\hat{a}^\dagger$ represent the field operators and ξ are complex variables, $(d^2\xi = d(Re\{\xi\}) \cdot d(\mathrm{Im}\{\xi\}))$,

$$P(\alpha) = \pi^{-2} \int e^{\alpha\xi^* - \alpha^*\xi} \chi_N(\xi) d^2\xi, \qquad \chi_N(\xi) = \mathrm{Tr}\left\{\rho e^{\xi a^\dagger} e^{-\xi^* a}\right\}; \quad (13.20)$$

$$Q(\alpha) = \pi^{-2} \int e^{\alpha\xi^* - \alpha^*\xi} \chi_A(\xi) d^2\xi, \qquad \chi_A(\xi) = \mathrm{Tr}\left\{\rho e^{-\xi^* a} e^{\xi a^\dagger}\right\}; \quad (13.21)$$

$$W(\alpha) = \pi^{-2} \int e^{\alpha\xi^* - \alpha^*\xi} \chi_W(\xi) d^2\xi, \qquad \chi_W(\xi) = \mathrm{Tr}\left\{\rho e^{\xi a^\dagger - \xi^* a}\right\}. \quad (13.22)$$

It is clear that the differences between the three functions P, Q, W are due exclusively to the noncommutativity of the operators $\hat{a}, \hat{a}^\dagger$. These functions are convenient for the evaluation of normal, antinormal or symmetrized products of the field operators, respectively, as shown by the following results:

$$\left\langle \hat{a}^{\dagger n} \hat{a}^m \right\rangle = \int d^2\alpha P(\alpha) \alpha^{*n} \alpha^m, \quad (13.23)$$

$$\left\langle \hat{a}^m \hat{a}^{\dagger n} \right\rangle = \int d^2\alpha Q(\alpha) \alpha^m \alpha^{*n}, \quad (13.24)$$

$$\left\langle S\left(\hat{a}^m \hat{a}^{\dagger n}\right) \right\rangle = \int d^2\alpha W(\alpha) \alpha^{*n} \alpha^m, \quad (13.25)$$

where S means that the expression is totally symmetrized, with the use of the commutation rule $[\hat{a}, \hat{a}^\dagger] = 1$.

The three functions are related through integral transformations; in particular,

$$Q(\alpha) = \frac{1}{\pi} \int d^2\beta P(\beta) \left|\langle\alpha|\beta\rangle\right|^2; \qquad W(\alpha) = \frac{2}{\pi} \int d^2\beta P(\beta) \left|\langle\alpha|\beta\rangle\right|^4. \quad (13.26)$$

Here $|\alpha\rangle$ and $|\beta\rangle$ represent coherent states of the field, so that $|\langle\alpha|\beta\rangle|^2 = \exp\left(-\left|\alpha - \beta\right|^2\right)$.[12] Thus, Q and W are both Gaussian convolutions of the P function, the Q distribution being wider than W. These expressions are frequently merged in a single one, by introducing the function W_ε, where to ε is assigned the values 0, 1/2 and 1 so that $W_0 = P$, $W_{1/2} = W$ and $W_1 = Q$. Then, in particular,[13]

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[11] Detailed discussions of these representations can be seen, e.g., in Hillery et al. (1984), Gardiner (1991) and Walls and Milburn (1994).

[12] Recall that the coherent states are not orthogonal.

$$Q(\alpha) = \frac{1}{\pi(1-\varepsilon)} \int d^2\beta e^{-|\alpha-\beta|^2/(1-\varepsilon)} W_\varepsilon(\beta). \tag{13.27}$$

Consider further the *displaced-number states* defined as

$$|\alpha, k\rangle = D(\alpha)\,|k\rangle = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}\,|k\rangle\,, \tag{13.28}$$

where $D(\alpha)$ is the displacement operator; then the function W_ε can be written in terms of displaced-number expectations of the density operator [Moya-Cessa and Knight 1993] ,

$$W_\varepsilon(\alpha) = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(\varepsilon-1)^k}{\varepsilon^{k+1}}\langle\alpha, k|\,\rho\,|\alpha, k\rangle\,. \tag{13.29}$$

The Q and W functions are uniformly continuous for any field state [Cahill and Glauber 1969], but the P function may become highly singular for certain states. The Q function is always positive, since it can be expressed as a coherent-state expectation of the density operator $\hat{\rho}$ in the form

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\,\hat{\rho}\,|\alpha\rangle\,; \tag{13.30}$$

because it is also bounded $(Q(\alpha) \leq 1/\pi)$ and normalized to unity, it has the properties of a probability density. Something approximately similar happens with the Wigner function; however, it can attain negative values, as does also the P function. The equation of evolution for these functions can be derived from the corresponding equation for the density matrix; the result is that only W satisfies a Liouville equation for Gaussian fields.[14]

13.3.2. ROLE OF THE WIGNER FUNCTION IN STOCHASTIC OPTICS

Both functions Q and W have been considered good candidates for describing the phase–space distributions in stochastic optics, but the specific properties of the Wigner function —particularly, that it obeys a Liouville equation without third-order terms, as opposed to the other two functions—have led to the conjecture that this function should describe the field states in stochastic optics, as is most clearly argued in Santos (1992b). The issue of the nonpositivity of the Wigner function becomes then crucial, and has been approached as follows. A detailed analysis of the problems of interest in quantum optics shows that $W > 0$ for practically all of them, the most important exception being the number states, represented by an expression similar to (7.133) in terms of Laguerre polynomials. However, pure number states cannot be constructed in the laboratory, and it is reasonable to assume that only mixtures of the vacuum state and a number state are realizable. It happens that in realistic conditions, such mixtures have a positive Wigner function, which seems to give support to the conjecture that all states which are realizable in the laboratory have positive Wigner densities. Rather than a real modification of quantum optics, this is a restrictive postulate, which points towards a method to discriminate physical from unphysical field states. Due to the importance of the point, it seems appropriate to discuss with some more detail a couple of examples.[15]

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[13] The details of the conventions may change from author to author. For a more detailed discussion see Graham et al. (1968).

[14] The formal problem for the free field corresponds just to the oscillator studied in chapter 7; refer to equation (7.43) with $\gamma = 0$. Marshall and Santos (1990) have added the observation that in the study of a reduced problem by means of a Fokker-Planck-like equation for the function W_ε (their equation (13)), there are strange diffusions (that may be even negative) if $\varepsilon \neq 0$, but the equation reduces to a genuine Fokker-Planck equation for $\varepsilon = 0$, i.e., for the Wigner function.

Consider first monochromatic coherent light of frequency ω, with its electric field decomposed into two quadratures, one proportional to $\cos\omega t$ and the other to $\sin\omega t$; both have the same fluctuations and their product minimizes the Heisenberg inequality. Now consider a *squeezed state*, in which the fluctuations of one quadrature are artificially reduced and those of the other one are correspondingly increased so that the product of the dispersions still minimizes Heisenberg's inequality. Depending on the selection of the parameters, the electric vibration can occur in any intermediate state from a minimum of indeterminacy in the phase and correspondingly highly fluctuating amplitude, to the inverse situation, as illustrated in figure 13.7. Formally, a squeezed state is obtained by first squeezing the vacuum field and then displacing it, so that it can be written as[16]

$$|\alpha,\varepsilon\rangle = D(\alpha)S(\varepsilon)\,|0\rangle = e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}}e^{(\varepsilon^*\hat{a}^2-\varepsilon\hat{a}^{\dagger 2})/2}\,|0\rangle\,. \tag{13.31}$$

The corresponding Wigner function can be constructed by applying the above definition to the minimum-dispersion packet constructed from the Wigner function (7.24). Denoting by x_1, x_2 the two quadratures and by a_1, a_2 the two displacements, a single-mode amplitude-squeezed coherent state

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[15] The role of the Wigner function in stochastic optics is discussed in detail, among other places, in Marshall (1990), Marshall and Santos (1991a, b, 1992), Marshall et al. (1994a, b) and Santos (1992b, d).

[16] In a representation of the vacuum field such as (4.35), the quadratures of interest may correspond to the variables q, p, as follows from equations (4.51). For a detailed account of the quantum theory of squeezed states see Walls and Milburn (1994), section 2.4 and passim; for a discussion from the point of view of stochastic optics see Marshall (1990) and Marshall and Santos (1990).

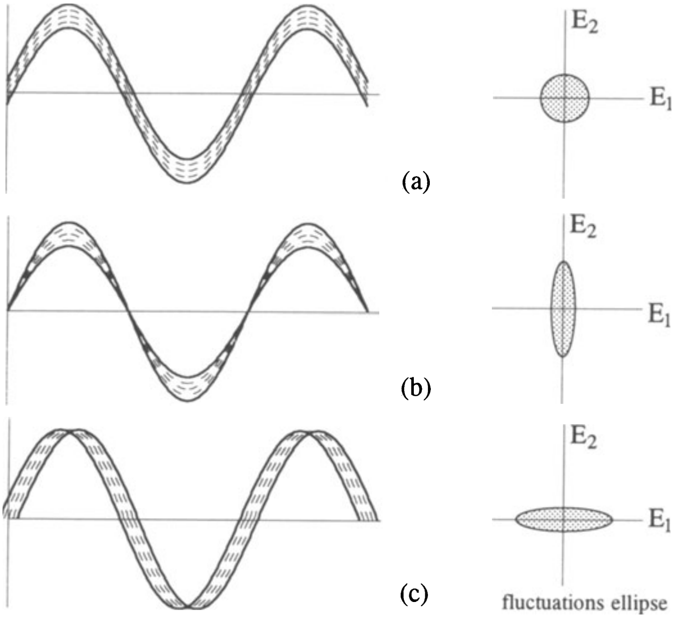

Figure 13.7. Phase and amplitude fluctuations of the electric field in: (a) a coherent state, (b) a squeezed state with reduced phase fluctuations, and (c) as (b), but with reduced amplitude fluctuations. [Adapted from Walls and Milburn (1994), §2.6.]

with squeezing $r = |\varepsilon|$ has the Wigner function

$$W_{r,a} = \frac{1}{2\pi\sigma_0^2} e^{-\frac{1}{2\sigma_0^2}\left[e^{2r}(x_1-a_1)^2+e^{-2r}(x_2-a_2)^2\right]} W', \tag{13.32}$$

where W' represents the probability density of the (unmodified) remaining modes of the field. The parametrization used for the squeezing is conventional, so that an entirely similar expression is used in quantum theory to represent the corresponding squeezed states [see, e.g., Walls and Milburn 1994, equation (4.43)]. Of course, there is no difficulty here with the positivity of the Wigner function.

As a second and more delicate example, consider a number state. Preparation of single-photon states can be achieved experimentally by absorption of one of the two correlated photons produced by parametric down-conversion; such a procedure was used for the first time in Hong and Mandel (1986). As discussed above, convincing evidence of the reality of such states is provided by measuring the anticorrelation counts in the two channels of a beam splitter. The n-photon field is conventionally described in quantum optics by its Q function,

$$Q_n(\alpha) = \frac{|\langle\alpha|n\rangle|^2}{\pi} = \frac{|\alpha|^{2n}}{\pi n!}e^{-|\alpha|^2}. \tag{13.33}$$

The corresponding Wigner function (cf. equation (7.133)),

$$W_n(\alpha) = \frac{1}{\pi}(-1)^n L_n(2|\alpha|^2)e^{-|\alpha|^2}, \tag{13.34}$$

is clearly nonpositive definite for $n > 0$; in particular, for the single-photon case one gets

$$W_1(\alpha) = \left(2|\alpha|^2 - 1\right)W_0(\alpha). \tag{13.35}$$

If, as was argued above, the one-photon state cannot exist in isolation, but coexists with the vacuum state, the physical state is neither $|1\rangle$ nor the entangled state $(1/\sqrt{2})\,[|0\rangle\,|1\rangle + |1\rangle\,|0\rangle]$ but the mixture $(1/2)\,[|0\rangle\,\langle 0| + |1\rangle\,\langle 1|]$ represented by the Wigner function

$$W_1'(\alpha) = \frac{1}{2}(W_0 + W_1) = |\alpha|^2\,W_0(\alpha), \tag{13.36}$$

which is positive for all values of α. A closer analysis reveals that the state described by this density behaves in practically all important respects like the state (13.35) [Marshall and Santos 1989a]. The conjecture about the physically realizable states implies then that independently of the differences, (13.36) gives the nearest description of a one-photon state that can be realized in the laboratory. Observe that W_1' is the same as the Q_1 representation for this single-photon state.[17]

An immediate consequence of the use of the Wigner distribution is the appearance of a threshold for the counting probabilities, which justifies its introduction in the previous discussion. This can be shown as follows. In quantum optics the single counting probability at time t of a light beam with a narrow frequency band around ω_0, for a window open during a time interval w and quantum efficiency of detection η, is defined as[18]

$$p_a = \frac{\eta}{\hbar\omega_0}\int_0^w ds\,\langle : I_a(t+s) : \rangle\,, \tag{13.37}$$

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[17] Although the equality of W_1' and Q_1 is a simple coincidence, there always exists a linear relationship between W_n and $Q_m, 0 \le m \le n$ and viceversa, which can be established with the help of equations (13.33) and (13.34). For details on the representations of the number states in quantum optics see, e.g., Walls and Milburn 1994, chapter 4.

[18] Note that this definition applies to absorption processes only; the inclusion of emission processes in the subsequent formulas would lead to difficulties, as becomes evident if one tries to include in equation (13.40) the spontaneous emissions.

For a discussion of the quantum theory of detection see Louisell (1973), chapter 3, and Peřina (1984), chapter 4.

where $I(t) \propto < \hat{E}^{(-)}\hat{E}^{(+)} >$; analogously, the coincidence probability is given by

$$p_{bc} = \left(\frac{\eta}{\hbar\omega_0}\right)^2 \int_0^w ds \int_0^w ds' \left\langle : I_b(t+s) I_c(t+s') : \right\rangle . \tag{13.38}$$

Now, when averaging over a Wigner function one may write

$$\begin{aligned} &\left\langle \hat{E}^{(-)}\hat{E}^{(+)} \right\rangle \\ &= \tfrac{1}{2}\left\langle \hat{E}^{(-)}\hat{E}^{(+)} + \hat{E}^{(+)}\hat{E}^{(-)} \right\rangle - \tfrac{1}{2}\left\langle 0 | \hat{E}^{(+)}\hat{E}^{(-)} - \hat{E}^{(-)}\hat{E}^{(+)} | 0 \right\rangle \\ &= \tfrac{1}{2}\left\langle \hat{E}^{(-)}\hat{E}^{(+)} + \hat{E}^{(+)}\hat{E}^{(-)} \right\rangle - \tfrac{1}{2}\left\langle 0 | \hat{E}^{(+)}\hat{E}^{(-)} + \hat{E}^{(-)}\hat{E}^{(+)} | 0 \right\rangle , \end{aligned} \tag{13.39}$$

where the first equality comes from the fact that the commutator is a c-number and the second one holds because the vacuum expectation of a normally ordered expression vanishes. The operators are now symmetrized and one can apply equation (13.25), from which it follows that

$$\langle : I(t) : \rangle = \langle I(t) \rangle - \langle 0 | I(t) | 0 \rangle = \langle I(t) \rangle - I_0 . \tag{13.40}$$

Similarly, one gets

$$\langle : I_b(t) I_c(t') : \rangle = \langle I_b(t) I_c(t') \rangle - I_0 \left[\langle I_b(t) \rangle + \langle I_c(t') \rangle \right] + I_0^2 . \tag{13.41}$$

Introducing these results into the single and joint probabilities, one obtains

$$p_a = \frac{\eta}{\hbar\omega_0} \int_0^w ds \, \langle I_a(t+s) - I_0 \rangle , \tag{13.42}$$

$$p_{bc} = \left(\frac{\eta}{\hbar\omega_0}\right)^2 \int_0^w ds \int_0^w ds' \left\langle [I_b(t+s) - I_0] \left[I_c(t+s') - I_0 \right] \right\rangle , \tag{13.43}$$

which shows that detectors are sensitive to radiation intensity above the zeropoint level only. These results provide a justification for equations (13.7) and (13.19), and explain why in anticorrelation experiments with very low intensities (such as those corresponding to one photon) only one detector can be fired at a time: the superposition with the zeropoint field in the beam splitter lowers the intensity below the threshold in one of the channels, while at the same time (by energy conservation) its value is well above threshold in the other channel.

The above ideas can be applied to the recombination experiment illustrated in figure 13.6 [Marshall et al. 1994b]. With the vacuum included, the fields in the arms b and c are

$$E_b(t) = \frac{1}{\sqrt{2}} \left[E_a(t) + i E_0(t) \right] ; \qquad E_c(t) = \frac{-1}{\sqrt{2}} \left[E_0(t) + i E_a(t) \right] . \tag{13.44}$$

Calling r the reflection coefficient of DP (which may be changed gradually between 0 and 1), and ϕ the phase shift introduced by PS, the field in arm f is

$$E_f(t) = \sqrt{1-r}E_b(t) + i\sqrt{r}E_c(t)e^{i\phi}, \tag{13.45}$$

assuming that the optical lengths of beams b and c are the same; otherwise a redefinition of ϕ takes care of the difference. It follows that the intensity I_f is given by

$$I_f = \frac{1}{2}\left[1 + 2\sqrt{r(1-r)}\cos\phi\right](I_a - I_0) + I_0, \tag{13.46}$$

so that equation (13.42) gives

$$p_r = \frac{\eta}{\hbar\omega_0}\langle I_a - I_0\rangle\left(\frac{1}{2} + \sqrt{r(1-r)}\cos\phi\right). \tag{13.47}$$

The visibility, given by

$$v = 2\sqrt{r(1-r)}, \tag{13.48}$$

reaches the 100% value for $r = 0.5$, as predicted by quantum theory, and decreases to zero for $r = 0$ or 1. These results are completely independent of the properties of the beam; it may be chaotic, coherent or a single-photon beam.

The assumption that the physically realizable states of light can always be described with a positive Wigner function is considered so fundamental, that it has led to a redefinition of stochastic optics as quantum optics in the Wigner representation, with the Wigner function taken as a true probability density. This postulate in combination with the demand of nonnegative probability densities implies the need to make some modifications in the quantum theory of detection, which in its present form involves negative probabilities when formulated in the Wigner representation.

13.4. The Bell inequalities and their optical tests

A considerable fraction of the effort on stochastic optics has been invested into the study of the optical experiments designed to test local realism by searching for violations of the Bell inequalities. The reason for such a sustained effort can be understood by considering that the conceptual framework of SED rests heavily on a realistic and objective conception of nature, in which instantaneous actions at a distance are ruled out. The assertion, so widely spread in the specialized and popular literature, that the experiments (with correlated photons) have refuted all local realistic hidden-variables theories, disqualifies a priori a theory as stochastic electrodynamics [see, e.g., Milonni 1994, p. 294]. As this affects directly the whole enterprise —not to say its motivations and conceptual foundations— the question becomes for SED one of principle, and its appropriate clarification demands full attention.

Strictly speaking the issue of the alleged experimental refutation of local realism is of concern not only to those interested in stochastic electrodynamics. By questioning the compatibility of our physical theories with philosophical realism, such a refutation transcends the particular field to which it technically pertains and can have vast consequences on all of contemporary science and on our conception of nature. It is not without captivation (and some trepidation) that we now hear of deep philosophical problems being settled in the laboratory.[19]

But whereas in a corner of modern research in physics, people insist that there are experimental proofs to the effect that realism is the wrong way to look at nature, most of contemporary philosophers continue to be seemingly unaware of such tests, which allegedly solve a millenary and central problem of philosophy, or, at least, of the philosophy of science. It could be argued that the real problem was born already with the advent of quantum mechanics itself, more than half a century ago, when physicists unveiled the essential nonseparability of the quantum world, and that the recent experiments have simply reaffirmed the old findings. Or in other words, that the psychological and philosophical impact of the quantum break is already a matter of history and the new experimental evidence brings no particularly significant changes to our world view.

In the case of quantum mechanics most physicists would accept that some of the surprising properties attributed to the quantum world are tightly dependent on the interpretation of the formalism, as was discussed in chapter 1. For instance, right from the beginning the EPR theorem was considered by the followers of the conventional interpretation as a demonstration of an essential wholeness of nature, whereas for their authors it conclusively demonstrated the incompleteness of the quantum description. And so, the most fundamental questions of interpretation have been a matter of unending discussion and controversy, well acknowledged and documented in both the scientific and philosophical circles (and frequently abused in the popular literature, which seems to be prone to the most extravagant and extreme stretchings of the current interpretation). It is argued, however, that the present-day situation is different in principle, because the modern experiments have been designed to test the Bell inequalities, which should be satisfied by *any* local realistic (hidden-variables) theory, in total independence from the specific details, so that there is no room for interpretative premises to fill any gap. They are the final test, and experiment has shown that quantum mechanics *is* incompatible with local realistic theories. So, we are told, the question has been settled once and for ever and the reason for the long-standing discussions has, finally, reached an end.

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[19] Comments of this sort stressing the intertwining of science and philosophy, can be found scattered in the literature regarding the Bell inequalities. As examples see Jarrett (1989) and de la Torre (1992), p. 85.

If this conclusion were true, the orthodox nonlocal and unrealistic interpretation of conventional quantum mechanics would really have captured the essential (and highly unconventional) gears of nature, and our sole remaining task would be to learn to live with them and take them as they are; to ask only questions that can be answered by the present theory and refrain from asking silly questions about its underpinnings. And then, efforts such as those reported in the present book would be altogether superfluous.

But things are not so simple, or we would be living in a schizophrenic world. And it is not just the schizophrenia between its macro- and micro-levels, since whereas the Bell practitioners theorize about the nonlocalities of the quantum world, the field-theoreticians continue to demand micro-causality (a concept not far from that of locality as defined in chapter 1) as a fundamental principle of the same quantum world. Thus, the effort made within stochastic optics to evaluate the real meaning and extent of the experiments designed to test local realism, is most fundamental for the elucidation of central aspects of present-day theoretical physics.

13.4.1. THE BELL INEQUALITIES

Bell's inequalities, which grew out of a variant of the EPR model, have acquired a towering importance, largely because despite the fact that they impose tight conditions on certain quantities accessible to experimental test, they are derivable from very simple and general theoretical assumptions. Of course, this is how things look in theory; in practice they are less clear, not only when it comes to the experimental testability of the inequalities, but also to some extent in what refers to the unrestrained validity of their premises. The literature on the subject is extremely vast, comprising hundreds of monographs and thousands of research papers, besides the countless popular expositions.[20]

There is a wide variety of Bell inequalities, derived from different assumptions about hidden variables, locality, causality, and so on. Adding to the prevailing confusion about their essential meaning, there are formal derivations of the inequalities that make no explicit appeal at all to the physical notions of realism, locality, and the like [see, e.g., Szabó 1994 and references therein]. Nevertheless, it is usually granted that any local-hidden-variables theory (of both the deterministic and the stochastic varieties) should satisfy them, whereas they are violated by both (nonrelativistic) quantum mechanics *and* experiment. So, it is concluded, the whole class of local hidden-variables theories has been definitively ruled out. The purpose of the discussion to follow is to show that things are much less clear and conclusive than is normally assumed, and that much work is still needed to reach a final conclusion.

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[20] We cite a small (and somewhat arbitrary) selection. In addition to the obligatory collection of papers by Bell himself (1987), some relevant monographs in which different points of view are expressed are: Belinfante (1973), Clauser and Shimony (1978), Selleri and Tarozzi (1981), d'Espagnat (1984), de Baere (1986), Mittelstaedt (1987), Redhead (1987), Ballentine (1988), Selleri (1988a, 1990), Tarozzi and van der Merwe (1988), Cushing and McMullin (1989), Home and Selleri (1991), Brody (1993) and Mermin (1993). The present exposition follows basically the treatments of Santos (1987a, 1992b, 1992f), Ferrero et al. (1990) and Marshall (1990).

The first impossibility proof (or so-called 'no-go' theorem) within quantum mechanics was von Neumann's (1932) theorem, which (apparently) establishes that it is impossible to construct a hidden-variables theory compatible with all the predictions of quantum mechanics. This result put a halt to the search for such theories for almost 20 years, until the reopening of the problem with the construction by Bohm (1952) of a hidden-variables model that reproduces quantum mechanics exactly. Later, a critical analysis by de Broglie (1953) drastically revalued von Neumann's result, and Feyerabend (1956) showed that it excludes only a restricted (and uninteresting) class of deterministic variables. However, it was John Bell (1966) who most contributed to clarify the subject, by extending the Feyerabend result and replacing von Neumann's theorem with one that excludes all theories containing *noncontextual* hidden variables, but allows *contextual* hidden variables (with no constraints on them) as always possible.[21] Further, Bell (1964) derived another theorem, this one excluding *local* hidden-variables theories from quantum mechanics.

Local realism

Consider the typical example of a compound system at rest with total angular momentum 0, which at time t_0 decays into two particles of spin 1/2 flying apart into widely separated regions of space I and II. There are two adjustable Stern-Gerlach apparatuses that measure the spin projections along given directions a and b at times t_I in region I and t_{II} in region II, respectively. The times of measurement are such that $|t_I - t_{II}| < R/c$, where R is the distance separating I and II, so that no causal influence can propagate between the particles during the actual measurement. Each measurement gives a pair of numbers $A(a), B(b)$, with possible values +1 or -1. In practice, most of the tests of Bell's inequalities have been made using a pair of photons from atomic sources (mainly atomic cascades and, more recently, parametric down conversion in nonlinear crystals); in such experiments the polarization correlation of photon pairs is measured. In all experiments except one [Hold and Pipkin 1974], the quantum predictions were verified and the Bell inequalities violated.

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[21] Consider a variable $A(\lambda)$, where λ represents the set of hidden variables that describe the subquantum- or micro-state and A may depend on both the quantity under measurement and the measuring apparatus. In a contextual theory it is assumed that the result of the measurement can be influenced by other apparatuses acting simultaneously, or in more general terms, by the *context* $C = \{\mu\}$; the contextual variables are then of the form $A(\lambda, \mu)$. Observe that the context includes, in particular, all observables being simultaneously measured, with no regard to the spatial separation between the measurements.

The inexistence of noncontextual hidden variables compatible with quantum mechanics was first proved by Gleason (1957) and independently by Kochen and Specker (1967); that contextual hidden variables are always possible was also demonstrated by Gudder (1970). A detailed analysis of the relationship between noncontextuality and separability, and between the last forms of Bell's theorem (the Greenberger-Horne-Zeilinger variant) and the Gleason and Kochen-Specker theorems, can be found in Mermin (1993).

We first discuss some properties of the quantities involved from the standpoint of local realism, to derive afterwards the Bell theorem. According to the notion of realism used in the EPR context (and discussed in section 1.4), physical systems possess definite properties (though not necessarily known), independently of their being observed. This idea is clearly posed in Einstein's famous question: Is the moon there when nobody looks at it? A measurement reveals a preexisting value of the measured property, even if the outcome may be affected by the measuring device. This realist view opposes the conventional quantum doctrine, according to which the outcome of a measurement is brought into being by the act of measurement itself (the Heisenberg transition from the possible to the actual).[22] Thus, if the set of variables λ specifies the state of the system at the moment of decay t_0, and the measuring apparatus contains the uncontrollable but pertinent set of variables μ (including *all* variables that are associated with the environment and thus define the experimental context), formally the realist expression for the result of a measurement is

$$A = A(a, \lambda, \mu). \tag{13.49}$$

The set of variables λ represents the objective properties of the particle prior to the measurement and includes the elements of reality of the EPR paper, although nowadays they are usually assimilated into the generic (and inappropriate) name of hidden variables (see §1.1.3). Notice the difference between the elements of reality (assumed to exist before the measurement) and the experimental results, such as A, which depend in principle on both the system and the measuring instrument. It is possible to suppress the reference to the environmental variables and write in place of equation

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[22] The discussion refers to preexisting *sharp* values; the objectivity of simultaneous unsharp enough values of noncommuting observables is compatible with quantum mechanics, as shown in Bush and Schroeck (1989).

(13.49)

$$P_A = P_A(a, \lambda), \tag{13.50}$$

where P_A is the probability of obtaining the value A when the measurement is made. To be more general and play the sure side, one may include contextual uncontrollable variables in this last expression and write

$$P_A = P_A(a, \lambda, \mu). \tag{13.51}$$

The basic quantity of interest in the Bell inequalities is the joint probability of obtaining a value A for a measurement on the particle in region I and state λ and a value B for its partner, also in state λ, with control parameters a and b, respectively, and in a given context specified by μ. Notice that the state variables λ are not being separated into two subsets λ_I and λ_{II}, because they may be common to both subsystems and simultaneously influence both results. Thus, if $w(\lambda, \mu|a, b)$ is the probability of λ and μ conditioned by a, b, the joint probability is

$$P_{AB}(a, b) = \int d\lambda d\mu P_A^I(a, \lambda, \mu) P_B^{II}(b, \lambda, \mu) w(\lambda, \mu|a, b). \tag{13.52}$$

Now, since the variables λ exist prior to any measurement, their distribution cannot depend in principle on the parameters a, b, μ that specify the measurements and the context, so that the demand of realism implies that $w(\lambda, \mu|a, b)$ can be written in the form $w(\lambda, \mu|a, b) = \rho(\lambda) r(\mu|\lambda, a, b)$, whence

$$P_{AB}(a, b) = \int d\lambda d\mu P_A^I(a, \lambda, \mu) P_B^{II}(b, \lambda, \mu) \rho(\lambda) r(\mu|\lambda, a, b). \tag{13.53}$$

Any theory that can be modelled using this joint probability is a *theory of contextual hidden variables.*

As a next step we introduce the requirement of locality (see section 1.3). An ambiguous locality condition would be that all influences decrease with distance so that actions at far distances have negligible effects; a more fundamental notion of locality is that an action at a place has no influence on events that are space-like separated from it. Since in the Bell experiment the measurements are carried out in principle in spatially separated regions,[23] locality is taken to mean that the measurement on I cannot be influenced by the control parameter b nor by the contextual parameters of

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[23] The experimenter indeed would like to make the measurements in space-like regions, although practical limitations frequently prevent the attainment of this goal. Only the Aspect et al. (1981, 1982a, b) experiments are widely recognized as satisfying this requirement, but even this belief is not entirely free of controversy.

II, and analogously for the measurement carried out on II, so that the set μ separates into two subsets μ_I, μ_{II} leading to the further factorization:

$$r(\mu|\lambda, a, b) = r_I(\mu_I|\lambda, a) r_{II}(\mu_{II}|\lambda, b). \tag{13.54}$$

Substituting in (13.53), we have

$$P_{AB}(a, b) = \int d\lambda Q_A^I(a, \lambda) Q_B^{II}(b, \lambda) \rho(\lambda) \tag{13.55}$$

where

$$Q_A^I(a, \lambda) = \int d\mu_I P_A^I(a, \lambda, \mu_I) r_I(\mu_I|\lambda, a), \tag{13.56}$$

and analogously for $Q_B^{II}(b, \lambda)$. The theories in which equation (13.55) holds are conventionally called *local realistic theories* [Clauser and Shimony 1978]; they are frequently considered to define *local realism*, or at least to constitute a necessary condition for any local (realistic) hidden-variables theory. In its turn, equation (13.54) is frequently taken as the definition (or condition) of *locality*.

Note that equation (13.55) is written in a time-independent form, under the assumption that events I and II are spatially separated. In the language introduced in section 1.3 this means that instead of locality we should speak of local action, or of separability. In particular, when the events in I and II are spatially separated —and the experimenter is supposed to guarantee this—, correlations due to direct (causal, subluminal) influences between measurements are excluded, so equation (13.55) is supposed to express the correlation between the outcomes A and B —through their respective probabilities— as established in their joint past. We will soon verify that quantum mechanics is incompatible with the structure of equation (13.55), which explains why it has been said that this theory is incompatible with the assumption of a common cause [Santos 1987a; Szabó 1994], although it would be better to say that quantum correlations do not fulfil transitivity as strictly as classical ones do [Santos 1986].

The Bell inequalities and Bell's theorem

To derive a Bell inequality we use the procedure of Clauser and Horne (1974). Observe that for any four numbers A, B, C and D in the interval $[0, 1]$,

$$AC - AD + BC + BD \leq B + C. \tag{13.57}$$

Now in place of these numbers we insert the probabilities Q_A^I, etc., and thus get

$$Q_A^I(a, \lambda) Q_B^{II}(b, \lambda) - Q_A^I(a, \lambda) Q_B^{II}(b', \lambda) + Q_A^I(a', \lambda) Q_B^{II}(b, \lambda)$$

$$+ Q_A^I(a',\lambda)Q_B^{II}(b',\lambda) \le Q_A^I(a',\lambda) + Q_B^{II}(b,\lambda). \tag{13.58}$$

Multiplying by $\rho(\lambda)$ and integrating over λ, this gives

$$P_{AB}(a,b) - P_{AB}(a,b') + P_{AB}(a',b) + P_{AB}(a',b') \le P_A(a') + P_B(b), \tag{13.59}$$

with $P_{A,B}(a) = \int d\lambda Q_{A,B}(a,\lambda)\rho(\lambda)$ and $P_{AB}(a,b)$ given by (13.55). This is a Bell inequality; it is a mathematical assertion that must be unconditionally fulfilled by any local hidden-variables theory.[24]

Bell's theorem in its turn establishes that all models based on a $P_{AB}(a,b)$ given by (13.55) are in contradiction with quantum mechanics [see, e.g., Mermin 1993]; this is proved as follows. Consider an atomic-cascade experiment, such as those realized by Aspect et al. (1981, 1982a, b) in which a jet of atoms is sent across a laser beam, so that some of the atoms become excited and decay by emitting within a short time interval two photons of wavelengths corresponding to green and blue, say, that travel in opposite directions. Sometimes each photon goes in the right direction to cross a polarizer and be detected at a photomultiplier. The coincidences within a sufficiently small time window are counted to determine the correlation as a function of the relative orientation of the polarizers, measured by the angle φ. These correlations can also be calculated, and the quantum prediction is [Clauser and Shimony 1978]

$$P_{AB}(\varphi) = \frac{1}{4}\eta_A\eta_B f\left(\varepsilon_A\varepsilon_B + F\varepsilon'_A\varepsilon'_B\cos 2\varphi\right). \tag{13.60}$$

For the single probabilities of detection, the prediction is

$$P_A = \frac{1}{2}\eta_A\varepsilon_A g_A, \qquad P_B = \frac{1}{2}\eta_B\varepsilon_B g_B. \tag{13.61}$$

In these expressions $\eta_{A,B}$ are the efficiencies of the detectors and the constants $\varepsilon_{A,B}$ characterize the polarizers; $g_{A,B}$ are geometric factors that determine the probability of a photon entering the collecting lens system (essentially, the aperture in steradians divided by 4π); f is a similar geometric factor determining the probability that the two light signals are collected by the respective lenses (essentially, g^2), and, finally, F is a depolarizing factor. In a typical experiment, $\eta \simeq 0.15$, $0.9 \lesssim \varepsilon' < \varepsilon < 1$, $g \simeq 0.1$, $f \simeq 0.01$, $0.95 \lesssim F \lesssim 0.99$.

Now consider an ideal atomic-cascade experiment, with $\eta = \varepsilon' = \varepsilon = g = f = F = 1$. The quantum prediction gives

$$P_A = P_B = 1/2, \qquad P_{AB}(\varphi) = \frac{1}{4}\left(1 + \cos 2\varphi_{AB}\right), \tag{13.62}$$

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[24] There are of course generalizations of the Bell inequalities to arbitrary spin and other variables and correlations; see, e.g., Garuccio and Selleri (1980), Cetto et al. (1985), Mermin (1986).

and Bell's inequality (13.59) becomes

$$\cos\varphi_{AB} - \cos\varphi_{AB'} + \cos\varphi_{A'B} + \cos\varphi_{A'B'} \leq 2. \tag{13.63}$$

It is easy to see that there are consistent choices of the four angles which violate this inequality; a conventional example is $\varphi_B - \varphi_A = \varphi_{A'} - \varphi_B = \varphi_{B'} - \varphi_{A'} = \pi/8$.

13.4.2. EXPERIMENTAL TESTS OF LOCAL REALISM

The above result explains the extended belief that local realism is incompatible with quantum mechanics. To cite Mermin (1994), "Bell proved that no assignment of preexisting properties could agree... with quantum mechanics." Moreover, since several experiments have reported violations of Bell inequalities, and these constitute a necessary condition for the existence of an underlying local hidden-variable theory, the experiment has refuted all such theories. However, this has been shown not to be the case by the explicit construction of ad hoc counterexamples based on (13.55) that nevertheless reproduce the predictions of quantum mechanics. A simple one is the following [Ferrero et al. 1990]. Assume λ to be a couple of angles λ_A, λ_B in $[0, \pi]$ and take

$$\rho(\lambda_A, \lambda_B) = \frac{1}{\pi^2}\left[1 + F\cos(2\lambda_A - 2\lambda_B)\right], \tag{13.64}$$

$$Q_K(\varphi_K, \lambda_K) = \begin{cases} \eta_K \varepsilon_K g_K / 4\beta_K, & \text{if } |\lambda_K - \varphi_K| \leq \beta_K \text{ or } |\lambda_K - \varphi_K \pm \pi| \leq \beta_K \\ 0, & \text{otherwise} \end{cases} \tag{13.65}$$

for $K = A, B$, and $g_A g_B = f$. Substituting in (13.55), the result agrees with the quantum correlation provided one sets

$$\frac{\sin 2\beta_K}{2\beta_K} = \frac{\varepsilon'_K}{\varepsilon_K}. \tag{13.66}$$

To recover equation (13.61) it suffices to set

$$P_K = \int d\lambda Q_K(\infty, \lambda)\rho(\lambda), \qquad Q_K(\infty, \lambda) = \frac{1}{2}\eta_K \varepsilon_K g_K. \tag{13.67}$$

The model is of course ad hoc and not intended to reproduce all quantum mechanical predictions; it is appropriate only for β_K big enough as to guarantee that Q_K is not greater than 1, which is however the case for the experimental values of the parameters and low detection efficiencies in nearly all the performed experiments.[25]

There exist other local realistic models for this experiment which agree with the quantum predictions, such as that suggested by Pascazio (1988), or those studied by Santos (1992d, 1995c), for which the agreement is total for the Aspect experiments, even assuming ideal setups and 100% efficient polarizers and detectors. The mere existence of such models shows that local realism has not been refuted by those experiments that confirm the quantum predictions.

It is important to understand the source of these contradictory claims, and how the models escape the conclusion that they should disagree with the quantum predictions, at least at high efficiencies. As an example, let us take the Santos model just cited and the Aspect experiments. In the experiments only the subensemble of pairs of photons such that both photons enter the apertures is considered; for 100% efficient detectors, all pairs in this 'passed subensemble' would be detected, and the figures reported would imply violation of the corresponding Bell inequality. An analysis of the local realistic model shows, however, that it does not allow defining such 'passed subensemble', so that the above construction fails. Indeed, the model is based on a purely wave theory of light, in which photons would be represented by wavepackets or the like, but not by indivisible particles; thus, its photons may partially enter the apertures, and the 'passed subensemble' does not exist. This observation has led Santos (1995b) to conclude that, contrary to the usual claims, it is the *corpuscular* local hidden-variables theories of light that have been refuted by Aspect's experiments, unless there is a conspiratorial behaviour of nature.[26]

Homogeneous and inhomogeneous Bell's inequalities

But there is more. Experiments testing Bell's theorem inequality (13.59), have actually never been performed; what has been tested are Bell inequalities of another species, called *homogeneous* (by Marshall and Santos) or *strong* (by Selleri). Such inequalities, first proposed by Clauser et al. (1969), were conceived to overcome the problem of the low detector efficiency. Indeed, the substitution of (13.60) and (13.61) into (13.59) gives a result that depends on the detection efficiencies, and it can be shown that no violation of the resulting inequalities can be attained with efficiencies below $2(\sqrt{2}-1) = 0.82...$ [Garg and Mermin 1987]. It was therefore proposed to replace the original (genuine, inhomogeneous or weak) Bell inequalities with their stronger relatives,

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[25] The agreement with quantum mechanics breaks down for the Holt and Pipkin (1974) experiment —the only one that has given results contrary to the predictions of quantum mechanics— because the value of β becomes too small. Details can be seen in Ferrero et al. (1990); see also Marshall and Santos (1989b) and Ferrero and Getino (1994).

[26] The conspiracy would mean that quantum mechanics works nicely for present-day inefficient set-ups, but will severely fail with the advent of more efficient set-ups.

$$\begin{aligned} P_{AB}(a,b) - P_{AB}(a,b') + P_{AB}(a',b) + P_{AB}(a',b') \\ \leq P_{AB}(a',\infty) + P_{AB}(\infty,b). \end{aligned} \tag{13.68}$$

The only difference lies in the probabilities $P_{AB}(a',\infty)$ and $P_{AB}(\infty,b)$ replacing $P_A(a)$ and $P_B(b)$, respectively, on the right-hand side; the ∞ means that the corresponding polarizer has been removed. Since all terms refer now to coincidence counts, the relation is efficiency-independent. This is the inequality that has been the subject of empirical tests and has been shown to be violated by quantum systems. It is known under the name of (second) CH inequality for the initials of its authors [Clauser and Horne 1974], and is similar to the CHSH inequality [Clauser et al. 1969], the difference being that the latter is written in terms of correlations instead of probabilities.

However, there is a further problem: to arrive at the homogeneous inequalities and to eliminate other functions that cannot be measured directly, the relation

$$Q_K(a,\lambda) \leq Q_K(\infty,\lambda) \tag{13.69}$$

was assumed to hold. This relation, known as the *no-enhancement assumption*, expresses the belief that the insertion of the polarizer cannot increase the probability of photodetection. The derivation of homogeneous Bell inequalities always requires the use of this or some other additional assumption concerning properties of the detection efficiencies as the polarizers are rotated or removed [Lepore and Selleri 1990]. Unfortunately, the extra assumption cannot itself be subject to direct empirical verification; thus one accepts it as being 'highly plausible' [Clauser and Shimony 1978], but its introduction certainly restricts the family of hidden–variables theories that are excluded by a violation of the inequality. When an experiment indicates a violation of equation (13.68), it is now uncertain whether this is due to a violation of (13.69) or of local realism. The hypothesis of no-enhancement is so much weaker and less general than the hypothesis of local realism, that it is more reasonable to consider (13.68) as a test of the former rather than the latter. No violation of a genuine (inhomogeneous) Bell inequality (valid for the whole family of local hidden-variables theories) has been reported until now. This gives support to the claim by some authors that local realism has not been refuted by experiment [see, e.g., Ferrero et al. 1990] and that atomic-cascade experiments are not suitable to test local realism [Clauser and Horne 1974, Santos 1987a, 1992d, Lepore and Selleri 1990].

To the above one should add that the quantum predictions for coincidence and single probabilities for photon pairs emitted by a 0-1-0 atomic cascade do not violate the inhomogeneous Bell inequality for any detector admittance, even with the efficiencies taken equal to 1 [Clauser and Horne 1974]. This is due to the three-body nature of the process: the recoil of the atom can use up enough momentum as to make the angle between the wave vectors of a photon pair differ from π so much as to appreciably reduce the polarization correlation and avoid any violation of the Bell inequality.[27]

About the reality of enhancement

For someone who 'holds absolutely fast' [Einstein 1949] to local realism, the conclusion drawn from the last section is that what cascade experiments show is that enhancement is a real phenomenon. It is therefore interesting to study this problem in some detail. To get a feeling of the possible origin of enhancement, recall the anticoincidence experiment sketched in figure 13.4. The beam splitter generates an entangled state of vacuum and one-photon states and the corresponding transmitted (b channel) and reflected (c channel) electric fields are

$$E_t = \frac{1}{\sqrt{2}}\left[E_1 + iE_0\right], \qquad E_r = \frac{1}{\sqrt{2}}\left[E_0 + iE_1\right]. \tag{13.70}$$

The intensity in the transmitted beam, say, given by

$$I_t \equiv |E_t|^2 = \frac{1}{2}\left[I_0 + I_1 + 2\mathrm{Re}\,(E_1 E_0^*)\right], \tag{13.71}$$

may be greater than the intensity I_1 of the incoming beam, for small relative phases between E_0 and E_1, since for one-photon states I_0 and I_1 are of equal order of magnitude. Basically the same analysis applies to a polarizer, so that it suffices to assume that the detection probability increases monotonically with the incoming intensity, to have an explanation of enhancement.[28]

[27] Although the three-body character of the problem was already recognized in Clauser and Horne (1974), the two-body approximation continues to be used [see, e.g., Walls and Milburn 1994, section 14.2]. Detailed discussions of some largely ignored aspects, such as the directional correlation mentioned here or the spatial evolution of the wave function, can be seen in Santos (1992d, 1995a). In Santos (1995c) it is shown that if the filters for the signal and idler photons are wide enough to insure a high probability of joint detection, the resulting loss of spatial coherence destroys entanglement, thus preventing the test of local realism. There is also a recent proposal for an experiment in which the loophole of the decreased angular correlation is addressed by detecting the atomic recoil [Huelga et al. 1994a, b]. For another loophole-free test, yet in an entirely different context, see Kwiat et al. (1994).

[28] More detailed discussions can be found in Marshall and Santos (1988a, b, 1989a), Marshall (1990, 1991b), Marshall et al. (1994b) and Ferrero and Marshall (1991b).

Let us now look at enhancement from the viewpoint of quantum optics.[29] The entangled two-photon state for the double emission in a calcium cascade $(J=0) \rightarrow (J=1) \rightarrow (J=0)$, in the limit when the collecting lens has zero aperture, is described with the matrix density $\rho = |\Psi\rangle\langle\Psi|$, where

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\hat{a}^{\dagger}_{1x}\hat{a}^{\dagger}_{2x} + \hat{a}^{\dagger}_{1y}\hat{a}^{\dagger}_{2y}\right]|0\rangle\,; \tag{13.72}$$

the indices 1,2 label the photons propagating on the z-axis and x, y refer to polarization along the xz- and yz- planes. To calculate the probabilities and correlations, it is convenient to use the Husimi Q function (see equation (13.21)) corresponding to this state; it is [Marshall 1990]

$$\rho_Q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = \frac{1}{2\pi^4}\left|\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2\right|^2 \exp(-\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_1^* - \boldsymbol{\alpha}_2 \cdot \boldsymbol{\alpha}_2^*), \tag{13.73}$$

where the vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ are the sets of Stokes parameters associated with the two beams. For the probabilities $Q^{I,II}$ (see equation (13.56)) one gets

$$Q^I(a, \boldsymbol{\alpha}_i) = \eta\left(\boldsymbol{\alpha}_{1a} \cdot \boldsymbol{\alpha}_{1a}^* - 1\right), \tag{13.74}$$

$$Q^I(\infty, \boldsymbol{\alpha}_1) = \eta\left(\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_1^* - 2\right), \tag{13.75}$$

and similarly for $Q^{II}(b, \boldsymbol{\alpha}_2)$, where the action of the polarizer has been represented by an orthogonal transformation of the modes; for example,

$$\alpha_{1a} = \alpha_{1x}\cos a + \alpha_{1y}\sin a \tag{13.76}$$

and so on. Further, the joint probability becomes

$$P(a, b) = \int d^4\alpha_1 d^4\alpha_2 Q^I(a, \boldsymbol{\alpha}_1) Q^{II}(b, \boldsymbol{\alpha}_2)\rho_Q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2). \tag{13.77}$$

We see that the Husimi Q function (13.73), used as a probability density for the problem, describes a pair of correlated systems, which agrees with what was stated in section 1.3, in the sense that entangled wave functions describe correlated systems. Further, the last formula formally coincides with that of local realistic theories, equation (13.55); here the hidden variables are the eight variables $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$ and locality appears as the separate action of the two polarizers at angles a,b on the two separate signals, $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$. However, the description is not local realistic, since $Q^I(a, \boldsymbol{\alpha}_1)$ can have negative values and hence cannot be interpreted as a probability. Further, from equations (13.74) and (13.75) we see that for values of the parameters such that $-\alpha_{1x}\sin\theta + \alpha_{1y}\cos\theta$ is close to zero, $|\boldsymbol{\alpha}_1|^2 \simeq |\boldsymbol{\alpha}_{1a}|^2$ and

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[29] A quantum optics study of the cascade experiments to test Bell's inequalities is given in Walls and Milburn (1994), chapter 14

$Q^I(a, \alpha_1) > Q^I(\infty, \alpha_1)$, which would mean enhancement, were it not for the possible negative values of these functions, which of course reinforces the doubts on the robustness of such interpretation.

13.4.3. ASSESSMENT OF THE PROBLEM OF QUANTUM THEORY AND LOCAL REALISM

It should not come as a surprise to find that quantum mechanics and local realism as encoded by equation (13.55) are incompatible; quantum mechanics contains nonlocalities and negative probabilities, in full contradiction with the intended contents of the definitions (13.54) and (13.55). This was just the kind of observation that led Bell to his inequalities, and it explains why purely formal probabilistic considerations allow us to derive such inequalities (in the form of probabilistic relations that violate quantum assertions) without the need to introduce any explicit consideration on locality, causality and so on.[30] However, the very fact that quantum mechanics is inconsistent with (13.55) opens a real problem by challenging fundamental notions such as locality, individuation of physical objects, and so on; we cannot be satisfied until its full elucidation is achieved. As we have just seen, despite the widespread belief on the contrary this issue is far from settled and, depending on personal convictions, several distinctive attitudes can be identified, some of which we briefly comment upon.

a) Quantum mechanics, which has shown itself to be an impressive and faithful description of nature, violates the Bell inequalities (without qualification). This, which is the most widespread opinion, implies that whatever is fundamental for the Bell inequalities, meaning local realism, should be abandoned as a misrepresentation of nature. This 'solves' one of the most long-standing problems of philosophy by dispatching local realism to the graveyard, at the price of opening the door to the spooky actions at a distance and leaving us with no picture of reality.[31]

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[30] The Bell inequalities can be shown to be a particular case of the condition that a correlation vector has a Kolmogorovian representation [Szabó 1994 and references therein]; in other words, they can be related to some metric inequalities well known in the theory of measure [see, e.g., Suppes and Zanotti 1981]. They thus turn out to be necessary conditions for the existence of a joint probability distribution of the observables (simultaneously measurable or not) and can be derived from the mere assumption of the existence of this distribution, without any appeal to locality. This observation has led to the statement that the Bell inequalities are irrelevant as regards physical questions on locality or causality. It can be replied to this that it is just locality which demands the existence of the joint probability distribution. Note that properties such as negative probabilities are immediate evidence of the non-Kolmogorovian structure of quantum theory.

[31] As experience shows, this opens the door also to an abusive disclosure of the magics and mystics of the quantum world in the popular press which, by the way, is becoming highly profitable, particularly among the followers of eastern philosophy. That this is practically the only opinion that has reached the popular press, seems to point towards an interesting subject of study for the sociology of science.

b) Local realism has not been correctly encoded. This is a complex subject, the full discussion of which would require in itself a whole chapter, so we restrict ourselves to some few illustrative comments.[32]

Various criticisms have been raised of the definitions (13.54) and (13.55) for locality and local realism, respectively. As examples, we recall that (13.55) has been considered too restrictive and not necessary for locality to hold [Popper 1988], and that the definition (13.54) of locality has been criticized on the basis that it is stronger than required to derive equation (13.55), assumed to be correct [Shimony et al. 1976]. Further, the form of the correlation (13.55) crucially depends on the factorizability condition (13.54); the existence of even one single common μ would destroy the factorizability and with it the validity of (13.55). Such is the case, for instance, if an intrinsic stochasticity is assumed to exist; hence, condition (13.55) is necessary for deterministic theories, but only sufficient for theories that are (intrinsically) indeterministic at the subquantum level.

An argument of a more general nature was advanced by Brody (1993, §16-18). For any bivariate distribution $p(x,y)$, an infinite family of conditional distributions can be shown to exist, such that it can be expressed in the form

$$p(x,y) = \int d\mu(z) q(x|z) r(y|z), \tag{13.78}$$

where the probabilities q and r and thus μ can depend on any set of parameters. Since the form (13.78) is always possible, factorizability itself cannot constitute a locality condition, and it cannot discriminate between local and nonlocal situations.

A further argument is that the transition from (13.57) to (13.59) is not innocuous, since it implies that for a given λ one may know all four values a, a', b, b'; this introduces the *joint-measurability assumption*, that is, the assumption that the spin projection of a particle can be measured in more than one direction without mutual intrusion [Brody 1993, §§16, 17, Brody and de la Peña 1979].[33]

c) Quantum mechanics must be modified. A local realist who accepts (13.55) as an appropriate definition, is obliged to conclude that quantum mechanics is in need of changes. Strictly speaking, probably nobody would reject this statement as a matter of principle and within a historical perspective, but there will certainly be disagreement on the reasons adduced and the kind of changes considered. Even if it may sound heretical to many, small modifications of present-day quantum mechanics are actively being explored as one of the ways to solve the decoherence problem of conventional quantum theory [see, e.g., Ghirardi 1993 and references in §1.4.2]. Further, a causal model with a rudimentary property of locality that 'acceptably' violates present-day quantum mechanics has been studied by Eberhard (1989). Of course, any change should be such as to preserve the empirical and practical integrity of the theory while at the same time correcting the undesired features (e.g., those contrary to local realism). This can give rise to big changes in the interpretation, however small the quantitative changes in the predictions (as the traditional example of classical mechanics and special relativity shows). Some of the early works on stochastic optics contain careful attempts to substantiate this line of research [see, e.g., Marshall and Santos 1988, 1989].

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[32] Rich and quite different critical analyses of the conditions of realism and locality are made by Redhead (1987) and Brody (1993). An analysis of Bell's inequalities as being independent of locality is given in Fine (1982). Other various studies of the concept of locality used in the Bell context are cited in Ballentine (1988). The factorizability condition has been discussed and critizised in detail by several workers, among them de Baere (1988), Garuccio (1988), and Selleri (1988b).

[33] Since the lack of joint measurability is a common situation, most often occurring for properties which are affected in a fundamental way by the measurement, there should be classical situations in which violations of corresponding Bell inequalities are predicted; accordingly, several examples have been provided [Scalera 1984, Notarrigo 1984, Brody 1993].

d) The scope of quantum mechanics must be revised. This is a variant of (c) above, in which the changes refer not to the formalism, but to a restriction on its range of applicability (scope). Though not normally appreciated as such, the definition of the scope of a theory constitutes one of the most important activities of scientific research.[34] In particular, a modification of the range of quantum mechanics can be envisaged, so as to make it compatible with local realism whilst preserving its present excellent agreement with experiment. Conventional quantum mechanics postulates that all vectors in Hilbert space correspond to physical states and all self-adjoint operators represent observables (superselection rules aside); Santos (1992b, d, f) has conjectured that fundamental constraints might exist in nature which entail a restriction on the subsets of quantum states and observables that are to be considered physical, and that these do not contradict local realism.[35]

e) The correspondence between physical variables and their quantum counterpart must be revised. In chapters 10 and 11 it was seen that the quantum variables, represented by operators, are quite abstract objects, that do not directly describe the properties of the given q-state; they encompass the whole set of possibilities for the system, not merely the actual motions. There seems to be no strong reason to worry about these entities

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[34] An illuminating discussion on this subject can be found in Brody (1993), chapter 5.

[35] These observations have been presented by Santos in the form of a research program. The fact that no experiment has discriminated up to now between quantum mechanics and the whole family of local hidden-variables theories means that the conjecture has not been disproved. The possibility of subjecting it to empirical test is discussed in Santos (1992f).

—mere artifacts for an approximate, partially-averaged description— not satisfying the locality condition (13.54).

This listing is not intended to be exhaustive. For instance, Bohm's interpretation of quantum mechanics is a variant of (a) that implies an explicit renunciation of Lorentz invariance, to allow for the quantum nonlocalities.[36] Further, there is the possibility to revise the concept of reality itself; an example of this extreme solution is the many-worlds interpretation originally proposed by Everett (1957) [see, e.g., Everett et al. 1973]. However, we are not going to enter into these matters, since they lie outside the scope of the present work.

Dictum (a) above and SED are incompatible, of course. But as to the remaining listed comments, they all contain possibilities that are still alive and should not be discarded by reasons of personal conviction or desire. All one can conclude at present is that the question is not yet settled; it is difficult even to ascertain whether we are at the beginning or at the end of the road. But the complexity of the path toward the solution means that we have much to learn in the search.

13.5. Description of the zeropoint field in Hilbert space

We have seen that the Wigner function is conjectured in stochastic optics to be the appropriate phase-space probability density for the radiation field. On the other hand, the Wigner distribution is obtained within the quantum formalism through the use of Weyl's rule (of total symmetrization) for the ordering of the operators.[37] Weyl's rule is uniquely determined within quantum mechanics by the sole requirements of Galileo invariance, unitarity, reality and normalization [Agarwal and Wolf 1970], which make it an attractive tool for quantum theory.

As mentioned in §4.3.4, Boyer has shown that under Weyl's correspondence rule all moments of the zeropoint field variables as predicted by SED for the Gaussian case coincide with those of QED. However, even if the two descriptions lead to such equivalent results, fundamental differences remain as long as one of the theories continues to be formulated in terms of a (c-number) stochastic field and the other in terms of noncommuting observables. For instance, recall that the use of the Wigner function has been suggested as a means to filter what is thought to correspond to physical states (with positive probability densities) from the set of all possible quantum states. If no further restrictions are imposed, quantum theory is by far richer than its stochastic counterpart, due to the non-equivalence of the different orderings.[38] The only possible way to put both theories into close correspondence is by reducing the number of physically significant variables, which is achieved by fixing an order for the observables in the quantum theory. Let us study in more detail the correspondence between the description of the Gaussian zeropoint field of SED and the quantum field of QED, under the Weyl rule of correspondence.[39]

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[36] The point is not whether the nonrelativistic theory violates Lorentz invariance, which of course everybody expects; but even its hypothetical relativistic version will contain nonlocal properties, i.e., noncausal connections between spatial events.

[37] Specifically, Weyl's rule puts all operators in symmetric order, so that, for instance, the equivalent of the stochastic expression $a_1 a_2^*$ is $\frac{1}{2}\left(\hat{a}_1 \hat{a}_2^\dagger + \hat{a}_2^\dagger \hat{a}_1\right)$, the equivalent of $a_1 a_2 a_3^* a_4^*$ becomes a sum of 24 terms, and so on (refer to footnote 23 in chapter 7).

13.5.1. FORMAL EQUIVALENCE OF THE STOCHASTIC AND QUANTUM DESCRIPTIONS FOR GAUSSIAN FIELDS

Our approach is based on Santos (1974) [see also Gadella and Santos 1981], but we choose a rather informal presentation. Let us consider the stochastic field described by the set of random amplitudes $\{a_1, a_2, \ldots, a_n^*\} \equiv A$, as is done in chapter 4; for simplicity we treat it as finite (this will be the case with a finite normalization volume and a cutoff; the limiting procedures to obtain the continuous case with no cutoff may be applied afterwards). We assume that the dynamical variables can be expressed as power series in the amplitudes, and that their expectation value is calculated as usual,

$$\langle f \rangle = E\left[f(A)\right]. \tag{13.79}$$

The purpose is to construct a mapping of the algebra of the stochastic variables onto a Hilbert-state space $\mathcal{H}$, associating with each state $E\left[\cdot\right]$ a (cyclic) normalized state vector $|\psi\rangle$ in $\mathcal{H}$, so that one can write for any variable f,

$$E\left[f\right] = \langle\psi|\, \hat{f}\, |\psi\rangle\,, \tag{13.80}$$

where $\hat{f}$ is an operator in $\mathcal{H}$, the image of f. Further, the complex conjugation of the stochastic variables is taken to correspond to Hermitian conjugation in $\mathcal{H}$. The linear mapping implies that sums of stochastic amplitudes transform into the corresponding operator sums; for products one must take, according to the previous discussion,

$$E\left[A^r A^{*s}\right] = \langle 0|\, S\left(\hat{a}^r \hat{a}^{\dagger s}\right) |0\rangle\,, \tag{13.81}$$

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[38] The counterargument has been advanced, namely that because SED predicts specific values for all correlations, whereas in QED the correlations may attain any desired value as long as a specific ordering is not postulated (which would be essentially arbitrary), the former is the most general of the two.

[39] At the end of chapter 4 another correspondence was studied, namely between the classical and the normally ordered quantum correlations. A detailed study of this formal equivalence from the point of view of classical and quantum optics is made in Peřina (1985), chapter 14.

where S is the operator that symmetrizes with respect to each $\hat{a}$ and $\hat{a}^\dagger$ introduced in equation (13.25), and $|0\rangle \in \mathcal{H}$ is the vector associated with the given probability state (distribution) of the random variables. Observe that the condition (13.80) may be possible for some probability distributions, but not for others; and conversely, not all vectors in $\mathcal{H}$ will correspond to probability distributions of the random variables, because positivity in the set of functions of the stochastic variables does not correspond in general to positivity in the set of operators in $\mathcal{H}$.

We establish the correspondence for random variables with a Gaussian distribution $P(A)$, taking the variance equal to $1/2$ for simplicity, so that (cf. equations (4.73) and (4.75))

$$E\left[A^r A^{*s}\right] \equiv \int dP A^r A^{*s} = \frac{r!}{2^r}\delta_{rs}, \tag{13.82}$$

whence

$$\langle 0|\, S\left(\hat{a}^r \hat{a}^{\dagger s}\right) |0\rangle = \frac{r!}{2^r}\delta_{rs}. \tag{13.83}$$

Although we already know from quantum theory how to achieve this, it is instructive to construct a suitable representation space step by step. Consider first the case $r = s$. The factor $r!$ may be interpreted either as the number of permutations of the r objects $\hat{a}$, or as the number of ways in which the r operators $\hat{a}$ can be paired with the r operators $\hat{a}^\dagger$ in $\hat{a}^r\hat{a}^{\dagger r}$ without changing the order of the factors. For instance, for $r = 2$ there are two different possible pairings $\hat{a}_\llcorner \hat{a}^\ulcorner \hat{a}^\dagger_\lrcorner \hat{a}^{\dagger\urcorner}$, $\hat{a}_\llcorner \hat{a}^\ulcorner \hat{a}^{\dagger\urcorner} \hat{a}^\dagger_\lrcorner$; for $r = 3$ there are six of them, and so on. Equation (13.83) will then be satisfied (for $r = s$) if each one of the $r!$ paired terms contributes to the average by an amount $1/2^r$. This can be achieved by assigning the expectation value 1 to the terms in which every operator $\hat{a}$ is to the left of its partner $\hat{a}^\dagger$, and the expectation value zero otherwise; only one out of every 2^r paired terms will thus survive. For example, out of the 24 terms to which the symmetrization of $\hat{a}^2\hat{a}^{\dagger 2}$ gives rise, the only three nonvanishing terms contribute thus with

$$\langle 0|\, S\left(\hat{a}^2\hat{a}^{\dagger 2}\right) |0\rangle = \frac{1}{6}\left(\hat{a}_\llcorner \hat{a}^\ulcorner \hat{a}^\dagger_\lrcorner \hat{a}^{\dagger\urcorner} + \hat{a}_\llcorner \hat{a}^\ulcorner \hat{a}^{\dagger\urcorner} \hat{a}^\dagger_\lrcorner + \hat{a}_\llcorner \hat{a}^\dagger_\lrcorner \hat{a}^\ulcorner \hat{a}^{\dagger\urcorner}\right) = \frac{1}{2} = \frac{2!}{2^2}, \tag{13.84}$$

in agreement with (13.83) for $r = s = 2$. It can be verified that the symmetrization rule is the only one leading to this result.

An equivalent form of expressing the above rules is by stating that the expectation of any product that begins with $\hat{a}^\dagger$ or ends with $\hat{a}$ is zero, because this means there is at least one pair with the wrong operator order. One is thus led to the rules

$$\hat{a}\,|0\rangle = 0, \qquad \langle 0|\, \hat{a}^\dagger = 0 \tag{13.85}$$

that define the vacuum vector (state) $|0\rangle \in \mathcal{H}$.

Now let $\hat{M}_1$ and $\hat{M}_2$ be two arbitrary ordered monomials in the $\hat{a}$ and $\hat{a}^\dagger$, and observe that

$$\langle 0| \hat{M}_1 \hat{a}\hat{a}^\dagger \hat{M}_2 |0\rangle = \langle 0| \hat{M}_1 \hat{a}^\dagger \hat{a} \hat{M}_2 |0\rangle + \langle 0| \hat{M}_1 \hat{M}_2 |0\rangle . \tag{13.86}$$

Indeed, the nonzero paired terms on the left-hand side are those contained in the first bracket on the right-hand side, plus those in which the product $\hat{a}\hat{a}^\dagger$ is paired, these being just the ones given by the last bracket. This equation holds for any $\hat{M}_1$ and $\hat{M}_2$, hence it is equivalent to the boson commutation rule

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1. \tag{13.87}$$

It can now be verified that the solution also holds good for $r \neq s$. In fact, in this case either an operator $\hat{a}$ can be moved to the right or an $\hat{a}^\dagger$ to the left by repeated application of the commutation rule, which gives only vanishing terms. It is also possible to generalize to any number of pairs of statistically independent Gaussian variables (a_i, a_i^*), $i = 1, 2, \ldots$; the solution is the same as above, with an index i attached to the corresponding operators $\hat{a}_i, \hat{a}_i^\dagger$, and

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij}, \qquad [\hat{a}_i, \hat{a}_j] = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0, \tag{13.88}$$

$$\hat{a}_i |0\rangle = 0, \qquad \langle 0| \hat{a}_i^\dagger = 0. \tag{13.89}$$

The new Hilbert space is the tensor product of the spaces $\mathcal{H}_i$.

Now suppose that in addition to the zeropoint field a certain amount of external radiation is present; for concreteness we consider the case of a single plane wave with parameters $b, \mathbf{k}_0, \omega_0, \sigma_0, \delta$ (real amplitude, wave vector, frequency, polarization and phase, respectively, in units such that b is a number similar to the a for the corresponding zeropoint field mode). This field is described exactly as before, except for the mode with additional radiation, whose operator becomes

$$\hat{a}_{k_0\sigma_0} \longrightarrow \hat{c}_{k_0\sigma_0} = \hat{a}_{k_0\sigma_0} + be^{i\delta} \equiv \hat{a}_{k_0\sigma_0} + \alpha \tag{13.90}$$

with $\alpha = be^{i\delta}$. A simpler description can be attained by changing to a new set of operators $\hat{a}'_{k\sigma}$, defined by

$$\hat{a}'_{\mathbf{k}_0\sigma_0} = \hat{a}_{\mathbf{k}_0\sigma_0} + be^{i\delta}, \qquad \hat{a}'_{\mathbf{k}\sigma} = \hat{a}_{\mathbf{k}\sigma} \quad (\mathbf{k} \neq \mathbf{k}_0, \sigma \neq \sigma_0) . \tag{13.91}$$

The primed operators still satisfy the commutation relations (13.88), but instead of (13.89) they satisfy the relations

$$\hat{a}'_{\mathbf{k}\sigma} |\alpha\rangle = 0 \quad \text{if } \mathbf{k} \neq \mathbf{k}_0,\ \sigma \neq \sigma_0, \qquad \hat{a}'_{\mathbf{k}_0\sigma_0} |\alpha\rangle = \alpha |\alpha\rangle \tag{13.92}$$

and their Hermitian conjugates. We have relabelled the vector $|0\rangle$ as $|\alpha\rangle$; it is convenient to simultaneously define a new vector $|0'\rangle$ such that for *all* **k** and σ,

$$\langle 0'|\,\hat{a}'^{\dagger}_{\mathbf{k}\sigma} = 0, \qquad \hat{a}'_{\mathbf{k}\sigma}\,|0'\rangle = 0. \tag{13.93}$$

The new operators and states describe a coherent state of (complex) amplitude α. A more formal procedure to arrive at the same results consists in making the transition from the unprimed description to the primed one by means of a unitary transformation; the unitary operator that does the trick is just the displacement operator $D(\alpha)$ such that [see, e.g., Walls and Milburn 1994, section 2.3]

$$|\alpha\rangle = D(\alpha)\,|0\rangle\,, \quad D^{\dagger}(\alpha)\hat{a}D(\alpha) = \hat{a} + \alpha, \quad D(\alpha) = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^{\dagger}}e^{-\alpha^*\hat{a}}, \tag{13.94}$$

in agreement with the previous results.

The result suggests considering the superposition of the zeropoint field and any nonrandom free field, expressed as a superposition of plane waves, as representing a pure state in stochastic optics, because it corresponds to a coherent state in quantum optics [Gadella and Santos 1981]. More generally, in stochastic optics all superpositions of coherent states with nonrandom coefficients correspond to pure states.

We may benefit from this procedure to relax to some extent the requirement of a Gaussian distribution. Since the above equivalence continues to hold after performing an arbitrary unitary transformation in Hilbert space (corresponding to a canonical transformation of the random variables), if the transformed basis is not linearly related to the old one, the transformed distribution is not Gaussian any more. However, under such a transformation both the commutation relations and the total symmetrization of the observables are preserved, so that these two properties can be taken as the general rules for the correspondence with Wigner functions.

For many applications it is more convenient to use the basis of number states, defined as

$$|n\rangle = \frac{1}{\sqrt{n}}\hat{a}^{\dagger n}\,|0\rangle\,, \tag{13.95}$$

instead of the nonorthogonal and overcomplete set of coherent states; it follows that

$$\langle r|s\rangle = \frac{1}{\sqrt{r!s!}}\,\langle 0|\,\hat{a}^{r}\hat{a}^{\dagger s}\,|0\rangle = \delta_{rs}. \tag{13.96}$$

Of course, these number states closely correspond to the q-states of the harmonic oscillator discussed in chapter 7; all comments made there (and elsewhere in the book) about their meaning and realizability apply here, *mutatis mutandis.*

13.5.2. DISCUSSION ON THE QUANTUM REPRESENTATION

It has thus been verified again that under the symmetrization rule, the quantum and the stochastic description of the Gaussian field are equivalent. This correspondence brings the two theories conceptually closer under the appropriate restrictions, but it does not assign discrete properties to the random field; it would perhaps be better to say that it serves to elucidate the stochastic nature of the quantum field. As the equivalent representation is applicable under the symmetrization rule only, rules valid for the usual Hilbert-space formulation of quantum mechanics but leading to contradictions under the symmetrization rule, hold no more. For instance, for a general dynamic variable Q, $Q \rightarrow \hat{Q}$ does not imply necessarily $Q^n \rightarrow \left(\hat{Q}\right)^n$. This explains apparently odd results, such as why there is no contradiction with the quantum results even though in the stochastic theory the ground state has a distributed energy (recall the similar discussion with regard to the harmonic oscillator in §7.4.1). The correct new rule, $Q^n \rightarrow S\left(\hat{Q}\right)^n$, leads to expectation values that may be quite different from those of $\hat{Q}^n$, particularly in the case of eigenstates of $\hat{Q}$.

The Hilbert-space formalism appears in this theory as a very efficient tool to deal with stochastic fields that satisfy linear equations of motion. But to reduce the passage to the quantum description to a mere formal step, devoid of any physical content, would be an oversimplification, the more so as only the free field has been considered here and all interaction with matter has been neglected. For example, the description of the field in terms of operators allows the use of different orderings, each one particularly suited to describe certain processes: normal order for absorption, antinormal for emission processes, symmetrical order for the description made in the present section, and so on, as was discussed in some detail in §13.3.1 in relation with the quantum distributions P, Q and W. So, laying the bridge between the two theories should not be a merely formal exercise, but it should be full of physical content, most of which is yet to be disclosed; this is particularly true with regard to such difficult questions as the meaning and nature of the photon, which remain basically as mysterious as ever. A more thorough analysis of this core problem is the main subject to be discussed in the concluding chapter.

CHAPTER 14

AN OUTLOOK AND SOME COROLLARIES

The theory thus far presented gives good support to the proposition that quantum phenomena of matter find their genesis in the zeropoint radiation field. Most important is the fact that the connection derived from the postulates of linear SED is only approximate and involves stochastic variables over which significant operations such as partial averagings and the like have been performed, which explains why an initially local and genuinely statistical theory eventually leads to a description much less stringent with respect to such (and other) physical properties, as is conventional quantum mechanics. Further, the principles of SED seem to allow the opening of a window into the undulatory behavior of matter, by taking into consideration the modulation wave produced by the Doppler shift of stationary carrier waves linked to the particle through radiation reaction.

We recall that two major problems were left open in the chapters regarding matter quantization. One is the need to resort to a new principle in order to extract information about the behaviour of the system from the original dynamical equations, as discussed in chapter 10. As already stated, we do not see a means for its solution within present-day physics, so we are forced to leave it aside. The second problem refers to an inconsistency —the schizophrenia discussed at the end of chapter 11— in the sense that such fine phenomena as the radiative corrections require a description of the zeropoint field different from that used to derive the zero-order behaviour of atomic matter. We now address this problem.

14.1. Is there a bridge connecting SED and QED?

Even though QED and SED are based on quite different principles, they are supposed to refer to the same portion of reality, each one in its own language. A most pertinent question is therefore whether it is possible to build a bridge to connect SED and QED, and whether it is needed.

From a pragmatic point of view, the expected response to the latter question would be: why worry at all about trying to establish such a connection, if QED is already in itself a fine theory which hardly requires any revision. However, from a more fundamental perspective, the reason for our endeavour would be at least threefold: on one hand, if quantum mechanics represents indeed a coarse, radiationless description of the mechanical part of the particle-field system in the quantum regime, QED should represent a more precise and complete description of the SED system in the same regime; any proof in this sense would be a crucial test for the whole scheme of SED. Secondly, in trying to construct the bridge that takes us to the domain of QED we should learn to identify the bricks that are needed for it, and thus become aware of any element or hypothesis that may be still lacking in SED for a more complete understanding of the quantum phenomenon. (This, at least has been a lesson learned in the passage from SED to quantum mechanics, as discussed mainly in chapters 10-12.) Last but not least, it is well recognized that the enormous merits of QED do not save it from deep conceptual problems, some of which even become manifest at the calculational level, as is the case e.g., with the divergencies;[1] our hope would be that the approach of SED could contribute to the elucidation of such kind of problems, and perhaps also to a definition of the domain of validity of the conventional quantum theory.

Now from our brief survey of stochastic optics we have learned about the difficulties of discriminating empirically at present between the formulations of optics in terms of random or quantum fields, on the basis of a certain family of problems, which only adds to the impossibility of extracting at present a definitive conclusion on these matters and makes the situation more uncertain. So the best we can do at this point is look in detail at the results obtained up to now, in an effort to get a better understanding of their meaning, implications and potentialities.

We recall that, according to linear SED, the stationary bound quantum states of the atomic system are determined by the k-averaged field, a field having certain components correlated among themselves and also to matter motions, and thus differing significantly from the free field. Alternatively, one can say that the particle performs a kind of coarse-graining or averaging over a large number of modes with nearly the same frequency. As a result, the description does not apply any more to a single realization, but to an ensemble of equivalent particles, governed in the quantum regime by equation (10.34), or (for each Cartesian component)

$$m\frac{d^2\tilde{x}_{\alpha\beta}}{dt^2} = \tilde{F}_{\alpha\beta}(t) + m\tau\frac{d^3\tilde{x}_{\alpha\beta}}{dt^3} + e\tilde{E}_{\alpha\beta}(t). \tag{14.1}$$

This equation, translated into matrix form and taken in the radiationless approximation, is just one of the possible forms of Heisenberg's equations of motion, namely,

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[1] As clearly expressed by Dirac himself, in one of his latest criticisms of the conceptual aspects of quantum theory [Dirac 1984], "just because the results (after renormalization) happen to be in agreement with observation does not prove your theory is correct". A detailed introductory discussion can be found in Pais (1986).

$$m\frac{d^2\hat{x}}{dt^2} = \hat{F}(\hat{x}(t)). \tag{14.2}$$

Now coming back to equation (14.1), its direct transcription into the operator language becomes

$$m\frac{d^2\hat{x}}{dt^2} = \hat{F}(\hat{x}(t)) + m\tau\frac{d^3\hat{x}}{dt^3} + e\hat{E}(t), \tag{14.3}$$

where $\hat{E}(t)$ is an appropriate operator representing the zeropoint radiation field. The problem with this equation is that, despite the fact that it is clear how to write a formal expression for $\hat{E}(t)$ using a perturbative approach to linear SED, it is unclear which is the correct field to be inserted here, since the matrix form given by equation (14.2) strictly speaking corresponds to just the zero-order theory, without the field. Indeed, in our previous work we have had opportunity to use at least three different forms for this field, each one within its own context that excludes the others. Let us comment briefly upon each one of these theories as an aid to our search for an answer to this question.

14.2. Comparison of the different theories considered

Radiative corrections in chapter 11. The calculations of chapter 11 led to a determination of atomic corrections (to the lowest required order) that pertain to QED, without the need to quantize the field. However, it is easy to see that this procedure cannot be followed indefinitely, due precisely to the poor description of the field used to perform the calculations.

When applying equation (14.1) to calculate perturbatively the corrections to the quantum mechanical equation (14.2) in chapter 11, the field $E(t)$ used there was the free zeropoint field, i.e., a field absolutely independent of the presence of matter. This selection was seen to be consistent with the requirement

$$\left\langle \dot{\mathbf{x}}^{(0)} \cdot \mathbf{E} \right\rangle = 0 \tag{14.4}$$

expressing the absence of correlation between the variables $\dot{\mathbf{x}}^{(0)}$ and $\mathbf{E}$. From a physical point of view this is quite understandable, since it means that the radiative corrections of interest are produced by all those field components that do not belong to the restricted class of 'relevant frequencies' and whose effect was totally neglected during the construction of the zero-order solution in chapter 10. However, it is clear that the relevant correlated field modes may also play some role in the determination of higher-order corrections, so that the present description should be considered only as an approximation, perhaps as a kind of first-order iteration of the complete theory. In such a hypothetical complete theory the field $E(t)$ cannot any more be just the free zeropoint field, but it must in some form contain the more complete information. In other words, neither the free field nor the correlated zero-order zeropoint field can be the definitive candidates to occupy the place of the exact $\hat{E}(t)$.

The Hilbert-space representation of the zeropoint field in chapter 13 (and also in §4.3.4). In section 13.5 it was shown that there exists a formal representation of the Gaussian random field in terms of an operator-valued field in a Hilbert space, under the Weyl-Wigner correspondence rule, with amplitudes satisfying the usual boson commutation rules

$$\left[\hat{a}_{\mathbf{k}\sigma}, \hat{a}^{\dagger}_{\mathbf{k}'\sigma'}\right] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}. \tag{14.5}$$

This quantum field was taken to be just an alternative description of the Gaussian zeropoint field under the appropriate restrictions, so that its introduction was considered a formal device to describe simply and efficiently the effects of the zeropoint field, assumed to retain its classical (c-number) nature for ever.

This viewpoint was reinforced with the study of some optical problems that are conventionally taken as definitive evidence of the quantum nature of light, and thus of the unavoidable need of the operator description for the zeropoint field, but which find a natural nonquantum explanation, subject to a couple of postulates concerning the interaction of matter and light, when the (active) presence of the zeropoint field is duly taken into account. However, the fact that stochastic optics still lacks of a full description of the matter-field interaction and is (provisionally) taking the quantum properties of matter for granted, makes it difficult to define at present without ambiguity the field that will be required to accomplish a full theory, free of phenomenological assumptions and valid to all orders of approximation (in principle, of course).

Now, only if matter is also described in terms of c-numbers is the assumed classical (c-number) nature of the field in stochastic optics consistent with the conclusion reached at the end of chapter 2, in the sense that due to the permanent absorption and emission of radiation by the particle, both field and matter *must* have the same c- or q- nature. A situation of this kind was met, for instance, in *conventional* SED, where everything is described in terms of c-numbers. However, we are trying to discuss the system once it has reached the quantum regime, and, at least for the time being, we know how to do this in general only in terms of q-numbers.

So once more we find ourselves in an uncertain situation: the field *is* stochastic and most probably possesses continuous properties, even when interacting with matter. But this does not exclude a Hilbert-space description as an appropriate tool to take into account, at least approximately, the properties acquired by the field through its interaction with matter — which has become quantized by this interaction. Now, accepting the second-quantized field as a useful representation leads to the emergence of another contradiction, because this step has sense only when matter is already quantized. Thus one must abandon the idea that matter quantization is due to the interaction with the field, and take it as a given, primitive phenomenon (which is a step backwards, conceptually speaking), so that one ends up with nonrelativistic QED to start with.

The equilibrium field studied in chapter 5. In chapter 5 it was found, without using any notion of discontinuity, that the spectrum $\rho(T)$ of the radiation field in equilibrium with matter at a given temperature T follows a Planck distribution, if at zero temperature there is a fluctuating field with $\rho(0) \sim \hbar\omega^3$. To arrive at this result only very general statistical considerations were needed. Thus, as was the case with matter in chapter 10, the quantum properties of the field emerge as a consequence of the field fluctuations at zero temperature; conventionally (and historically), however, this result is considered to prove that the equilibrium field *is* quantized.

The discussion was somewhat extended in chapter 5 with the result that other typically quantum aspects of the field, such as the directed momentum associated with atomic transitions, or the fundamentals of Compton dispersion, were understood as consequences of the zeropoint field. In more general terms, the 'particle' term in the Einstein fluctuations formula (5.35) was explained within the continuous stochastic framework as arising from the additional fluctuations due to the interference between the vacuum and the thermal component of the field.

From this picture it follows that the equilibrium field is not any more the free field, but one modified by its continuous interaction with matter. We have found that the differences can be expressed, in the approximation leading to usual quantum mechanics, by means of the modified statistical properties of the field components, such as those studied in chapter 10, which should be conveniently extended to arbitrary temperature. On the other hand we have the usual quantum formalism, which accounts for the difference by means of the quantization of the field. Thus one finds oneself in an ambivalent situation very similar to that discussed in relation to stochastic optics; apparently one must take quantization for granted, but the random source is there.

The metaclassical theory of Sokolov and Tumanov in chapter 2. In this theory, by construction, $\hat{E}(t)$ stands for the second-quantized radiation field, whereas initially the particle is described by classical variables which as a result of the interaction with the field become operators satisfying the usual commutation rule $[\hat{x}, \hat{p}] = i\hbar$.

This result seems to lead very close to nonrelativistic QED expressed in terms of the Heisenberg equations of motion.[2] However, despite its simplicity (or perhaps, due to it) the method is not devoid of difficulties. An obvious one is the lack of an equivalent theory for the general non-linear case. A second one is the abstract nature of the procedure and unclearness of its empirical meaning; the second-quantized field encloses already the quantum mysteries one is attempting to unravel. A major technical difficulty seems to be that all operators act on the same Hilbert space, whereas in the usual quantum description field and particle have their own Hilbert space, and the equation of motion operates on the product space. The common space implies, in particular, correlations between the field and particle variables that strongly affect the behaviour of the latter; for instance, there is in principle no reason to expect that detailed balance will hold in the general nonlinear case or to assume that particle and field operators commute at equal times. Thus, despite the appearances, we have not arrived at QED, even if the field is second-quantized.

There is still another difference between the two theories, of a more fundamental conceptual nature. In the usual quantum theory, both quantization of matter and of field are primitive and enter into the theory on an equal footing. In the theory of Sokolov and Tumanov, in contrast, the field produces the quantization of matter, which thus becomes a derived (or acquired) property. Indeed, it seems more appropriate to consider that this theory leads to a *second* quantized theory of the elementary matter oscillators, rather than to a first quantization, as was assumed in our discussion of chapter 2 (and also by its authors and elsewhere), since the nonquantized field leads by itself to a (first) quantization of matter.

Incidentally, a remarkable feature of the Sokolov-Tumanov theory is that it is a local-realistic hidden-variables theory from the standpoint of the particle, since the latter obeys a causal equation, the interaction is local and the field determines uniquely the variables of the particle. The nonrealistic behaviour characteristic of the quantum description comes from the fact that the particle is connected to an operator-valued field, so that no values can be assigned to the variables, neither of the field nor of the particle. This is a kind of variable (able to destroy preexistence, and even existence as they do here) not considered in the discussions of hidden-variables theories within the usual Bell context, the nearest thing to them being the intrinsically stochastic hidden variables (those which can destroy factorizability) that were taken into account in section 13.4 with the introduction of the set of variables μ.

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[2] Equation (14.3) with $\hat{E}(t)$ representing the quantized electromagnetic field is a well-known (approximate) form of the Heisenberg equations of motion in nonrelativistic QED, including the radiation force [see, e.g., Milonni 1994]. Early versions of it for the harmonic oscillator are given in Renne (1971) and Boyer (1975b), and a more general discussion within QED is that of Dalibard et al. (1982). A very early precursor of (14.3) not including the radiation field, and thus representing an attempted modification to quantum mechanics to take into account radiation reaction, appears in Fermi (1927).

14.3. Looking for the missing links

The mere replacement of the stochastic field $E(t)$ by its quantum version gives rise to two opposing results. On one side, a fundamental theory in which quantization is a derived property, is replaced by another one that assumes a primitive quantization. However, this loss is somehow compensated by the fact that to accomplish full consistency, matter should be second-quantized (there is actually no reason to stop after only the first loop in a series of successive approximations), and this process incorporates into the description a *new* element, namely the statistics obeyed by matter. In other words, far from being a mere formal step, the substitution of the continuous random field $E(t)$ by its quantized counterpart $\hat{E}(t)$ implies deep changes in the description of the system, including a full account of the statistics.

This observation points towards a major missing link, one whose absence is apparently obstructing the development of a finer description of the quantum regime, namely, the Pauli principle. Formally, one could guarantee its enforcement by resorting to a second-quantized description of the particle, which follows automatically from the introduction of the quantized field, as just said; since this step is consistent with Planck's law, every piece of the puzzle fits into place, and the task is accomplished ... assuming one is ready to pay for it the price of the highly abstract formulation, the obscure empirical meaning of the 'new' description, and all the associated conceptual difficulties. In brief, we conclude that for the relation between SED and QED to be established in more precise terms it is necessary for the former to find an answer to the perplexing question of the Pauli principle, which, for the time being, seems to be far away.[3]

In addition to the automatic account of the statistics, there is another important meaning to be attributed to a second-quantized electron theory. While in the quantum-mechanical description $E = \int d^3x \psi^* \hat{H} \psi$ is a nonrandom number, the corresponding quantity in the second-quantized version leads to a much finer description in which additional fluctuations due to the random behaviour of the electrons are allowed. For instance, for the total energy within the normalization volume one gets, with $\hat{a}_\beta \equiv \hat{a}_{\mathbf{k}\lambda}$, $\Psi = \sum \psi_\beta \hat{a}_\beta$ and $\hat{H}_0\psi_\beta = E_\beta\psi_\beta$,

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[3] In fact, quantum statistics is another subject that has received almost no attention in SED. The discussion in §12.4.4 suggests, however, that two electrons belonging to the same system and in the same state of motion (including spin) would be associated with the same de Broglie modulation wave (including polarization), and could thus have indirect interactions leading to the exclusion principle. This opens up the interesting possibility to understand the Pauli principle —and more generally, quantum statistics— as another result of the fine-scale effects of the zeropoint field on matter; no wonder, then, that in quantum mechanics this principle must be introduced *ad hoc*.

$$\hat{H} = \int d^3x \Psi^\dagger \hat{H}_0 \Psi = \sum_\beta E_\beta \hat{a}^\dagger_\beta \hat{a}_\beta, \tag{14.6}$$

which is clearly a dispersive quantity. This shows that the second-quantized theory restores part of the fluctuations that were lost in the k-averaging process. Now the only constant energy is the average over the realizations of the electron field, itself transformed into an operator (or a random field, in the language of stochastic theory).

A final consideration as regards the formal transition $E(t) \longrightarrow \hat{E}(t)$ concerns condition (14.4) (the earlier equation (9.4)), demanding that the 'new' field $\hat{E}(t)$ be uncorrelated from the zero-order atomic motions. This demand leads to yet another of the postulates of nonrelativistic QED, namely, that field and particle observables commute at equal times, each of these parts having its own Hilbert space. Another piece that fits the puzzle. We have now a free field, but one consistent with the whole series of properties of the system, including those acquired by the field through the interaction (quantization of matter, A and B coefficients, Planck's distribution, Compton rules, etc.). This formal process, however, is merely a balsam, unable to provide a cure for the whole pack of conceptual difficulties of quantum theory, of the kind discussed in chapter 1. Not only do these problems remain untouched in the 'new' description, but they are translated to the field domain, and new ones are generated with the uncritical transcription of the quantum rules of matter to the field and the insertion of the operator-valued field in the place that legitimately should correspond to the random equilibrium (co-determined) field.[4]

Translated into plain words, the (formal) transition $E(t) \longrightarrow \hat{E}(t)$ leading to QED cannot be taken as the definitive answer to the problem; on the contrary, it results in a theory of the FAPP species, which is useful to describe the *end* result but leaves aside important aspects of the underlying physics. It would be a mistake to discard the possibility that in the future a better theory of the quantum world will be formulated, one that avoids the undesirable properties of the present-day quantum description and simultaneously brings us closer to a satisfactory explanation of its underlying physics. The present work suggests that the tools provided by SED may help to unfold such a theory. For the moment, the least we have learned is that according to SED, the nonlocal and nonrealistic, sometimes almost magical, properties of our present-day theories are a result of the description, not a feature of nature.

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[4] A comment on the ubiquitous problem of the divergencies seems to be in place. Along the present work the need to consider the particle as endowed with some kind of structure has been noted once and again. The structure not only accounts for the cutoff required to avoid the divergencies that arise from point-like objects but, what is more important in the present context, it generates (in association with radiation reaction) the zitterbewegung-like oscillations that are a basic quantum feature of particles as explained in chapter 12. From a conceptual point of view this is in accord with present-day trends in field theory when attention is directed to strings as the fundamental objects of nature. Strings are of course the simplest fundamental way to introduce structure into the theory and avoid problems with the divergencies. (That string theory is simultaneously an attempt to construct a unified description of the different forces of nature, is incidental to our present purposes, even if it represents its main objective.)

14.4. Second-level theories

The matter-field interaction has been shown to lead to a very complex evolution of the system that transforms in a nontrivial way the initial 'classical' equations into their quantum counterpart. This is not just a matter of mere evolution of the forms, but the interplay between the parts of the system is such that the rules of their behaviour become qualitatively transformed.

We may give a picture of what we have in mind, by comparing the qualitative meaning of this process to that which took place billions of years ago on the surface of the earth. First there was an inanimate world, where physics and chemistry were absolute rulers. Suddenly, natural processes at random led somehow, somewhere, to an outbreak of life; from that moment, everything started to change, the inanimate world left its place to the animated one, and new phenomena, rules, laws and orders, the biological and biochemical ones, not subsumable into the original ones, emerged and took control of the situation in the biosphere. A higher-order world has appeared. Nature has built upon itself.

Something analogous seems to describe the evolution of the SED world: the initial classical laws leave their place to the quantum ones as soon as the space becomes occupied by the zeropoint field and conditions arise that lead to the possibility of reaching detailed balance. This nontrivial evolution results in a second-level theory emerging from first-level theories. Such a perspective helps to describe the complex process leading to qualitative changes in the behaviour of the constituents, and thereby understand the origins of the higher-level theory; but when the emphasis is to be put on the end result and not on the process itself, the emerging laws simply replace the original ones. Once the situation demanding a higher-level description takes place, going backwards to the primary elements would be equivalent to throwing away the baby with the bath water. It would be a poor approach to try to reconstruct in each instance the full process step by step, if there is the possibility of a synthetic approach to deal with the general case.

All this may sound almost naive, considering that the development of synthetic theories is one of the central tasks of science. For instance, within theoretical physics, classical statistical physics is a simple example of a second-level construct, in which the fundamental equations of classical mechanics have been replaced by statistical laws that are of an entirely different nature, but very apt to describe the elements of interest in the complex situations for which the theory is designed. Also and in a slightly different line of reasoning, we have the examples of well-established studies in modern thermodynamics, self-organization, morphogenesis, etc., or more generally speaking, of complex systems in which dramatic qualitative changes of structures occur, giving rise to new orders, concepts and even laws [see, e.g., Serra et al. 1986, Mainzer 1994].

It is well known that complexity (of which nonlinearity is an important ingredient) makes it impossible to analyze such systems by simply studying their separate parts and adding the descriptions; this, which is conventional wisdom in biology, sociology, etc. and even in philosophy, is however a nonreductionist view still quite uncommon within theoretical physics. To insist on this point therefore makes sense, since it is entirely alien to quantum theory, at least in the view of those who take it for the most primitive possible description of matter on the microscopic level, not derivable from anything else.

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